

Billiard and arithmetic

Nicolas Bedaride

Laboratoire d'Analyse Topologie Probabilités,
Université Paul Cézanne.

Complexity

If v is an infinite word, we define the **complexity** function $p(n, v)$ as the number of different words of length n inside v .

$$p : \mathbb{N}^* \rightarrow \mathbb{N}$$

$$p : n \mapsto p(n, v)$$

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Three examples :

abbbababaaaaaaaaa . . .

abaababaabaababaababa . . .

abaaabbabbaaaaababa . . .

Sturmian word I

Theorem [Morse-Hedlund 1940.] Let v be an infinite word, assume there exists n such that $p(n, v) \leq n$. Then v is an ultimately periodic word.

A word v such that $p(n, v) = n + 1$ for all integer n , is called a Sturmian word.

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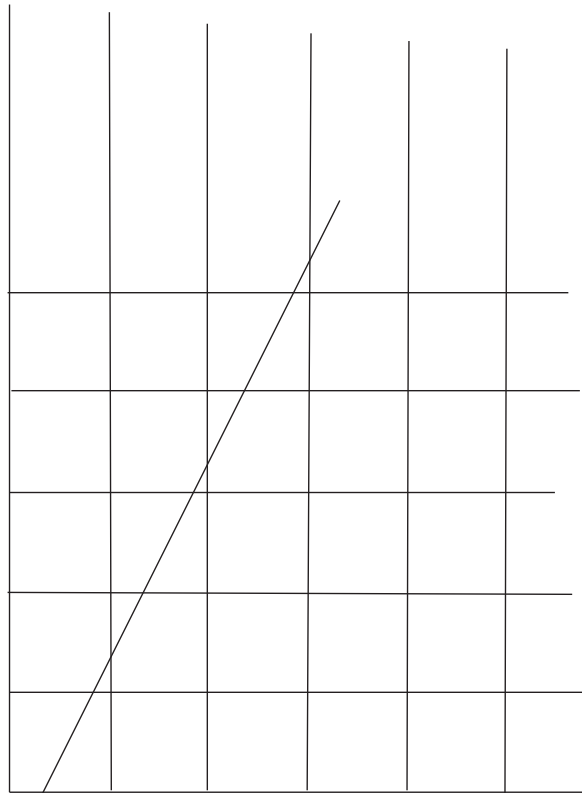
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Theorem [Morse-Hedlund 1940] We code the lattice \mathbb{Z}^2 with two letters. Let v be a sturmian word, then there exists

m, ω in \mathbb{R}^2 such that $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in \mathbb{R}^2$, $\frac{\omega_2}{\omega_1} \notin \mathbb{Q}$,

$$\phi(m, \omega) = v.$$

Sturmian word II



$v = aabaabaab \dots$

Let v be a Sturmian word, and \mathcal{L}_v its language. Consider the following language

$$\mathcal{L} = \bigcup_v \mathcal{L}_v.$$

$$p(n) = \text{card}\mathcal{L}(n).$$

Computation of $p(n)$.

Proof of :

- Tarannikov, Lipatov.
- Mignosi.
- Berstel-Pocchiola.
- Cassaigne-Hubert-Troubetzkoy.

Theorem For all integer n :

$$p(n) = \sum_{i=0}^n \phi(i)(n+1-i) \sim Cn^3,$$

where ϕ is the Euler function.

Generalisation ?

Number Theory

$$\mathcal{L} = \bigcup_{\alpha} \mathcal{L}_{\alpha}.$$

Let w be a word of length n :

$$v_i(w) = \min(a \in \mathcal{F}_i(w)).$$

$$\|w\| = \max\left(\frac{v_i(w) - 1}{i}, i \leq n\right).$$

$$B_n(x) = \{v \in \mathcal{L}(n), \|v\| \leq x\}.$$

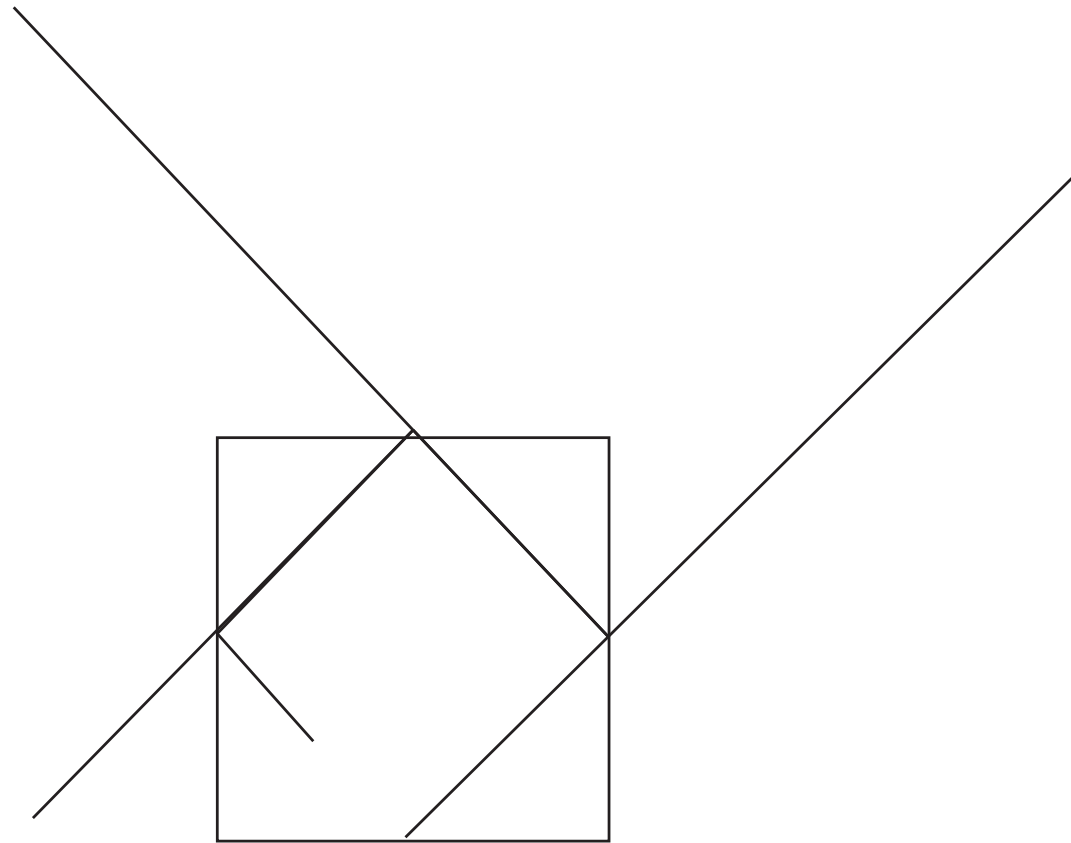
$$B_n(x) = \bigcup_{\alpha \leq x} \mathcal{L}_{\alpha}(n).$$

Riemann hypothesis

$$C_{n-1}(x) = \text{card}B_n(x) - \text{card}B_{n-1}(x).$$

$$R_n(x) = \frac{C_{n-1}(x)}{C_{n-1}(1)} - x.$$

$$\int_0^1 R_n(x)^2 dx = o\left(\frac{1}{n^{3/2-\varepsilon}}\right) \quad \forall \varepsilon > 0.$$



Billiard orbit inside the square and Sturmian word.

Billiard map

Let P be a polyhedron, $m \in \partial P$ and $\omega \in \mathbb{P}\mathbb{R}^d$.

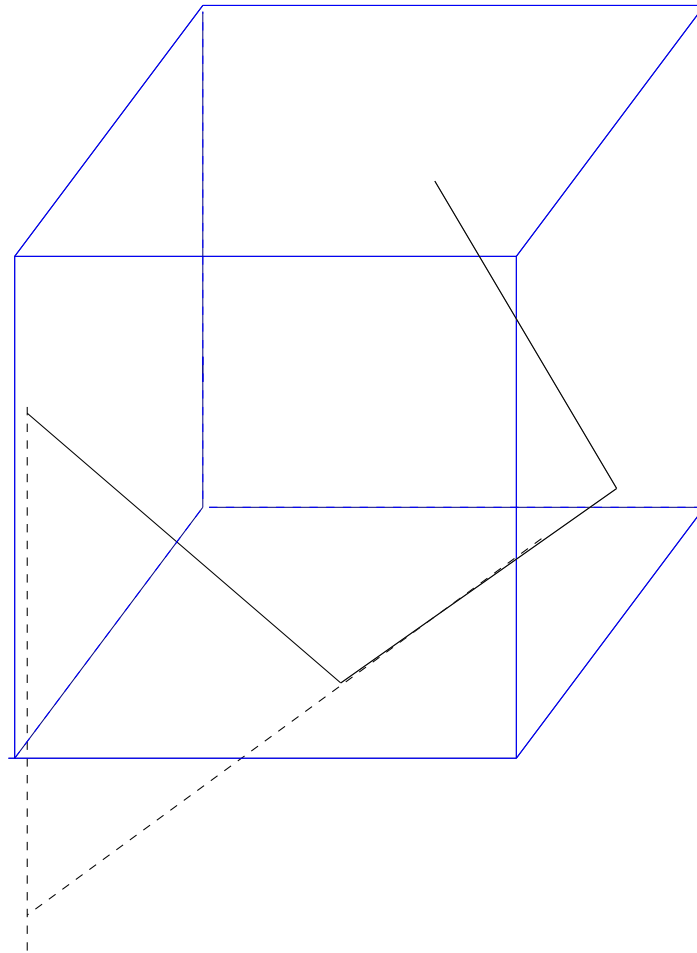
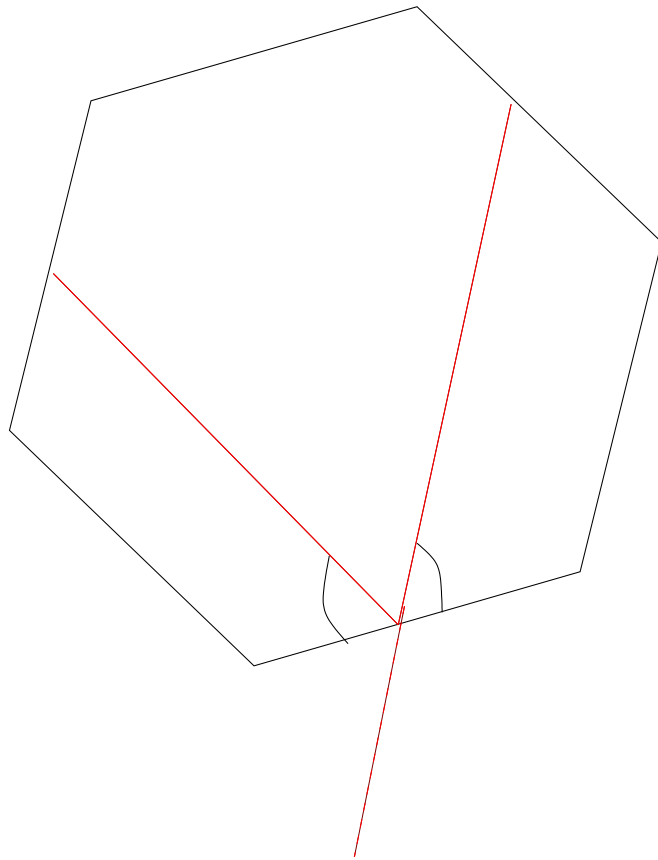
The point moves along a straight line until it reaches the boundary of P .

On the face : orthogonal reflection of the line over the plane of the face.

$$T : X \longrightarrow \partial P \times \mathbb{P}\mathbb{R}^d.$$

If a trajectory hits an edge, it stops.

Trajectories



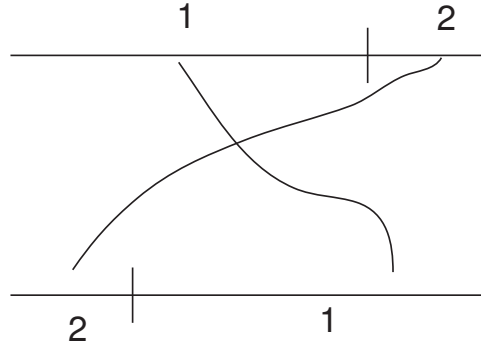
Reflections and billiard.

Sturmian words III

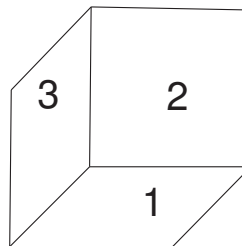
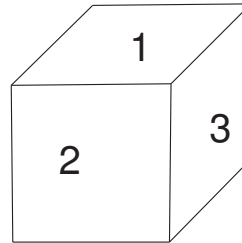
Consider the first return map of the billiard flow on a transverse set (interval, plane ...):

It is a rotation on the torus.

- Dimension one :



- Dimension two :



Sturmian word.



Rotation on the torus \mathbb{T}^1 .



Coding of a billiard trajectory inside the square.

Rotation on the torus \mathbb{T}^d .



Coding of a billiard trajectory inside the cube of \mathbb{R}^{d+1} .

If v codes a rotation of vector $\omega \in \mathbb{R}^d$ we denote

$$p(n, v) = p(n, \omega).$$

Dimension d

Arnoux-Mauduit-Shiokawa-Tamura ; Baryshnikov ; B

Theorem

Under some hypothesis on ω :

- In \mathbb{R}^3 , $p(n, \omega) = n^2 + n + 1$.
- In \mathbb{R}^{d+1} , $p(n, \omega) \sim n^d$.

$p(n)$?

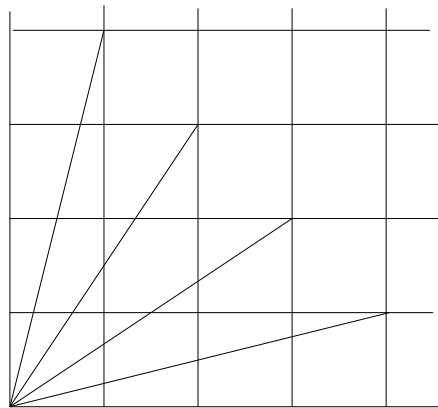
Theorem [B-Hubert] Consider the cube of \mathbb{R}^{d+1} coded with $d + 1$ letters, then there exists $a, b > 0$ such that

$$a \leq \frac{p(n)}{n^{3d}} \leq b.$$

Diagonals

Definition Consider a polyhedron of \mathbb{R}^3 .

- A diagonal between two edges A, B is the union of all billiard trajectories between A and B .
- We say it is of length n if it intersects n faces between the two edges.
- Let $N(n, P)$ be the number of generalized diagonals inside P .



Diagonals of the square.

- Geometry of the billiard words.
- Link between words and diagonals.
- Estimate.
- Induction on the dimension.

Key point

Theorem [Cassaigne-Hubert-Troubetzkoy ;B] If P is a polygon, then

$$s(n + 1) - s(n) = N(n, P).$$

For the square we have

$$N(n, P) = \phi(n).$$

Lemma For the cube we have

$$s(n + 1) - s(n) = \sum_{\gamma \in \text{Diag}(n)} \sum_{v \in \gamma} i(v).$$

Number theory

Lemma For all integer $p \geq 1$ there exists $C > 0$ such that for all n : If we denote $S_l = \sum_{\substack{m \leq l \\ \gcd(m,l)=1}} m^p$.

We have

$$\sum_{l \leq n} S_l \geq Cn^{p+2}.$$

For $p = 0$, we have $S_l = \phi(l)$.

Exact formula

Some problems appear :

- Exact formula for the arithmetic sums.
- Estimate for the constants in the induction.
- Counting problem, see Sturmian words.

Count the number of Sturmian words v of length n such that

$$m \leq |v|_a \leq p.$$

Rotation

Consider a rotation on the torus \mathbb{T}^2 . Some questions are natural :

- Change the partition, compute $p(n)$?
- Is there a three length theorem ?
- Find an algorithm to compute the coding of an orbit.

Discrete line

- Billiard orbit inside a cube
- Discrete line in \mathbb{R}^3 .

Projection of the billiard word : billiard word inside the square.