

Tree Automata Make Ordinal Theory Easy

Thierry CACHAT

LIAFA, Université Paris VII

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Plan:

Motivations, background

First Order theory of $(\omega^\omega, +)$, extension to $(\omega^{\omega^i}, +)$

Monadic Second Order theory of $(\omega^2, <)$, extension to $(\omega^i, <)$

Comparisons

Motivations

Automata based decision procedure are efficient and simple.

e.g. FO, MSO / \mathbb{N} , binary trees, ...

Ordinal theories: original proof uses ordinal automata [Büchi65].

Complexity improved using a complicated coding by finite words

[Maurin97].

“We have a proof without theorem”

Wolfgang Thomas, 2002

(closure of tree automata under boolean operations and projection)

Presburger arithmetic: $\text{FO}(\mathbb{N}, +)$

[Presburger1930, Skolem1931]: quantifier elimination

[Büchi1960]: using finite automata

integer $x \in \mathbb{N}$	\leftrightarrow	$u \in \{0, 1\}^*$ in binary
relation $x + y = z$	\leftrightarrow	automaton over $\{0, 1\}^3$
models: set $X \subseteq \mathbb{N}$	\leftrightarrow	automaton over $\{0, 1\}$
formula $\psi(x_1, \dots, x_k)$	\leftrightarrow	automaton over $\{0, 1\}^k$
quantification \exists	\leftrightarrow	projection (\rightsquigarrow non-determinism)
\vee, \wedge	\leftrightarrow	union, product
negation	\leftrightarrow	complementation (\rightsquigarrow costly)

Induction on the structure of the formula.

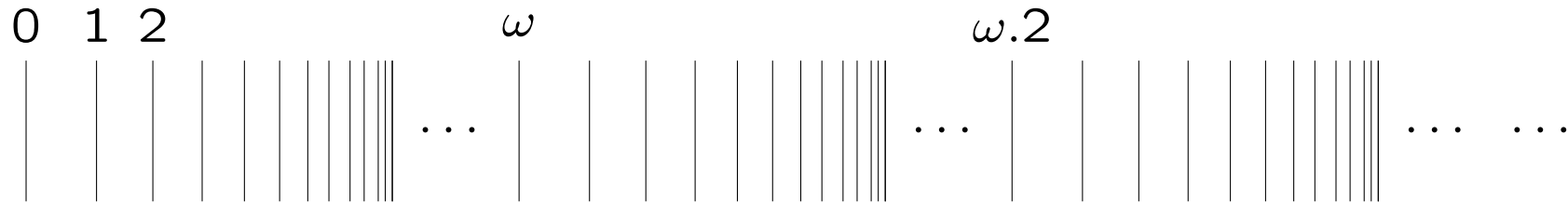
Emptiness is decidable

see [Klaedtke04]

Ordinal arithmetic

$$1 + \omega = \omega \neq \omega + 1$$

$$2 \cdot \omega = \omega \neq \omega \cdot 2$$



\forall ordinal α : $\alpha = \{\beta : \beta < \alpha\}$

$$\begin{aligned} \omega^i + \omega^j &= \omega^j && \text{if } i < j \\ &= \omega^j \cdot 2 && \text{if } i = j \end{aligned}$$

Cantor Normal Form

$\forall 0 < \alpha < \omega^\omega, \exists$ unique integers p, n_0, n_1, \dots, n_p , s.t.

$$n_p > 0 \text{ and } \alpha = \omega^p n_p + \omega^{p-1} n_{p-1} + \dots + \omega^1 n_1 + n_0$$

Given $\alpha = \omega^p n_p + \dots + \omega^1 n_1 + n_0$

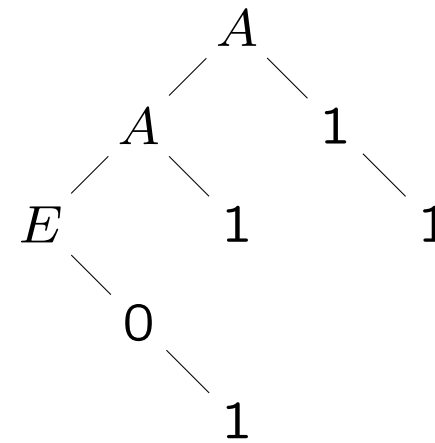
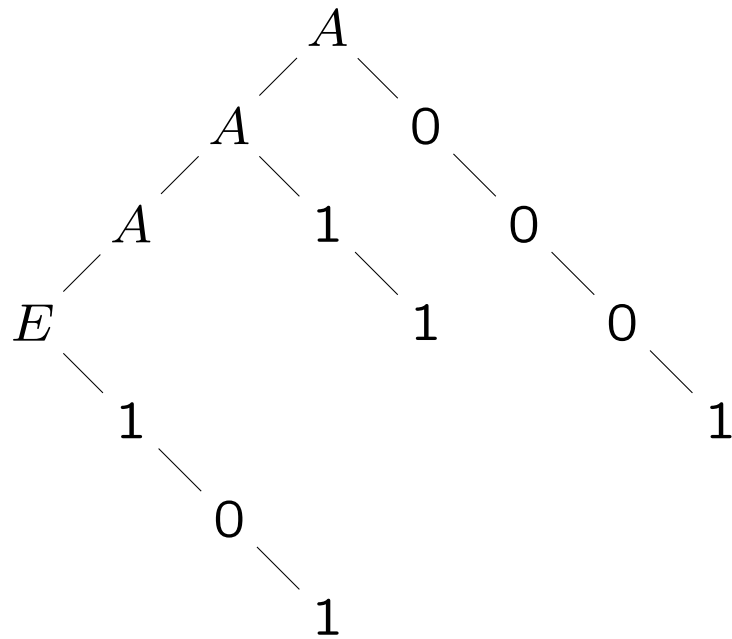
and $\alpha' = \omega^{p'} n'_{p'} + \dots + \omega^1 n'_1 + n'_0$ in CNF,

$$\alpha + \alpha' = \omega^p n_p + \dots + \omega^{p'} (n_{p'} + n'_{p'}) + \dots + \omega^1 n'_1 + n'_0$$

Representation by finite trees

$$\alpha = \omega^3.5 + \omega.3 + 8$$

$$\alpha' = \omega^2.2 + \omega.1 + 3$$



- | | | |
|--------------------------|-------------------|--|
| addition | \leftrightarrow | tree-automaton over $\{E, A, 0, 1\}^3$ |
| boolean operations | \leftrightarrow | closure properties |
| \exists quantification | \leftrightarrow | projection |

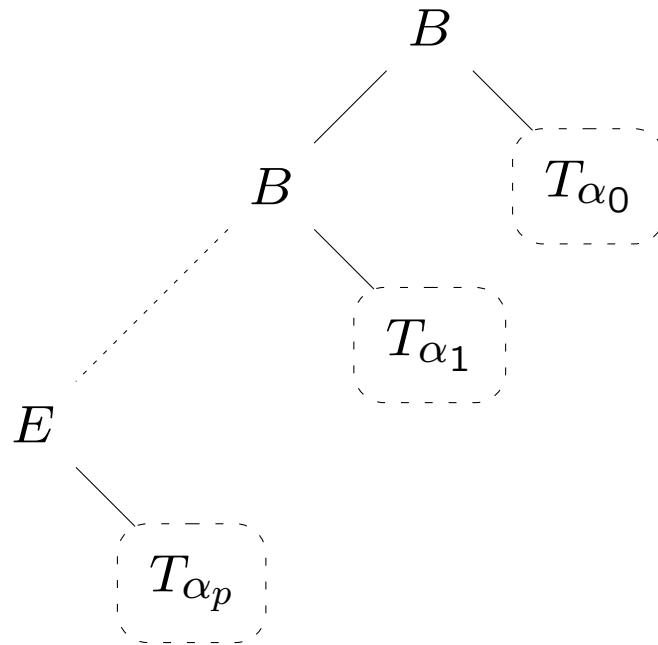
Induction on the formula. Emptiness decidable

Add dummy symbols $\#$

Extension to ω^{ω^i}

Cantor Normal Form \rightsquigarrow Any $\beta < \omega^{\omega^2}$ is uniquely written as

$$\omega^{\omega \cdot p} \alpha_p + \dots + \omega^{\omega \cdot 2} \alpha_2 + \omega^{\omega} \alpha_1 + \alpha_0, \text{ where } p < \omega, \alpha_i < \omega^{\omega}, \alpha_p > 0.$$



and so on by induction ...

Complexity

Let $\text{Tower}(0, n) = n$ and $\text{Tower}(k + 1, n) = 2^{\text{Tower}(k, n)}$.

Thm: $\forall i < \omega \exists c_i$ such that $\text{FO}(\omega^{\omega^i}, +)$ is decidable in time $\mathcal{O}(\text{Tower}(n, c_i))$, where n is the length of the formula.

Best known result:

via reduction to Weak Monadic Second Order logic $(\omega^\omega, <)$, in turn decidable in time $\mathcal{O}(\text{Tower}(6n, c'))$ [Maurin96].

lower bound?

Application: Similar technique for $\text{FO}(\mathbb{N}, *)$, via prime number decomposition.

Monadic Second Order ($\mathbb{N}, <$) [Büchi62]

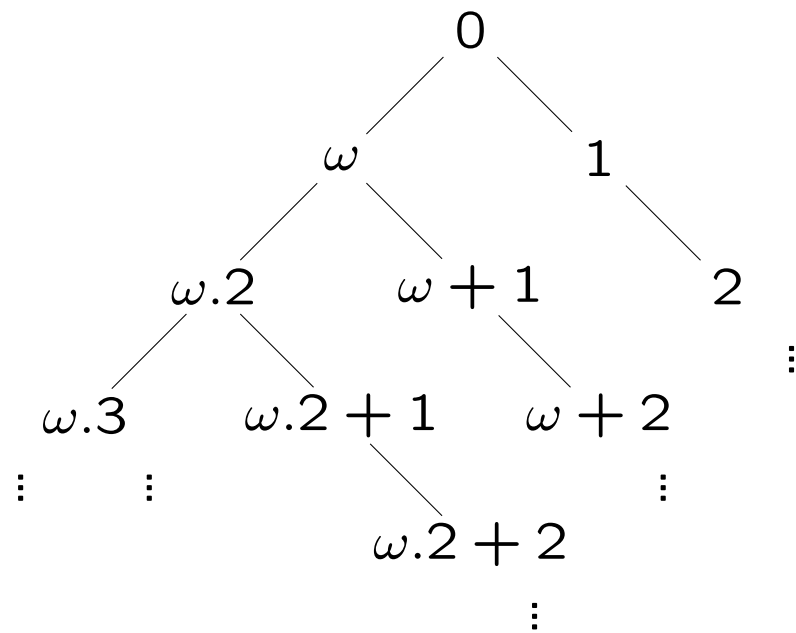


set $X \subseteq \mathbb{N}$			\leftrightarrow	infinite word $v \in \{0, 1\}^\omega$
$x \in \mathbb{N}$	\leftrightarrow	$\{x\} \subseteq \mathbb{N}$	\leftrightarrow	$0^x 1 0^\omega$
$x \in X$			\leftrightarrow	Büchi automaton over $\{0, 1\}^2$
$x < y$			\leftrightarrow	Büchi automaton over $\{0, 1\}^2$
formula $\psi(x_1, \dots, x_k)$			\leftrightarrow	Büchi automaton over $\{0, 1\}^k$
quantification \exists			\leftrightarrow	projection (\rightsquigarrow non-determinism)
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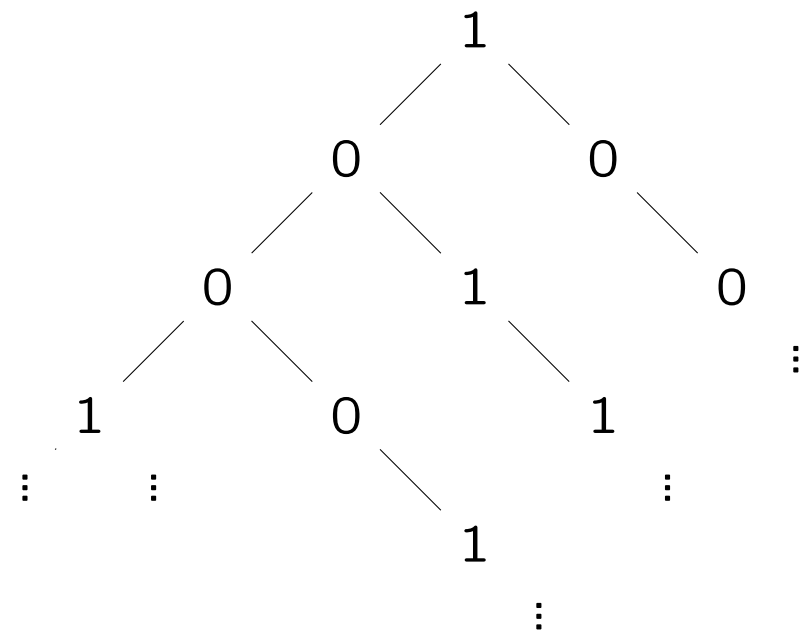
Non-elementary lower bound

Representation of any $X \subseteq \omega^2$ by an infinite tree

positions



labels $\in \{0, 1\}$



$\{0, \omega + 1, \omega + 2, \omega.2 + 2, \omega.3\}$

Use Rabin tree-automaton (or Muller or parity)
cannot be determinized yet complemented

Example of automata

Test X is $\{x\}$: non-deterministically search the unique node labeled 1

Test $x < y$: either on the same branch, or on different branches ...

Other cases are easy and similar to $(\mathbb{N}, <)$

Extension to ω^i : as in the case of $\text{FO}(\omega^{\omega^i}, +)$ with e.g.

$$X \subseteq \omega^3 \text{ is written } \bigcup_{j < \omega} (\omega^2 \cdot j + X_j) \text{ where } \forall j : X_j \subseteq \omega^2$$

Thm: $\forall i < \omega \exists d_i$ such that $\text{MSO}(\omega^i, <)$ is decidable in time $\mathcal{O}(\text{Tower}(n, d_i))$, where n is the length of the formula.

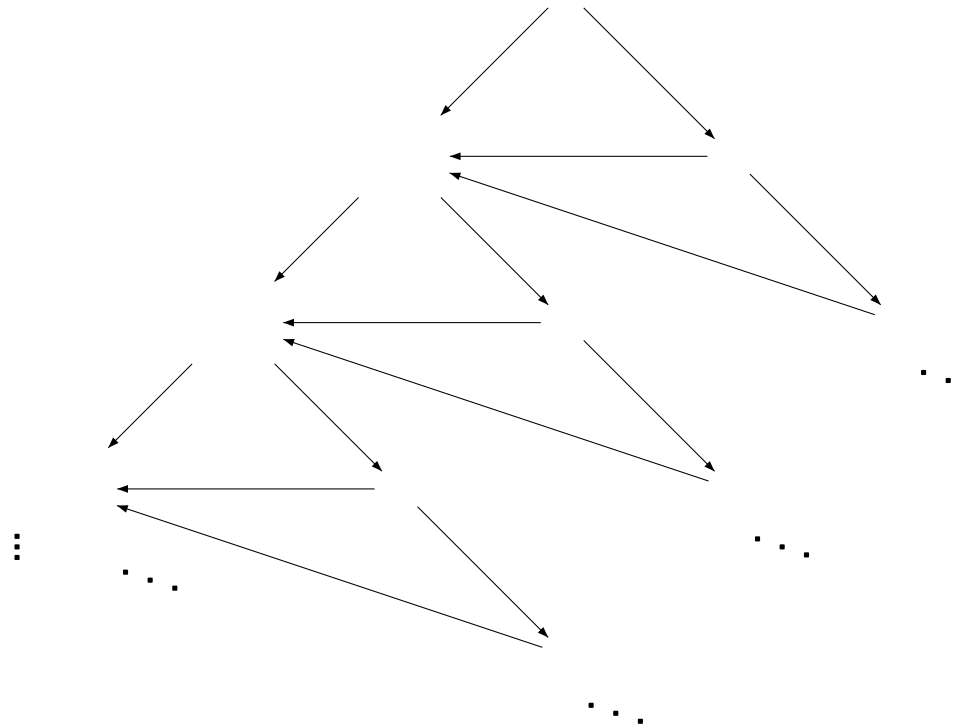
Lower bound in $\text{Tower}(n, c)$ even for $i = 1$

[Maurin96] WeakMSO $\mathcal{O}(\text{Tower}(6n, d'))$

MSO interpretation

The graph of $(\omega^i, <)$ is prefix recognizable, ie at level 1 in the Caucal Hierarchy [Caucal02]

Transitive closure of



Open problem: higher ordinals at higher levels of the hierarchy?

Weak MSO and FO

Weak MSO: same syntax as MSO, but sets are interpreted as **finite**

Any ordinal β is uniquely written as

$$2^{\gamma_{n-1}} + \dots + 2^{\gamma_0} \text{ where } (\gamma_{n-1} > \gamma_{n-2} > \dots > \gamma_0)$$

Examples: $2^\omega = \omega$, $2^{\omega \cdot i + j} = 2^{\omega \cdot i} \cdot 2^j = \omega^i \cdot 2^j$, $2^{\omega^2} = (2^\omega)^\omega = \omega^\omega$.

[Buechi65] theories $WMSO(\alpha, <)$ and $FO(2^\alpha, +, E)$ are equireducible in linear time. $\rightsquigarrow WMSO(\omega^{i+1}, <)$ and $FO(\omega^{\omega^i}, +, E)$

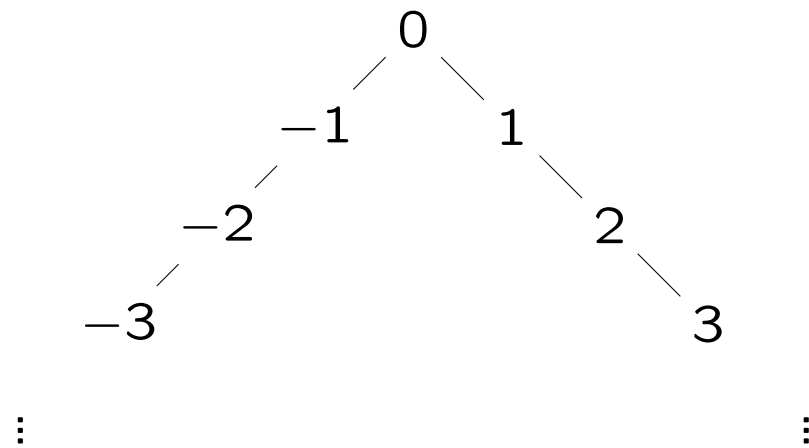
Any ordinal β can yet be written in a unique way in the form

$$\alpha = \gamma \cdot \omega^\omega + \omega^p n_p + \omega^{p-1} n_{p-1} + \dots + \omega^1 n_1 + n_0 \quad \text{where } n_p > 0$$

$WMSO(\alpha, <)$ depends only on $(\gamma > 0), n_p, \dots, n_0)$

Extension to other linear orderings

$\text{MSO}(\mathbb{Z}, <)$



Consider $-\omega + \omega$ $(-\omega).\omega$ \dots

Conjecture: new proof of [\[CartonRispal04\]](#)

Open problems: linear orders of infinite rank, ordinals higher than ω^ω .