What is a digital computer?

What is computational universality ?

- What is a universal Turing machine?
  - We do know
- What is a universal cellular automaton?
  - Do we know?
- What is a universal symbolic system?
  - We want to know
- What properties has a universal symbolic system?
  - We want to know

- What is a universal analog system?
  - Not here

Some answers

- A digital computer
  - is a ‘universal symbolic system’
  - must have ‘some’ chaos
  - can be Devaney-chaotic

Computing on symbolic spaces

Jean-Charles Delvenne (Imperial College of London)
Petr Kůrka (Université Charles de Prague)
Vincent D. Blondel (Université catholique de Louvain)

Plan of the talk

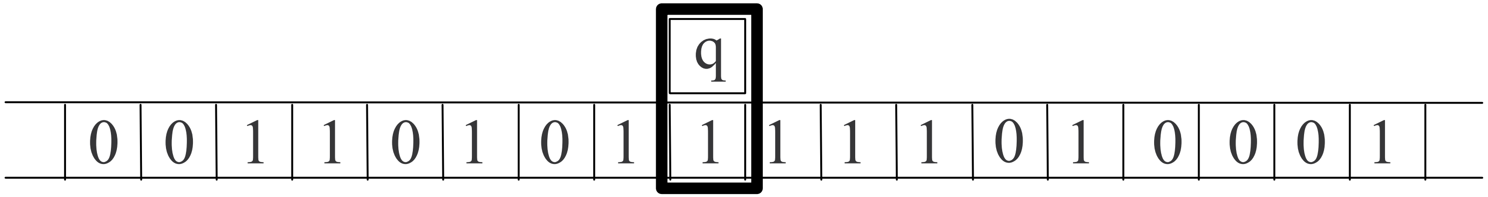
- A definition of universality in symbolic systems.
    - Building the definition
    - Examples
- Universality and dynamical properties
    - Minimality, equicontinuity, shadowing property
    - Chaos and “edge of chaos”
- Conclusions

Computational universality...

- ... is defined for *Turing machines*, counter machines, tag systems, etc. (many variants, but all equivalent)
- ... is not uniquely defined for *cellular automata* (many definitions, but not equivalent)
- ... is not uniquely defined for *analog* systems (many definitions, but not equivalent)
- Can we find a *general* definition?

- In this talk: discrete time, symbolic space, deterministic = continuous self-map of $2^{\mathbb{N}}$

Turing machines



- Finite or infinite tape
- At every step,
  - change currently read symbol
  - change state of the head
  - shift the tape to the left or to the right
- Map on the set of configurations $Q \times (2^* \cup 2^{\mathbf{Z}})$

What is a dynamical system?

- Cantor space = $A^{\mathbb{N}}$ (or $2^{\mathbb{N}}$ or $A^{\mathbb{Z}}$ or $A^{\mathbb{Z}^2}$ or ...)
- Metric: $d\left(w0\ldots,w1\ldots\right)=2^{-n}$
  (*w* is a word of length *n*)
- *Symbolic dynamical system = continuous self-map of (a closed subset of) the Cantor space*
- *The self-map must be computable.*
- Turing machines, cellular automata, shift, subshifts
- Shift: $A^{\mathbb{N}} \to A^{\mathbb{N}}$: $a_0a_1a_2a_3\ldots \to a_1a_2a_3\ldots$

Topology of the Cantor set

- Cylinder = $w2^{\mathrm{N}}$ (*w* is a finite word)
- Cylinders = balls
- Closed open sets = finite unions of cylinders
- *Clopen sets = countable basis of topology*
- *Clopen set = finite number of bits are specified*
- Clopen set = finitely described

Back to Turing

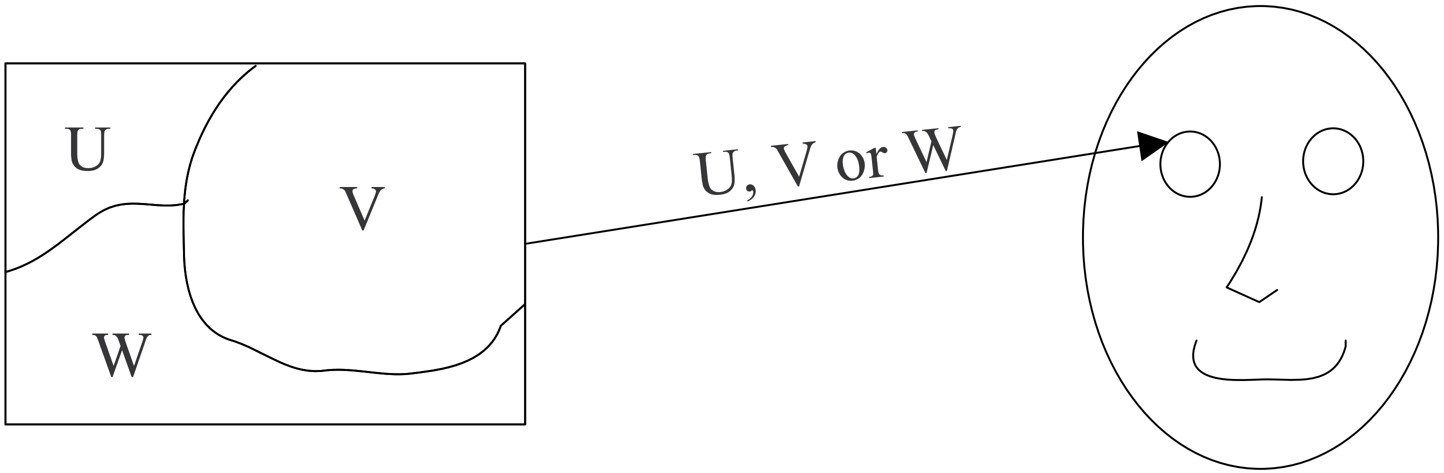
- Computation = Brain + paper + pencil
- Brain = finite automaton (head)
- ‘I am starting to compute’ = initial state
- ‘I have ended the computation’ = final state
- Paper = tape
- *Computation = Turing machine*
- Universal Turing machine =

  Turing machine with r.e.-complete halting problem (Davis)

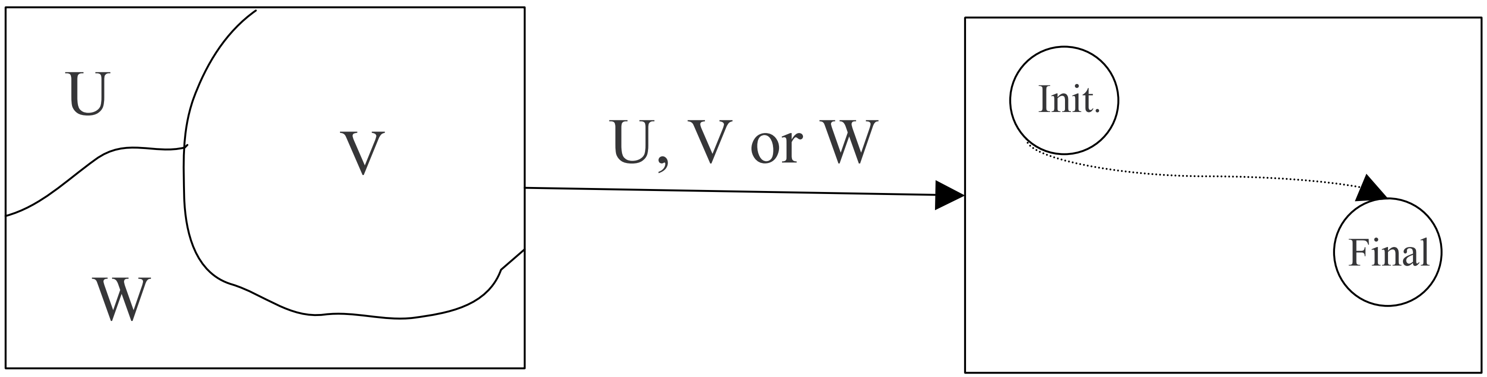
Universal symbolic system

- Computation = brain + dynamical system
- The brain observes, does not control the system
- *Brain = finite automaton*
- ‘I am starting to compute’ = initial state
- ‘I have ended the computation’ = final state
- *Observe* the system through *finite partition*
- Symbolic system: *clopen partition*

Universal symbolic system



Is there a trajectory leading the initial state to the final state of observer automaton?



Universal symbolic systems

- *Universal* if ‘Is there a trajectory leading the initial state to the final state of observer automaton ?’ is r.e.-complete
- We can express

  ‘Is there a trajectory from clopen U to clopen V ?’

  ‘Is there a trajectory from U to V that avoids W ?’

  etc.
- Robustness to noise on initial condition
- Non-deterministic computation
- Preserved by (effective) conjugacies and extensions
- Model-checking; variant with Büchi automata

Examples

- Universal Turing machines are universal
- A universal *cellular automaton*
- A universal *subshift* (=subsystem of the shift)
- A *chaotic* universal symbolic system

- Game of Life?
- $3n$+1 map?
- Analytic functions on *p*-adic integers? On adelic integers?

Another attempt: point to point

- Turing machine: *universal* iff point-to-point reachability is r.e.-complete (Davis)
- Dynamical system: universal iff point-to-point reachability is r.e.-complete?
- But, in a general dynamical system:
  - uncountably many points
  - choice of a countable set of points is arbitrary
  - physically unrealizable (noise, finite precision)
  - shift with countable initial points is universal

Linking computation to dynamics

- Wolfram: universal cellular automata are thought to be in “class IV”

- Edge of chaos: a universal system is
  - not too predictible (asymptotically stable fixed point)
  - not too unpredictible (chaos, full shift)

A universal system...

- ... has a *proper subsystem* (is not minimal)
- ... has a *sensitive* point (small perturbations can be amplified, butterfly effect)
- ... has not the *shadowing property* (cannot be ‘simulated’)
- Examples:

  minimal subshifts, full shift, subshifts of finite type, identity are not universal

Shadowing property

- A system *f* has the *shadowing property* if for all ε there is a δ such that every δ-pseudo-orbit is ε-shadowed by some point
- A δ-pseudo-orbit is a sequence $(x_n)_n$ such that

$$\forall n : d\left(x_{n+1}, f\left(x_n\right)\right) < \ddot{a}$$

- A sequence $(x_n)_n$ is ε-shadowed by *y* if

$$\forall n : d\left(x_n, f^n\left(y\right)\right) < \mathring{a}$$

- Intuition: a numerical trajectory is very close to an actual trajectory of the system

Chaos

- Devaney: a dynamical system is *chaotic* if
  - it is *sensitive to initial conditions*:

$$\exists \mathring{a} \quad \forall \ddot{a}, x \quad \exists y, n : d(x,y) < \ddot{a} \quad \text{and} \quad d\left(f^n(x), f^n(y)\right) > \mathring{a}$$

  - and periodic points are dense
  - and it is transitive: it is possible to go from any open set to any open set

- Chaos has often shadowing property, not always

Chaos

- *There is a chaotic universal system*
- Undecidable property = ‘Can we go from U to V and avoid W ?’
- *‘Edge of chaos’ is false in this setting*
- ‘Edge of shadowing property’

Conclusions

- A general *definition* of universality for symbolic systems
- Sufficient *conditions* for non-universality: minimality, equicontinuity, shadowing property, but not chaoticity
- Open problems:
  - extension of definition and results to *analog* systems, e.g., is a three-body system universal?
  - *complexity* theory, e.g., P=NP ?

Universal Turing machines are universal

- Space of configurations = $Q \times (2^* \cup 2^{\mathbf{Z}})$
- Can be recoded as a closed subset of $2^{\mathbf{N}}$
- Finite words = isolated points = clopen sets
- Halting problem = “Is there a trajectory from U to V ?” for U isolated clopen set

Analog computation

- How to choose a countable family of open sets ?
- How to encode a real number into a sequence of bits ?
- All choices equivalent ?
- Give a special role to rationals ?

Is the full shift universal?

- Shift: $A^{\mathbf{N}} \to A^{\mathbf{N}} : a_0 a_1 a_2 a_3 \ldots \to a_1 a_2 a_3 \ldots$ (A finite)
- Initial configuration:
  - $1^n 0^t 2^\infty$ if data *n* halts on the UTM in time *t*
  - $1^n 0^\infty$ if data *n* doesn't halt on the UTM
- *Computable*
- The halting problem: “Will we reach symbol 2?” is r.e.-complete
- Hard part of the computation hidden in encoding the initial condition