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Local Rules and Global Order, Or Aperiodic Tilings

Can local rules impose a global order? If yes, when and how? This is a philosophical question that could be asked in many cases. How does local interaction of atoms create crystals (or quasicrystals)? How does one living cell manage to develop into a pine cone whose seeds form spirals (and the number of spirals usually is a Fibonacci number)? Is it possible to program locally connected computers in such a way that the network is still functional if a small fraction of the nodes is corrupted? Is it possible for a big team of people (or ants), each trying to reach private goals, to behave reasonably?

These questions range from theology to "political science" and are rather difficult. In mathematics the most prominent example of this kind is the so-called *Berger theorem on aperiodic tilings* (exact statement below). It was proved by Berger in 1966 [1].[1] In 1971 the proof was simplified by Robinson [7], who invented the well-known "Robinson tiles" that can tile the entire plane but only in an aperiodic way (Fig. 1).

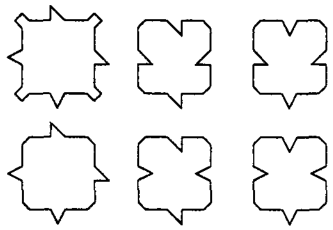

Fig. 1. The Robinson tiles [reflections and rotations are allowed].

Since then many similar constructions have been invented (see, e.g., [3, 6]); some other proofs were based on different ideas (e.g., [4]). However, we did not manage to find a publication which provides a short but complete proof of the theorem: Robinson tiles look simple, but when you start to analyze them you have to deal with many technical details. ("This argument is a bit long and is not used in the remainder of the text, so it could be skipped on first reading," says C. Radin in [6] about the proof.)

It's a pity, however, to skip the proof of a nice theorem whose statement can be understood by a high school student (unlike the Fermat Theorem, you don't even need to know anything about exponentiation). We try to fill this gap

[1]In fact, the motivation at that time was related to the undecidability of a specific class of first-order formulas, see [2].

and provide a simple construction of an aperiodic tiling with a complete proof, making the argument as simple as possible (at the cost of increasing the number of tiles).

Of course, simplicity is a matter of taste, so we can only hope you will find this argument simple and nice. If not, you can look at an alternative approach in [5].

Definitions

Let A be a finite (nonempty) alphabet. A *configuration* is an infinite cell paper where each cell is occupied by a letter from A; formally, the configuration is a mapping of type $\mathbb{Z}^2 \to A$. A *local rule* is an arbitrary subset $L \subset A^4$ whose elements are considered as 2×2 squares: $\langle a_1, a_2, a_3, a_4\rangle \in L$ is a square

a_1	a_2
a_3	a_4

We say that these squares are *allowed* by rule L. A configuration τ *satisfies* local rule L if all 2×2 squares in it are allowed by L. Formally this means that

$$\langle \tau(i,j),\ \tau(i+1,j),\ \tau(i,j+1),\ \tau(i+1,j+1)\rangle \in L$$

for any $i,j \in \mathbb{Z}$. A non-zero integer vector $t = (t_1,t_2)$ is a *period* of τ if the t-shift preserves τ, i.e.,

$$\tau(x_1 + t_1,\ x_2 + t_2) = \tau(x_1,\ x_2)$$

for any $x_1,x_2 \in \mathbb{Z}$.

The Aperiodic Tilings theorem

Theorem (Berger): *There exist an alphabet A and a local rule L such that*

(1) *there are tilings that satisfy L;*
(2) *any tiling satisfying L has no period.*

To prove the theorem we need some auxiliary definitions.

Substitution Mappings

A *substitution* is a mapping s of type $A \to A^4$ whose values are considered as 2×2 squares:

$$s\colon a \mapsto \begin{array}{|c|c|}\hline s_1(a) & s_2(a) \\ \hline s_3(a) & s_4(a) \\ \hline \end{array}$$

We say that a substitution s *matches* local rule L if two conditions are satisfied:

(a) all values of s belong to L;

(b) taking any square from L and replacing each of the four cells by its s-image, we get a 4×4 square that satisfies L (this means that all nine 2×2 squares inside it belong to L).

Remark. Consider a square X of any size $N \times N$ (filled with letters from A) satisfying L. Apply substitution s to each letter in X and obtain a square Y of size $2N \times 2N$. If the substitution s matches L, then Y satisfies L. Indeed, any 2×2 square in Y is covered by an image of some 2×2 square in X.

This is true also for (infinite) configurations: applying a substitution to each cell of a configuration that satisfies L, we get a new configuration that satisfies L (assuming that the substitution matches L).

Proposition 1. If a substitution s matches a local rule L, there exists a configuration τ that satisfies L.

Proof. Take any letter $a \in A$ and apply s to it. We get a 2×2 square $s(a)$ that belongs to L. Then apply s to all letters in $s(a)$ and get a 4×4 square $s(s(a))$ that satisfies L. Next is an 8×8 square $s(s(s(a)))$ that satisfies L, etc. Using a compactness argument, we conclude that there exists an infinite configuration that satisfies L.

Here is a direct proof not referring to compactness. Assume that substitution s is fixed. A letter a' is a *descendant* of a letter a, if a' appears in the interior part of some square $s(s(\ldots s(a)\ldots))$ obtained from a. Each letter has at least one descendant, and the descendent relation is transitive (if a' is a descendant of a and a'' is a descendant of a', then a'' is a descendant of a). Therefore, some letter is a descendant of itself (start from any letter and consider descendants until you get a loop). If a appears in the interior part of $s^{(n)}(a)$, then $s^{(n)}(a)$ appears in the interior part of $s^{(2n)}(a)$, which appears (in its turn) in the interior part of $s^{(3n)}(a)$, and so on. Now we get a increasing sequence of squares that extend each other and together form a configuration. (Here we use that a appears in the *interior* part of the square obtained from a.)

Proposition 1 is proved.

Now we formulate requirements for substitution s and local rule L which guarantee that any configuration satisfying L is aperiodic. They can be called "self-similarity" requirements, and guarantee that any configuration satisfying L can be uniquely divided (by vertical and horizontal lines) into 2×2 squares that are images of some letters under s, and that these pre-image letters form a configuration that satisfies L. Here is the exact formulation of the requirements:

(a) s is injective (different letters are mapped into different squares);

(b) the ranges of mappings $s_1,s_2,s_3,s_4 : A \to A$ (that correspond to the positions in a 2×2 square, see above) are disjoint;

(c) any configuration satisfying L can be split by horizontal and vertical lines into 2×2 squares that belong to the range of s, and pre-images of these squares form a configuration that satisfies L.

The requirement (b) guarantees that there is only one way to divide the configuration into 2×2 squares; the requirement (a) then guarantees that each square has a unique preimage.

Proposition 2. Assume that substitution s and local rule L satisfy requirements (a), (b) and (c). Then any configuration satisfying L is aperiodic.

Proof. Let τ be a configuration satisfying L and let $t = (t_1,t_2)$ be its period. Both t_1 and t_2 are even numbers. Indeed, (c) guarantees that τ can be split into 2×2 squares, and then (b) guarantees that the t-shift preserves these squares (since, say, an upper left corner of a square must go to another upper left corner).

Then (a) guarantees that pre-images of these 2×2 squares form a configuration that satisfies L and has period $t/2$. Therefore, for each periodic L-configuration with period t we have found another periodic L-configuration with period $t/2$. An induction argument shows that there are no periodic L-configurations.

Proposition 2 is proved.

Using Propositions 1 and 2 we conclude that to prove the Aperiodic Tilings theorem it is enough to construct a local rule L and substitution s matching L that satisfy (a), (b) and (c). This we now do.

Construction: An Alphabet

Letters of A are considered as square tiles with some drawings on them. We describe a local rule and substitution in terms of these drawings.

Each of the four sides of a tile

(1) is dark or light (has one of two possible *colors*);

(2) has one of two possible *directions*, indicated by arrows;

(3) has one of two possible *orientations*; this means that one of two possible orthogonal vectors is fixed; we say that this orthogonal vector goes "from inside to outside". (Our drawings show the orientation by a gray shading inside.)

In this way we get three bits per side, i.e., 12 bits for each tile. In addition to these 12 bits, a tile carries two more bits, so the size of our alphabet is $2^{14} = 8192$. These two additional bits are graphically represented as follows: we draw a cross (Fig. 2) in one of four versions (which differ by a rotation).

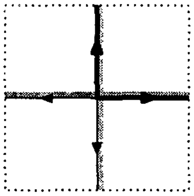

Fig. 2. One version of cross.

It is convenient to assign color, direction, and orientation to the segments that forming a cross. Namely, two neighboring sides of a cross are dark, the other two are light. The direction arrows go from the center outward, and the orientation is shown by a gray stripe that shows the "inside" part as indicated in the picture (gray stripes are inside the dark angle).

This will be important when we define the substitution.

Substitution

To perform the substitutions, we cut a tile into four tiles. The middle lines of the tile become sides of the new (smaller) tiles, with the same color, direction and orientation. Before cutting we draw crosses on the small tiles in such a way that the dark angles form a square as shown (Fig. 3).

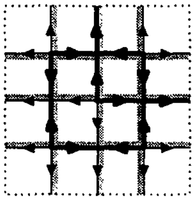

Fig. 3. A tile split in four parts.

It is immediately clear that conditions (a) and (b) of Proposition 2 are satisfied. Indeed, to reconstruct tile x from its four parts, it is enough to erase some lines, and the position of a tile in $s(x)$ is uniquely determined by the orientation of its central cross. The condition (c) will be checked later after the local rule is defined.

Local Rule

The local rule (L) is formulated in terms of lines and their crossings. There are two types of crossings that appear when tiles meet each other. First, a crossing appears at the point where corners of four tiles meet; crossing lines are formed by the tile sides. Second, a crossing appears at the middle of tile sides, where middle lines of tiles meet the tile side. First of all, the following requirement is put:

if two tiles have a common side, this shared side has the same color, direction, and orientation in both tiles.

Therefore, we can speak about the color, direction and orientation of a boundary line between two tiles without specifying which of the two tiles is considered.

We also require that

all crossings (of both types) are either crosses or meeting points. A *cross* is formed by four outgoing arrows that have colors and orientation as shown in Fig. 4 (up to a rotation, so there are four types of crosses). In a *meeting point*, two arrows of the same color, the same orientation, but opposite directions, meet "face to face," and the orthogonal line goes through this meeting point without change in color, direction, or orientation. One more restriction is put: if two dark arrows meet, then the orthogonal line goes "outward" (its *direction* agrees with the orientation of the arrows).

Our local rule is formulated in terms of restrictions saying which crossings are allowed when lines meet. Formally speaking, the local rule is a set of all quadruples of tiles where these restrictions are not violated. Fig. 4 shows the first type of allowed crossing, a cross, in one of four possible versions (which differ by a rotation). The second type

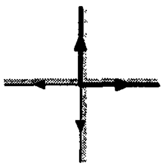

Fig. 4. A cross formed by outgoing arrows.

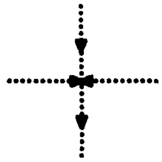

Fig. 5. Arrows meet.

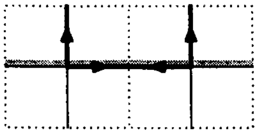

Fig. 7. Two neighbor tiles.

of allowed crossings (symbolically shown in Fig. 5) has more variations: (a) the meeting arrows can be horizontal or vertical; (b) the vertical line can have two orientations; (c) the horizontal line can have two orientations; (d) the vertical line can have one of two colors; (e) the horizontal line can have one of two colors; and finally (f) if two light arrows meet, the perpendicular line can go in either of two directions. So we get $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 48$ variations in this way.

Remark. The Local rule ensures that the orientation of any horizontal or vertical line remains unchanged along the whole line. (Indeed, the orientation does not change at crosses or meeting points.)

Substitution and Local Rule

We have to check that the substitution matches the local rule. Indeed, when tiles are split into groups of four, the old lines still form the same crossings as before, but new crossings appear. These new crossings appear (a) in the centers of new tiles (where new lines cross new ones) and (b) at the midpoints of sides of new tiles (where new lines cross old ones). In case (a) we have legal crosses by definition. In case (b) it is easy to see that two arrows meet creating a legal meeting point. See Fig. 6, which shows a tile split into four tiles, with all possible meeting places of new and old lines circled. The orientation matches because the orientation of the new crosses is fixed by s; all other requirements are fulfilled, too.

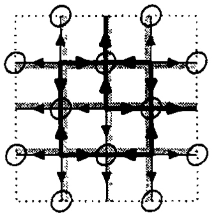

Fig. 6. New lines meet old lines.

Self-similarity Condition

It remains to check condition (c) of Proposition 2. Assume that we have a configuration that satisfies the local rule.

Step 1. Tiles are grouped by fours.

Consider an arbitrary tile in this configuration and a dark arrow that goes outward. It meets another arrow from a neighboring tile, and this arrow must be dark by the local rule. These two arrows must have the same orientation, therefore we get half of a dark square (Fig. 7), not a Z-shape. Repeating this argument, we conclude that tiles form groups of four tiles whose central lines form a dark square (Fig. 8).

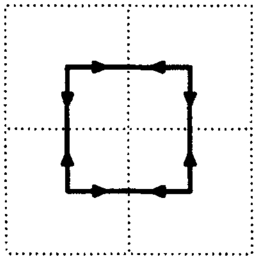

Fig. 8. Four adjacent tiles.

Step 2. These 2×2 squares are aligned.

If two groups (each forming a 2×2 square) were wrongly aligned, as shown in Fig. 9, then the orientation of one of the lines (in our example, the horizontal line) would change along the line (recall that all crosses have fixed orientation of lines). Therefore, 2×2 squares are aligned.

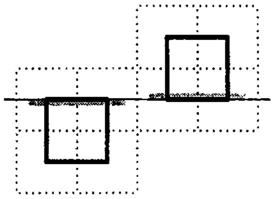

Fig. 9. Bad placement.

Step 3. Each group has a cross in the middle.

What can be in the group center? The middle points of the sides of the dark square are meeting points for dark arrows. Therefore, according to the local rule, an outgoing arrow should be between them. So a meeting point cannot appear in the center of a 2×2 group, and the only possibility is a cross.

Step 4. Uniform colors on sides.

To finish the proof that each group belongs to the range of the substitution, it remains to show that the color, direction, and orientation do not change at the midpoint of a side of a 2×2 group. This is because this midpoint is a meeting point for arrows perpendicular to the side.

Step 5. Pre-image tiles satisfy the local rule.

This is evident: the substitution adds new lines. So taking the pre-image just means that some lines are deleted, and no violation of the local rule can happen.

The Aperiodic Tilings theorem is proved.

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