Analyse récursive vue avec des fonctions réelles récursives

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Discrete Case

- There are several models for computation over integers
  - Recursive functions
  - Turing machines
  - Circuits
  - λ-calculus
  - ...
- But those models are "equivalent".

Church-Turing thesis

All reasonable powerful enough discrete models of computation compute exactly the same functions.

Approaches to analog computation

Several different devices

- Differential analyzer [Bush 31]
- Neural networks [Hopefield 84]
- Operational Amplifiers
- ...

Several different models:

- General Purpose Analog Computer (GPAC) [Shannon 41]
- Computable Analysis [Turing 36]
- BSS model [Blum Shub Smale 89]
- ...

However, contrarily to the digital case, few connections between these models are known.

How to compare such models

- Comparing two models that compute over $\mathbb{R}^k$:
  Two models are equivalent if they compute the same functions

  e. g. $\quad$ GPAC-computable $\Leftrightarrow \mathcal{R}ec(\mathbb{R})$

- Comparing classes of functions $\mathcal{F} \subset \mathbb{N}^k \to \mathbb{N}^k$ and $\mathcal{G} \subset \mathbb{R}^k \to \mathbb{R}^k$:
  - Compare $DP(\mathcal{G})$ with $\mathcal{F}$
  - $DP(\mathcal{G})$ is the set $\{f_{|\mathbb{N}}; f \in \mathcal{G}, f(\mathbb{N}) \subset \mathbb{N}\}$.

  e. g. $\quad \mathcal{R}ec(\mathbb{N}) = DP(\mathcal{H})$

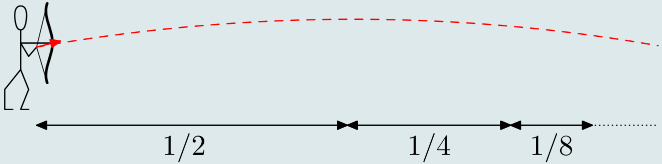
Linking models of “real” computation

- The models of computable analysis and $\mathbb{R}$-recursive functions deal with similar functions but lack relations between their classes.
- Investigating such links can help giving an analog characterization of what may be considered reasonable in computation over the reals.
- A step towards a Church Thesis for computation over the reals?
- A way to characterize the algorithmic complexity of some problems on dynamical systems?

"Real-time" computing yields unreasonable results

Zeno paradox (Ζήνον ὁ Ελεάτης)

At any time between its launch and arrival, an arrow has first to cover one half of the distance towards its goal.



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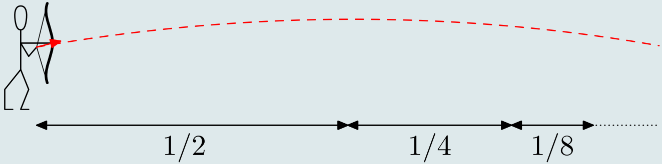
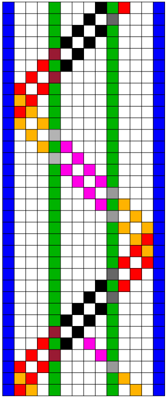


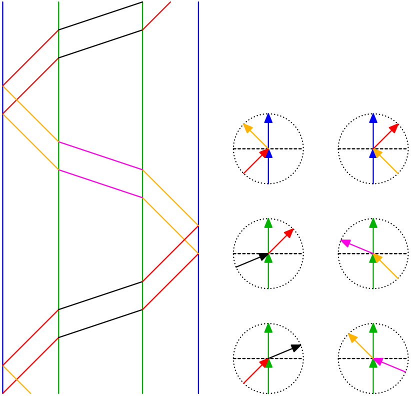
Accelerating Turing machine

An ATM achieves its first computing step in time $\frac{1}{2}$, its second step in time $\frac{1}{4}$, its n-th step in time $\frac{1}{2^n}$.
At time 1, this machine has done an infinity of computation steps.

Zeno phenomenon in signal machines

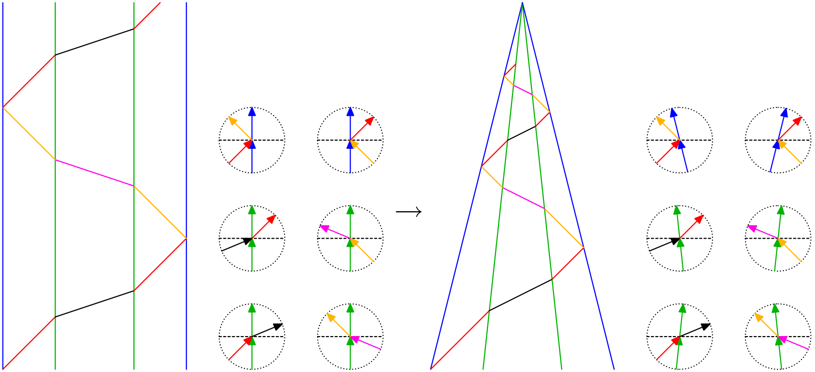
Signal machines [Durand-Lose 03] are a continuous counterpart to cellular automata:





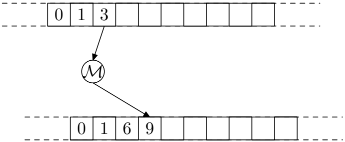
Zeno phenomenon in signal machines (2)

It is possible to reduce the time taken to do a computation by changing the slopes of the signals:

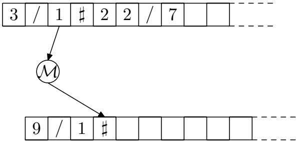


Setting

Turing Machine:



Recursive analysis:



Recursive functions:
$[0, S, U; COMP, REC, \mu]$

$\mathbb{R}$-recursive functions:
$[0, 1, U; \mathrm{COMP}, \mathrm{INT}, \mu_{\mathbb{R}}]$

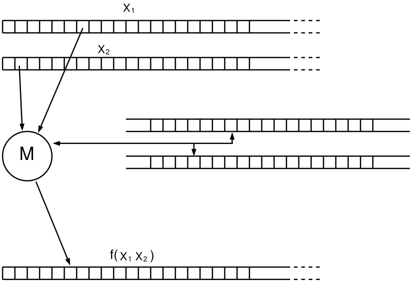
Recursive and Sub-recursive functions

$$
\begin{array}{ccc}
\mathcal{R}ec(\mathbb{N}) & = & [0, S, U; COMP, REC, \mu] \\
\cup\!\nmid & & \\
\mathcal{PR}(\mathbb{N}) & = & [0, S, U; COMP, REC] \\
\cup\!\nmid & & \\
\mathcal{E}_n(\mathbb{N}) & = & [0, S, U, \ominus, E_{n-1}; COMP, B\Sigma, B\Pi] \\
\cup\!\nmid & & \\
\mathcal{E}_3(\mathbb{N}) = \mathcal{E}(\mathbb{N}) & = & [0, S, U, \ominus; COMP, B\Sigma, B\Pi]
\end{array}
$$

Recursive and Sub-recursive functions

$$
\begin{array}{lcl}
\mathcal{R}ec(\mathbb{N}) & \sim & \text{Turing machines} \\
\cup\!\!\!\!\!\not\,\, & & \\
\mathcal{PR}(\mathbb{N}) & \sim & \text{For programs (no while)} \\
\cup\!\!\!\!\!\not\,\, & & \\
\mathcal{E}_n(\mathbb{N}) & & \text{Grzegorczyk's hierarchy} \\
\cup\!\!\!\!\!\not\,\, & & \\
\mathcal{E}_3(\mathbb{N}) = \mathcal{E}(\mathbb{N}) & \sim & \text{Time bounded by a } 2^{2^{2^{\cdots^{n}}}}
\end{array}
$$

Recursive analysis: type 2 machines



A tape represents a real number

Let $\nu_{\mathbb{Q}}$ be a representation of the rational numbers.
$(x_n) \rightsquigarrow x$ iff $\forall i, |x - \nu_{\mathbb{Q}}(x_i)| < \frac{1}{2^i}$

M behaves like a Turing machine

Write-only one-way output tape.

Computable functions

Definition [Computable functions]

A function $f : [a, b] \rightarrow \mathbb{R}$ with $a, b \in \mathbb{Q}$ is computable (resp: elementarily computable) iff there exists $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ recursive (resp: elementary) such that

$$\forall X \rightsquigarrow x, (\phi(X)) \rightsquigarrow f(x).$$

$\mathbb{R}$-recursive functions [Moore 96]

Definition [$\mathcal{G}$]

$\mathcal{G} = [0, 1, U; \mathrm{COMP}, \mathrm{INT}, \mu_{\mathbb{R}}]$

$$\begin{aligned} REC &: f, g \mapsto h \\ h(x, 0) &= f(x) \\ h(x, S(n)) &= g(x, n, h(x, n)) \end{aligned}$$

$$\begin{aligned} \mathrm{INT} &: f, g \mapsto h \\ h(x, 0) &= f(x) \\ \frac{\partial h}{\partial y}(x, y) &= g(x, y, h(x, y)) \end{aligned}$$

Problems with $\mathcal{G}$

- Not always well defined ($0 \times +\infty = 0$, non integrable functions).
  - [Mycka Costa 04] presents well-defined operator (differential recursion and infinite limits) that have the same power as $\mathcal{G}$.
- Presents time compression phenomenon (Zeno's paradox).
- Contains unwanted functions (in particular $\chi_{\mathbb{Q}}$ or functions that decide the halting problem of Turing machines).

Setting

Turing Machines	Recursive Analysis
Recursive functions	$\mathbb{R}$-recursive functions
$\mathcal{R}ec(\mathbb{N}) = [0, S, U; COMP, REC, \mu]$	$\mathcal{G} = [0, 1, U; \mathrm{COMP}, \mathrm{INT}, \mu_{\mathbb{R}}]$
$\mathcal{E}(\mathbb{N}) = [0, S, U, \ominus; COMP, B\Sigma, B\Pi]$	$\mathcal{L} = [0, 1, -1, \pi, U, \theta_3 \; \mathrm{COMP}, \mathrm{CLI}]$

$DP(\mathcal{G}) \supsetneq \mathcal{R}ec(\mathbb{N})$.
$DP(\mathcal{L}) = \mathcal{E}(\mathbb{N})$.

- There exists an operator UMU such that

$$DP(\mathcal{L} + \mathrm{UMU}) = \mathcal{R}ec(\mathbb{N})$$

Real recursive functions [Campagnolo Moore Costa 00]

Definition [$\mathcal{L}$]

$$\mathcal{L} = [0, 1, -1, \pi, U, \theta_3; \mathrm{COMP}, \mathrm{LI}]$$

With

- U : projections
- $\theta_3 : \begin{cases} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \max(0, x^3) \end{cases}$
- COMP: composition
- LI: given g, h. $f = \mathrm{LI}(g, h)$ is the maximal solution of
  $$\begin{array}{rcl} f(\overrightarrow{x}, 0) & = & g(\overrightarrow{x}) \\ \frac{\partial f}{\partial y}(\overrightarrow{x}, y) & = & h(\overrightarrow{x}, y) f(\overrightarrow{x}, y) \end{array}$$

Properties of $\mathcal{L}$

Theorem [Campagnolo Moore Costa 00]

$$DP(\mathcal{L}) = \mathcal{E}(\mathbb{N})$$

Theorem [Campagnolo Moore Costa 00]

$$DP(\mathcal{L}_n) = \mathcal{E}_n(\mathbb{N})$$

Extension to recursive functions

- This result gives a characterization of $\mathcal{E}(\mathbb{N})$ (and has been extended to all levels of the Grzegorczyk hierarchy).
- We introduce an operator UMU to obtain

$$DP(\mathcal{L} + \mathrm{UMU}) = \mathcal{R}ec(\mathbb{N}).$$

A real μ operator

Remark: A naive "return the smallest root" operator yields unwanted functions (see [Moore 96]).

Definition

Given $f : \mathcal{D} \times \mathcal{I} \subset \mathbb{R}^{k+1} \to \mathbb{R}$ differentiable such that:

- $\forall \overrightarrow{x} \in \mathcal{D}$, the function $g_{\overrightarrow{x}} : y \mapsto f(\overrightarrow{x}, y)$ is non decreasing,
- $g_{\overrightarrow{x}}$ has a unique root $y_{\overrightarrow{x}} \in \mathring{\mathcal{I}}$,
- $\frac{\partial f}{\partial y}(\overrightarrow{x}, y_{\overrightarrow{x}}) > 0$.

$$\mathrm{UMU}(f) = \begin{cases} \mathbb{R}^k & \longrightarrow & \mathbb{R} \\ \overrightarrow{x} & \mapsto & y \text{ such that } f(\overrightarrow{x}, y) = 0 \end{cases}$$

$\mathcal{H} = \mathcal{L} + \mathrm{UMU}$

Definition [$\mathcal{H}$]

$$\mathcal{H} = [0, 1, U, \theta_3; \mathrm{COMP}, \mathrm{CLI}, \mathrm{UMU}]$$

Proposition

$$\mathcal{H} = \mathcal{L} + \mathrm{UMU}$$

Proof:

- $-1 = \mathrm{UMU}(x \mapsto x + 1)$
- $x \mapsto \frac{1}{1+x^2} = \mathrm{UMU}\left(x, y \mapsto (1 + x^2)y - 1\right)$;
  $\arctan(0) = 0$ and $\arctan'(x) = \frac{1}{1+x^2}$;
  $\pi = 4\arctan(1)$

Result: $DP(\mathcal{H}) = \mathcal{R}ec(\mathbb{N})$

Theorem

$$DP(\mathcal{H}) = \mathcal{R}ec(\mathbb{N})$$

Where $\mathcal{R}ec(\mathbb{N})$ denotes the set of discrete partial recursive functions.

Proof: we have to demonstrate both directions.

- $DP(\mathcal{H}) \subset \mathcal{R}ec(\mathbb{N})$ comes from the fact that UMU preserves computability (in the sense of recursive analysis).
- $\mathcal{R}ec(\mathbb{N}) \subset DP(\mathcal{H})$ can be proven using a normal form theorem in $\mathcal{R}ec(\mathbb{N})$ and translating the discrete μ into our UMU.

Consequences

Corollary

$$\mathcal{L} \subsetneq \mathcal{H}$$

Theorem [Normal Form]

A function from $\mathcal{H}$ can be written with at most 3 nested UMU.

We may need 2 UMU to obtain π and -1. The other UMU comes from the simulation of the discrete μ.

Setting

	Recursive analysis
Turing machines	Type-2 machines
Recursive functions	$\mathbb{R}$-recursive functions
	$\mathcal{G}$, $\mathcal{L}$, $\mathcal{H}$

To characterize in an algebraic way the real recursive functions, we will define a limit operator.

Characterizing computable analysis classes

- Previous results give analog characterizations of $\mathcal{E}(\mathbb{N})$ and $\mathcal{R}ec(\mathbb{N})$.
- With a limit operator, we can extend those characterizations to obtain characterizations of $\mathcal{E}(\mathbb{R})$ and $\mathcal{R}ec(\mathbb{R})$.

$$\mathcal{H} + \mathrm{LIM} = \mathcal{R}ec(\mathbb{R})$$

- From [Mycka Costa 04], we know that a natural limit operator is as powerful as Moore's $\mu_{\mathbb{R}}$.

Operator LIM

Definition

Given $f : \mathbb{R} \times \mathcal{D} \subset \mathbb{R}^{k+1} \to \mathbb{R}^l$,

- if there are $K : \mathcal{D} \to \mathbb{R}$ and $\beta : \mathcal{D} \to \mathbb{R}$ *polynomials* such that

$$\forall \overrightarrow{x}, \forall t \geq \|\overrightarrow{x}\|, \|\frac{\partial f}{\partial t}(t, \overrightarrow{x})\| \leq K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x})),$$

- if $\overrightarrow{x} \mapsto \lim_{t\to+\infty} f(t, \overrightarrow{x})$ is $\mathcal{C}^2$.

Then, $F = \mathrm{LIM}(f, K, \beta)$ is defined by

$$F(\overrightarrow{x}) = \lim_{t\to\infty} f(t, \overrightarrow{x}).$$

Theorems

We will write $\mathcal{C}^\star$ where $\mathcal{C} = [\mathcal{F}; \mathcal{O}]$ to denote the class $[\mathcal{F}; \mathcal{O}, \mathrm{LIM}]$.

Theorem

For functions of class $\mathcal{C}^2$ defined on a compact domain,

$$\mathcal{L}^\star = \mathcal{E}(\mathbb{R}).$$

Theorems

Theorem

For functions of class $\mathcal{C}^2$ defined on a compact domain,

$$\mathcal{H}^\star = \mathcal{R}ec(\mathbb{R}).$$

Consequences

Theorem [Normal Form]

A function from $\mathcal{L}^\star$ or $\mathcal{H}^\star$ can be written with at most 2 nested LIM

One limit to obtain $1/x$ and another from the limit mechanism.

Proposition

Let $\bar{D} = [0, 1, -1, U; \mathrm{COMP}, \bar{I}]$.

$$(\bar{D} + \theta_3)^* \supseteq \mathcal{PR}(\mathbb{R}).$$

Results

$$\mathcal{R}ec(\mathbb{N}) = DP(\mathcal{H})$$

$$\mathcal{E}(\mathbb{R}) = \mathcal{L}^\star$$
$$\mathcal{E}_n(\mathbb{R}) = \mathcal{L}_n^\star$$
$$\mathcal{PR}(\mathbb{R}) \subseteq (\bar{\mathcal{D}} + \theta_3)^\star$$
$$\mathcal{R}ec(\mathbb{R}) = \mathcal{H}^\star$$

Results

- Link between discrete and real classes of functions.
- Machine-independent characterizations of classes from Recursive analysis.
- Equivalence between what can be computed by a GPAC and by recursive analysis. ([Bournez Campagnolo Graça H.])
- Can we label $\mathcal{R}ec(\mathbb{R})$ as what is reasonable?