Discrete Rotations and Symbolic Dynamics

Valérie Berthé,

LIRMM – UMR 5506 – Univ. Montpellier II, 161 rue Ada, 34392 Montpellier Cedex 05, France

Bertrand Nouvel,

LIP –UMR 5668 – ENS Lyon, 49, Allée d'Italie, 69364 Lyon Cedex 7, France

Abstract

The aim of this paper is to study local configurations for discrete rotations. The algorithm of discrete rotation we consider is the following: a discretized rotation is defined as the composition of a Euclidean rotation with a rounding operation, as studied in [NR03,NR04,NR05]. It is possible to encode all the information concerning a discrete rotation as two multidimensional words C_α and C'_α that we call configurations. We introduce here two discrete dynamical systems defined by a $\mathbb{Z}^2$-action on the two-dimensional torus that allow us via a suitable symbolic coding to describe the configurations C_α and C'_α; we then deduce various combinatorial properties for both configurations, and in particular, results concerning densities of occurrence of symbols.

Key words: Discrete rotations, discrete geometry, word combinatorics, two-dimensional words, symbolic dynamics.

1 Introduction

Symbolic dynamics and more generally, discrete dynamical systems have natural and deep interactions with combinatorics on words. This interaction is particularly well-illustrated in the Sturmian case, see e.g. [Lot02,Fog02]. The combinatorial objects involved are the Sturmian words, while the dynamical systems are the irrational rotations of the torus $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. A Sturmian word is indeed a coding with respect to a particular two-interval partition of

Email addresses: berthe@lirmm.fr (Valérie Berthé), bertrand.nouvel@ens-lyon.fr (Bertrand Nouvel).

the one-dimensional torus $\mathbb{T}^1$ of the orbit of a point under the action of an irrational rotation. This point of view allows one to deduce many combinatorial properties of Sturmian words, as discussed in [BFZ05], such as, e.g., the densities of occurrences of factors that can be computed thanks to the equidistribution properties of irrational rotations, or such as powers of factors in Sturmian words [Van00], or the characterization of Sturmian words fixed points of substitutions [BEIR].

Several attempts of generalization of this fruitful interaction have been proposed. For more details, see the survey [BFZ05]. One of the first idea which comes to mind is a rotation of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. As an example, the Tribonacci word, that is, the fixed point of the substitution $1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ codes the orbit of a point of the torus $\mathbb{T}^2$ under the action of a translation in $\mathbb{T}^2$ with respect to a partition of $\mathbb{T}^2$ into three pieces with fractal boundary [Rau82,Lot05]. More generally, fixed points of Arnoux-Rauzy sequences over n letters [AR91] code orbits of points of the torus $\mathbb{T}^{n-1}$ under the action of a translation in $\mathbb{T}^{n-1}$ with respect to a partition of $\mathbb{T}^{n-1}$ into n pieces with fractal boundary [AI01].

A second approach, which is dual to the previous one, consists in working with two rotations of $\mathbb{T}^1$. It is indeed convenient to describe arithmetic discrete planes in the sense of [Rev91] by use of the coding with respect to a three-interval partition of a $\mathbb{Z}^2$-action by two irrational rotations on $\mathbb{T}^1$ [BFJP]. One thus gets two-dimensional words over a three-letter alphabet that can be considered as two-dimensional Sturmian words [BV00]. The study of the underlying dynamical system allows one here to obtain a better understanding of the combinatorial and geometric properties of arithmetic discrete planes, such as the enumeration of some local configurations, the so-called (m, n)-cubes, as well as their densities of occurrence, or their centrosymmetry properties [BFJP].

In all these cases connections between word combinatorics, symbolic dynamics, arithmetics and discrete geometry prove to be natural and enligthening. We consider in the present paper a further generalization motivated by discrete geometry, and more precisely, arithmetic discrete geometry, in the sense of [Rev91]. We study indeed configurations associated with a discrete rotation; there exists several extensions of the the notion of Euclidean rotation in discrete geometry, such as reviewed in [And92]. We consider here discrete rotations defined as the composition of a Euclidean rotation with a rounding operation. It is possible to encode all the information concerning a discrete rotation as two multidimensional words C_α and C'_α that we call configurations. These configurations have been introduced and studied in [NR03,NR05,NR05]. The main purpose of the present paper is to prove that both configurations are codings of a $\mathbb{Z}^2$-action by two rotations on $\mathbb{T}^2$ with respect to a partition into a finite number of rectangles. We then deduce in particular results concerning the density of each symbol in C_α and C'_α.

This paper is organized as follows. We introduce the first definitions and conventions in Section 2. Section 3 is devoted to the dynamical study of the configuration C_α, from which combinatorial properties are deduced in Section 4. A similat study for C'_α is performed in Section 5. Let us note that results presented here extend those of [BN05].

2 Definitions and conventions

We work in the *discrete plane* $\mathbb{Z}^2$. For each point $\mathbf{v} \in \mathbb{Z}^2$, $x_\mathbf{v}$ stands for its horizontal coordinate and $y_\mathbf{v}$ for its vertical coordinate.

Let x be a real number. We recall that the floor function $x \mapsto \lfloor x \rfloor$ is defined as the greatest integer less or equal to x. The *rounding function* is defined as $[x] := \lfloor x + 0.5 \rfloor$ and $\{x\} := x - [x]$. These applications can be extended to vectors in $\mathbb{Z}^2$, by independent application on each component.

The *discretization cell* of the point $\mathbf{v} \in \mathbb{Z}^2$ is defined as the set of elements $\mathbf{w}$ in $\mathbb{R}^2$ which have the same image by discretization as $\mathbf{v}$, i.e., $[\mathbf{v}] = [\mathbf{w}]$. Hence the discretization cell of $\mathbf{v}$ is defined as the half-opened unit square centered at $[\mathbf{v}]$.

We use the canonical bijection between the torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the square $\{\mathbf{v} \in \mathbb{R}^2; x_\mathbf{v} \in [-\frac{1}{2}, \frac{1}{2}[$ and $y_\mathbf{v} \in [-\frac{1}{2}, \frac{1}{2}[\}$, i.e., the discretization cell of 0. By abuse of notation, we also denote by $\{\mathbf{v}\}$ the image under the canonical projection from $\mathbb{R}^2$ onto $\mathbb{T}^2$ of a point $\mathbf{v} \in \mathbb{R}^2$. Let us stress the fact that the map $x \mapsto \{x\}$ is thus an additive morphism from $\mathbb{R}^2$ onto $\mathbb{T}^2$.

Without loss of generality, we assume throughout this paper that $\alpha \in [0, \pi/4]$: the arguments used here can easily be extended to the case of any other octant. We denote by r_α the Euclidean rotation of angle α:

$$r_\alpha : \mathbb{R}^2 \to \mathbb{R}^2, \ \mathbf{v} \mapsto \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \mathbf{v}.$$

The *discrete rotation* $[r_\alpha]$ is defined as

$$[r_\alpha] : \mathbb{Z}^2 \to \mathbb{Z}^2, \ \mathbf{v} \mapsto [r_\alpha(\mathbf{v})].$$

By $\{r_\alpha\}$ we mean the map

$$\{r_\alpha\} : \mathbb{Z}^2 \to \mathbb{T}^2, \ \mathbf{v} \mapsto \{r_\alpha(\mathbf{v})\}.$$

We denote by $(\mathbf{i},\mathbf{j})$ the canonical basis of the Euclidean space $\mathbb{R}^2$. We set $\mathbf{i}_\alpha := r_\alpha(\mathbf{i})$ and $\mathbf{j}_\alpha := r_\alpha(\mathbf{j})$.

Let Q be a finite set called alphabet. A two-dimensional word in $Q^{\mathbb{Z}^2}$ is called a *configuration* over Q. An application from $\{0,1,\cdots,m-1\}\times\{0,1,\cdots,n-1\}$ to Q is called a *pattern* of size $[m,n]$. Let C be a configuration in $Q^{\mathbb{Z}^2}$. A pattern χ of size $[m,n]$ occurs at position $\mathbf{v}$ in C if $C(\mathbf{v}+\mathbf{p})=\chi(\mathbf{v})$, for all $\mathbf{p}$ with $x_\mathbf{p}, y_\mathbf{p} \in \{0,1,\cdots,m-1\}\times\{0,1,\cdots,n-1\}$. The rectangular complexity function of the configuration C is defined as the function $p_C\colon \mathbb{N}^2 \to \mathbb{N}$, that counts the number of patterns of size $[m,n]$ in C.

The *density* of the symbol $p \in Q$ in the configuration $C \in Q^{\mathbb{Z}^2}$ is defined as the following limit (if it exists):

$$\eta_C(p) = \lim_{N\to\infty} \frac{\mathrm{Card}\{\mathbf{v}\in\mathbb{Z}^2,\ x_v, y_v \in \{-N,\cdots,N\} \text{ and } C(\mathbf{v}) = p\}}{(2N+1)^2}.$$

We similalry define the density of a pattern χ in the configuration C as the following limit (if it exists):

$$\eta_C(\chi) = \lim_{N\to\infty} \frac{\mathrm{Card}\{\mathbf{v}\in\mathbb{Z}^2,\ x_v, y_v \in \{-N,\cdots,N\} \text{ and } \chi \text{ occurs at position } \mathbf{v}\}}{(2N+1)^2}.$$

A *dynamical system* (X,T) is defined as the action of a continuous and onto map T on a compact space X. Given two continuous and onto maps T_1 and T_2 acting on X and satisfying $T_1 \circ T_2 = T_2 \circ T_1$, the $\mathbb{Z}^2$*-action* by T_1 and T_2 on X, that we denote (X,T_1,T_2), is defined by

$$\forall (m,n)\in\mathbb{Z}^2,\ \forall x \in X,\quad (m,n)\cdot x = T_1^m \circ T_2^n(x).$$

It is natural to associate with two-dimensional symbolic dynamical system to the triple (X,T_1,T_2) by coding the orbits of the points of X under the $\mathbb{Z}^2$-action as follows: given $x_0 \in X$ and given a *labelling function* l defined on X with values in a finite set Q that takes constant values on the atoms of a finite partition of X, the configuration C defined by

$$\forall (m,n)\in\mathbb{Z}^2,\ C(m,n) = l(T_1^m \circ T_2^n(x_0))$$

is called the coding of the orbit of x_0 under the $\mathbb{Z}^2$-action (X,T_1,T_2) with respect to the labelling function l.

3 Dynamical system associated with C_α

According to [NR03], we associate a first configuration C_α with the discrete rotation $[r_\alpha]$ that encodes local information concerning the discrete rotation: the configuration C_α is defined at point $\mathbf{v} \in \mathbb{Z}^2$ according to the action of the discrete rotation on the 4-neighbours of $\mathbf{v}$; furthermore, there exists a planar transducer that uses the configuration C_α as input and gradually computes the action of the discrete rotation [NR05].

More precisely, for a given $\mathbf{v} \in \mathbb{Z}^2$, we denote by $\mathcal{V}_4(\mathbf{v})$ the set of 4-neighbours of $\mathbf{v}$, that is, $\mathcal{V}_4(\mathbf{v}) = \{\mathbf{v}+\mathbf{i},\ \mathbf{v}+\mathbf{j},\ \mathbf{v}-\mathbf{i},\ \mathbf{v}-\mathbf{j}\}$. The configuration C_α maps each point $\mathbf{v}$ of $\mathbb{Z}^2$ to the set $[r_\alpha](\mathcal{V}_4(\mathbf{v})) - [r_\alpha][\mathbf{v}]$, that is,

$C_\alpha(\mathbf{v}) := \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ with $(a_k = [r_\alpha(\mathbf{v}+r^k_{\pi/2}(\mathbf{i}))] - [r_\alpha(\mathbf{v})]$, for $k = 0, \cdots, 3)$.

One easily checks that C_α contains either 3 or 4 non-zero elements; for a detailed proof, see [NR03]. Let Q_α stand for the finite set of values taken by C_α.

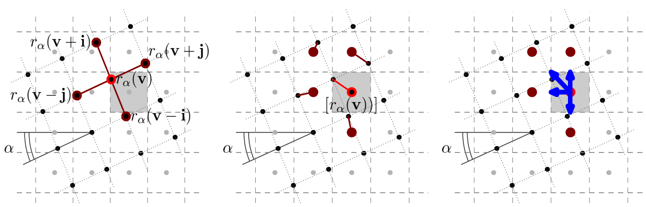


Figure 1. A progressive construction of the configuration C_α: we represent the set of vectors that leads to the relative positions of the 4-neighbors of $\mathbf{v}$ after the action of the discrete rotation.

We define a *frame* of the torus $\mathbb{T}^2 \equiv [-\frac{1}{2}, \frac{1}{2}[\times[-\frac{1}{2}, \frac{1}{2}[$ as a rectangle of the form $[a, b[\times[c, d[$, with $-\frac{1}{2} \leq a \leq b < \frac{1}{2}$ and $-\frac{1}{2} \leq c \leq b < \frac{1}{2}$. The interpretation of C_α as a coding a $\mathbb{Z}^2$-action is based on the following result:

Theorem 1 ([NR05]) *There exists a partition $P_\alpha = \{I_p,\ p \in Q_\alpha\}$ of the torus $\mathbb{T}^2$ into a finite number of frames such that*

$$\forall \mathbf{v} \in \mathbb{Z}^2,\ C_\alpha(\mathbf{v}) = p \Longleftrightarrow \{r_\alpha(\mathbf{v})\} \in I_p.$$

More precisely, the partition P_α is defined as follows: if $\alpha \in [0, \pi/6]$ (resp. $[\pi/6, \pi/4[$), then the torus is divided into at most 25 frames, delimited by the (at most) 10 lines with equation $x = -\frac{1}{2}$, $x = \frac{1}{2} - \cos(\alpha)$, $x = \sin(\alpha) - \frac{1}{2}$, $x = \frac{1}{2} - \sin(\alpha)$, $x = \cos(\alpha) - \frac{1}{2}, \frac{1}{2}$, $y = -\frac{1}{2}$, $y = \frac{1}{2} - \cos(\alpha)$, $y = \sin(\alpha) - \frac{1}{2}$, $y = \frac{1}{2} - \sin(\alpha)$, $y = \cos(\alpha) - \frac{1}{2}, \frac{1}{2}$, (resp. $x, y = -\frac{1}{2}$, $\frac{1}{2} - \cos(\alpha)$, $\frac{1}{2} - \sin(\alpha)$, $\sin(\alpha) - \frac{1}{2}$, $\cos(\alpha) - \frac{1}{2}[$, $\frac{1}{2}$). More precisely, the alphabet Q_α has exactly 25 elements if $\alpha \neq 0, \pi/4, \pi/6$, 16 elements if $\alpha = \pi/6$, and 9, if $\alpha = \pi/4$.

$(0,0) \mapsto$	$(1,0) \mapsto$	$(2,0) \mapsto$	$(3,0) \mapsto$	$(4,0) \mapsto$	$(5,0) \mapsto$
$(0,1) \mapsto$	$(1,1) \mapsto$	$(2,1) \mapsto$	$(3,1) \mapsto$	$(4,1) \mapsto$	$(5,1) \mapsto$
$(0,2) \mapsto$	$(1,2) \mapsto$	$(2,2) \mapsto$	$(3,2) \mapsto$	$(4,2) \mapsto$	
$(0,3) \mapsto$	$(1,3) \mapsto$	$(2,3) \mapsto$	$(3,3) \mapsto$	$(4,3) \mapsto$	$(5,3) \mapsto$
$(0,4) \mapsto$	$(1,4) \mapsto$	$(2,4) \mapsto$	$(3,4) \mapsto$	$(4,4) \mapsto$	$(5,4) \mapsto$
$(0,5) \mapsto$	$(1,5) \mapsto$		$(3,5) \mapsto$	$(4,5) \mapsto$	$(5,5) \mapsto$

Figure 2. Table describing the action of ϕ_c. The symbols represent the directions of the vectors of $C_\alpha(\mathbf{v})$.

Consider now the following two actions

$$T_{\mathbf{i}_\alpha} \colon \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + \{\mathbf{i}_\alpha\},\ T_{\mathbf{j}_\alpha} \colon \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + \{\mathbf{j}_\alpha\}.$$

One has

$$\forall \mathbf{v} \in \mathbb{Z}^2,\ \{r_\alpha\}(\mathbf{v}) = T_{\mathbf{i}_\alpha}^{x_\mathbf{v}} \circ T_{\mathbf{i}_\alpha}^{y_\mathbf{v}}(\mathbf{0}).$$

We then associate with the partition P_α the labelling function

$$l_{C_\alpha} \colon \mathbb{T}^2 \to Q_\alpha,\ \mathbf{v} \mapsto \phi_c(f_{C_\alpha}(x_\mathbf{v}), f_{C_\alpha}(y_\mathbf{v})),$$

where $\phi_c : \{0,1,2,3,4\}^2 \to Q_\alpha$ if $\alpha \in [0, \pi/6]$ (resp. $\phi_c : \{0,1,3,4,5\}^2 \to Q_\alpha$ if $\alpha \in [\pi/6, \pi/4]$) is described in Figure 2, and $f_{C_\alpha} \colon [-1/2, 1/2] \to \{0,1,2,3,4,5\}$ is defined by

if $\alpha \in [0, \pi/6]$:

$$\left| \begin{array}{ll} [-\frac{1}{2}, \frac{1}{2} - \cos(\alpha)[& \mapsto 0 \\ [\frac{1}{2} - \cos(\alpha), \sin(\alpha) - \frac{1}{2}[& \mapsto 1 \\ [\sin(\alpha) - \frac{1}{2}, \frac{1}{2} - \sin(\alpha)[& \mapsto 2 \\ [\frac{1}{2} - \sin(\alpha), \cos(\alpha) - \frac{1}{2}[& \mapsto 3 \\ [\cos(\alpha) - \frac{1}{2}, \frac{1}{2}[& \mapsto 4 \end{array} \right|$$

if $\alpha \in [\pi/6, \pi/4]$:

$$\left| \begin{array}{ll} [-\frac{1}{2}, \frac{1}{2} - \cos(\alpha)[& \mapsto 0 \\ [\frac{1}{2} - \cos(\alpha), \frac{1}{2} - \sin(\alpha)[& \mapsto 1 \\ [\frac{1}{2} - \sin(\alpha), \sin(\alpha) - \frac{1}{2}[& \mapsto 5 \\ [\sin(\alpha) - \frac{1}{2}, \cos(\alpha) - \frac{1}{2}[& \mapsto 3 \\ [\cos(\alpha) - \frac{1}{2}, \frac{1}{2}[& \mapsto 4 \end{array} \right|$$

The values taken by C_α, i.e., the elements of Q_α are depicted in Figure 2 according to the directions of the vectors of $C_\alpha(\mathbf{v})$, for $\mathbf{v} \in \mathbb{Z}^2$.

Theorem 1 can then be reformulated as follows:

Corollary 2 *Let C_α be the configuration associated with the discrete rotation $[r_\alpha]$. We use the notation introduced above. The configuration C_α is the coding of the orbit of $\mathbf{0}$ under the $\mathbb{Z}^2$-action $(\mathbb{T}^2, T_{\mathbf{i}_\alpha}, T_{\mathbf{i}_\alpha})$ with respect to the labelling function l_{C_α}.*

Corollary 2 means that the position, in the discretization cell of a point $\mathbf{v} \in \mathbb{Z}^2$, of the point $\{r_\alpha\}(\mathbf{v})$ of the lattice $\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ determines the directions of the images of the neighbours of $\mathbf{v}$ under the action of the discrete rotation.

Example : the case $\alpha = \pi/4$

We detail here the case $\alpha = \pi/4$. In this case, the alphabet $Q_{\pi/4}$ has 9 elements. Consider the sequences in lines of the two-dimensional word $C_{\pi/4}$. One has $m\mathbf{i}_{\pi/4} = m(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, for $m \in \mathbb{Z}$. One easily checks that the one-dimensional words $(C_{\pi/4}(m, n_0))_{m \in \mathbb{Z}^2}$ are codings of the rotation $R_{1/\sqrt{2}}\colon \mathbb{R}/(\sqrt{2}\mathbb{Z}) \to \mathbb{R}/(\sqrt{2}\mathbb{Z})$, $x \mapsto x + \frac{1}{\sqrt{2}}$, with respect to the three intervals $[-1/2, -3/2+\sqrt{2}[$, $[-3/2+\sqrt{2}, 1/2[$, $[1/2, \sqrt{2}-1/2[$. By renormalizing by $\frac{1}{\sqrt{2}}$, one obtains a coding of the rotation by $1/2$ over $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ with respect to three intervals of length $1 - 1/\sqrt{2}$, $\sqrt{2} - 1$, and $1 - 1/\sqrt{2}$. One obtains a similar result for the sequences in columns. Furthermore, the two-dimensional word $C_{\pi/4}$ presents some intriguing self-similarity properties studied in [Nou]. We plan to explore them by exploiting the self-similarity of the underlying dynamical system provided by Corollary 2, such as illustrated in Fig. 3, and by exhibiting a two-dimensional substitution generating the two-dimensional word $C_{\pi/4}$.

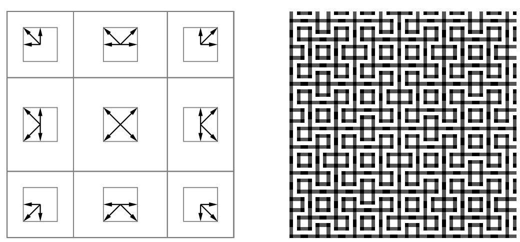


Figure 3. Left: the partition $P_{\pi/4}$. Right: an illustration of the self-similarity of $C_{\pi/4}$.

4 Distribution of symbols in C_α

We can now deduce from the $\mathbb{Z}^2$-action introduced in Section 3 combinatorial properties of the two-dimensional word C_α, and in particular, results concerning densities of symbols, by using classical tools from symbolic dynamics and ergodic theory.

Let $G_\alpha \subseteq \mathbb{T}^2$ stand for the orbit of $\mathbf{0}$ under the $\mathbb{Z}^2$-action $(\mathbb{T}^2, T_{\mathbf{i}_\alpha}, T_{\mathbf{i}_\alpha})$ with respect to the labelling function l_{C_α}: this very orbit is the orbit coded by the configuration C_α. In other words, G_α is the image by the canonical projection

$x \mapsto \{x\}$ onto $\mathbb{T}^2$ of the lattice $L_\alpha := \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ of $\mathbb{R}^2$; G_α is invariant by rotation by $\pi/2$.

Let us recall that an angle α is said *Pythagorean* if $\cos\alpha$ and $\sin\alpha$ are both rational. The density of G_α is a key ingredient of our combinatorial study. Let us distinguish two cases according to the fact that α is Pythagorean or not.

Lemma 3 *The group G_α is dense in $\mathbb{T}^2$ if and only if α is not Pythagorean. If α is not Pythagorean, then the two-dimensional sequence $(u_{m,n})_{(m,n)\in\mathbb{Z}^2}$, defined by $u_{m,n} := T^m_{\mathbf{i}_\alpha} \circ T^n_{\mathbf{j}_\alpha}(\mathbf{0})$ is equidistributed in $\mathbb{T}^2$. If α is Pythagorean, then the configuration C_α is periodic, and its lattice of periods has dimension two.*

PROOF. Let us assume that α is not Pythagorean. We prove the equidistribution of the two-dimensional sequence $(u_{m,n})_{m,n\in\mathbb{Z}^2}$ in $\mathbb{T}^2$ by using a classical argument on Weyl sums. Indeed, for $p,q \in \mathbb{Z}^2$, we set $f_{p,q}\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto e^{2i\pi(px+qy)}$. One first checks that $\iint_{[0,1]^2} f_{p,q}(x,y)dx\,dy \neq 0$ if and only if $p = q = 0$. Furthermore one has

$$
\begin{aligned}
f_{p,q}(u_{m,n}) &= e^{2i\pi p(m\cos\alpha - n\sin\alpha)} \cdot e^{2i\pi q(m\sin\alpha + n\cos\alpha)} \\
&= e^{2i\pi m(p\cos\alpha + q\sin\alpha)} \cdot e^{2i\pi n(-p\sin\alpha + q\cos\alpha)}.
\end{aligned}
$$

By hypothesis, one has either $\cos(\alpha)$ or $\sin(\alpha)$ irrational. Then one cannot have simultaneously $p\cos(\alpha) + q\sin(\alpha) \in \mathbb{Z}$ and $-p\sin(\alpha) + q\cos(\alpha) \in \mathbb{Z}$. One thus gets that for $(p,q) \in \mathbb{Z}^2$, $(p,q) \neq (0,0)$, then

$$
\lim_{n\to+\infty} \frac{1}{(2N+1)^2} \sum_{|m|,|n|\leq N} f_{p,q}(u_{m,n}) = 0,
$$

which yields the equidistribution of $(u_{m,n})_{m,n\in\mathbb{Z}^2}$.

We assume now that α is a Pythagorean angle. There exists a unique prime Pythagorean triple $(a,b,c) \in \mathbb{N}^3$ that satisfies $1 \leq b \leq a \leq c$, $gcd(a,b,c) = 1$, $\cos(\alpha) = \frac{a}{c}$, $\sin(\alpha) = \frac{b}{c}$, and hence $a^2 + b^2 = c^2$. Let $u, v \in \mathbb{Z}^2$ such that $ua - bv = \gcd(a,b)$. The vector $u\mathbf{i}_\alpha + v\mathbf{j}_\alpha$ generates G_α, which is hence a finite cyclic group of order c. Moreover, the vectors $q\mathbf{i}_\alpha$ and $q\mathbf{j}_\alpha$ are period vectors for C_α, hence the lattice of periods of C_α has dimension two. This ends the proof. □

Let us note that more information on rotations with Pythagorean angles can be found in [NR04]. We can now decduce from Lemma 3 density results for C_α.

Theorem 4 *Let C_α be the configuration associated with the discrete rotation $[r_\alpha]$. For every symbol $p \in Q_\alpha$, its density $\eta_{C_\alpha}(p)$ in C_α exists and is equal to*

- *the area of the frame I_p defined in Theorem 1, if α is not Pytahgorean,*
- *and to $1/c \cdot Card\ (G_\alpha \cap I_p)$, if α is Pythagorean, where c stands for the order of the group G_α.*

PROOF. By definition, one has

$$\eta_{C_\alpha}(p) = \lim_{N\to\infty}(\{r_\alpha\}(\{-N,\cdots,N\}^2)\cap I_p)/(2N+1)^2.$$

If α is not Pythagorean, then the result comes directly from Lemma 3.

Let us assume now α Pythagorean. One first checks that $\eta_{C_\alpha}(p) = \lim_{N\to\infty}(\{r_\alpha\}(\{-c\lfloor N/c\rfloor,...,c\lfloor N/c\rfloor\}^2)\cap I_p)/(2N+1)^2)$. But as G_α is cyclic and of order c, then $\eta_{C_\alpha}(p) = \frac{\{r_\alpha\}(\{0,...,c-1\}^2)\cap I_p}{c^2} = \frac{\text{Card}\ (G_\alpha\cap I_p)}{c}$.□

We can similarly deduce the following combinatorial properties of the two-dimensional word C_α. Let us note that we have focused here on the statistical properties of repartition of the symbols in Q_α because of their interest for the study of the discrete rotation $[r_\alpha]$.

Theorem 5 *Let C_α be the configuration associated with the discrete rotation $[r_\alpha]$. The density of rectangular patterns exists in C_α for every pattern χ that occurs in C_α. The two-dimensional word C_α is uniformly recurrent, i.e., for every positive integer n, there exists a positive integer N such that every square pattern of size $[N,N]$ of C_α contains every square patternof size $[n,n]$ of C_α. Furthermore, there exists a positive constant A such that the rectangular complexity function of C_α satisfies*

$$\forall m,n,\ p_{C_\alpha}(m,n) \leq A\cdot mn.$$

PROOF. We first deduce from Corollary 2 that given two positive integers m,n, there exists a finite partition of $\mathbb{T}^2$ into finite unions of frames $P_\alpha^{[m,n]} = \{J_\chi,\ \chi \text{ pattern of size } [m,n] \text{ of } C_\alpha\}$ such that χ occurs at position $\mathbf{v}$ in C_α if and only if $\{r_\alpha\}(\mathbf{v}) \in I_\chi$. Let us stress the fact that the sets J_χ are not necessarily frames, nor even connected sets; indeed, they are obtained as finite intersections of frames I_p associated with symbols $p\in Q_\alpha$. More precisely, I_χ is obtained as follows:

$$I_\chi = \cap_{0\leq k\leq m-1,\ 0\leq \ell\leq n-1} T^k_{\mathbf{i}_\alpha}\circ T^\ell_{\mathbf{j}_\alpha} I_{\chi(k,l)}.$$

This allows us to deduce the existence of densities for all rectangular patterns of C_α. We thus obtain analogously as for Theorem 4 that they are equal to the measure of I_χ, in the non-Pythagorean case, and to the cardinality of the intersection of G_α with I_χ, in the Pythagorean case.

Let us assume that α is non-Pythagorean. We assume w.l.o.g. that $\cos(\alpha)\notin\mathbb{Q}$.

According to [Sla67], given any interval I of $\mathbb{T}^1$, there exists n_0 such that among any finite sequence of points $\{k\cos(\alpha)\}, \{(k+1)\cos(\alpha)\}, \cdots, \{(k+n_0)\cos(\alpha)\}$, at least of them belongs to I. Let us fix a pattern χ and a position $\mathbf{v} \in \mathbb{Z}^2$. We apply the previous result to the interval $I_\chi \cap [-1/2, 1/2[$, and to the sequence $(T^k_{\mathbf{i}_\alpha}(\mathbf{v}) \cap [-1/2, 1/2[)_{k\in\mathbb{Z}} = (x_{\mathbf{v}} + k\cos(\alpha))_{k\in\mathbb{Z}}$. Hence given any $\mathbf{v} \in \mathbb{Z}^2$, the pattern χ occurs at position $\mathbf{v} + k(1,0)$, for some k with $0 \leq k \leq n_0$, of the configuration C_α, which yields the uniform recurrence. If α is Pythagorean, then the uniform recurrence follows from the fact that C_α has a lattice of periods of rank 2.

We obtain an upper bound on the complexity function by counting the connected components of the sets obtained by taking intersections of the form $\bigcap_{0\leq k\leq m-1,\ 0\leq \ell\leq n-1} T^k_{\mathbf{i}_\alpha} \circ T^\ell_{\mathbf{j}_\alpha} I_{\chi(k,l)}$. We thus get $P_{C_\alpha}(m+1,n) - P_{C_\alpha}(m,n) \leq 5n$, for all $n \in \mathbb{N}$, which yields the desired result by a simple induction. □

Remark 6 *Let us note that we deduce from Lemma 3 that the symbols that appear in C_α at indices of the form $2\mathbf{v}$, for $\mathbf{v} \in \mathbb{Z}^2$ are exactly the elements of Q_α. Indeed, in the non-Pythagorean case, the sequence $(u_{2m,2n})_{(m,n)\in\mathbb{Z}^2}$ is still dense. Otherwise, we use the fact that the Pythagorean triple (a,b,c) introduced in the proof of Lemma 3 is assumed to be a prime triple, i.e., $\gcd(a,b,c) = 1$. We will use this remark hereafter.*

5 Distribution of Symbols in C'_α

We consider now a second configuration C'_α studied, e.g., in [NR05]:

$$\forall \mathbf{v} \in \mathbb{Z}^2,\ C'_\alpha(\mathbf{v}) := \bigcup_{\mathbf{w} \text{ such that } [r_\alpha(\mathbf{w})] = \mathbf{v}} C_\alpha(\mathbf{w}).$$

The configuration C'_α codes the action of $[r_\alpha]$ on the 4-neighbours of preiamages of points of $\mathbb{Z}^2$.

Let Q'_α stand for the set of values taken by C'_α. We want to state a result analogous to Theorem 1 in order, first, to interpret the configuration C'_α as a coding of a symbolic dynamical system, and second, to compute the densities of the symbols in C'_α. Let us note that Corollary 1 in [NR05] does not directly yield a dynamical interpretation of C'_α.

Let us note that there exist elements $\mathbf{v} \in \mathbb{Z}^2$ that have no antecedent by $[r_\alpha]$. Such an element is called a *hole*. An example of a hole is depicted in Figure 5 below. According to [NR04], two holes can never be adjacent, i.e., if $\mathbf{v}$ is a hole, then neither $\mathbf{v}+\mathbf{i}$, nor $\mathbf{v}+\mathbf{j}$ is a hole. Our strategy in order to describe C'_α as a coding of a $\mathbb{Z}^2$-action is thus to create a "block configuration" by working with patterns of size $[2,2]$ that occur in C'_α. According to Remark 6, there is no restriction in working with even indices, rather than with odd indices.

We then introduce a particular domain of $\mathbb{R}^2$ that is a fundamental domain for the lattice $\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$, such that if we know the projection of a point $\mathbf{p} \in \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ in that domain, then we can recover the symbols that appear in the block configuration; therefore we find out what are the symbols that appear in C'_α. We thus deduce a symbolic dynamical system for the block configuration. Finally, we use this dynamical system, in order to get the density of the symbols both in the block configuration and in C'_α.

5.1 Dynamical system for $C'_{B\alpha}$

We denote by $(Q'_\alpha)^{[2,2]}$ the set of patterns of size $[2,2]$ that occur in C'_α. Let $(C'_\alpha)^{[2,2]}$ be the configuration with values in the finite alphabet $(Q'_\alpha)^{[2,2]}$ that maps $\mathbf{v}$ to to the pattern of size $[2,2]$ that occurs at position $2\mathbf{v}$ in C'_α. Since $(C'_\alpha)^{[2,2]}(\mathbf{v})$ is an application that returns patterns of size $[2,2]$, then $C'_\alpha(\mathbf{v})$ is obtained by taking the value at position $(x_\mathbf{v} \bmod 2, y_\mathbf{v} \bmod 2)$ in the $[2,2]$ pattern $(C'_\alpha)^{[2,2]}(\lfloor x_\mathbf{v}/2 \rfloor, \lfloor y_\mathbf{v}/2 \rfloor)$.

For any $\mathbf{v} \in \mathbb{Z}^2$, one sets

$$F_B(\mathbf{v}) = [x_\mathbf{v} - \frac{1}{2}, x_\mathbf{v} + \frac{3}{2}[\times[y_\mathbf{v} - \frac{1}{2}, y_\mathbf{v} + \frac{3}{2}[.$$

Let

$$F_{D_\alpha} := \Big([-\frac{1}{2}, \cos\alpha - \frac{1}{2}[\Big)^2 \cup \Big([\cos\alpha - \frac{1}{2}, \cos\alpha + \sin\alpha - \frac{1}{2}[\times[-\frac{1}{2}, \sin\alpha - \frac{1}{2}[\Big).$$

The set F_{D_α} is a fundamental domain for the lattice $L_\alpha = \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ (see Figure 4), i.e., $\cup_{\gamma \in L_\alpha} F_{D_\alpha} + \gamma$ is a partition of $\mathbb{R}^2$. We thus set $\mathbb{T}^2_\alpha := \mathbb{R}^2/(\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha)$. Furthermore, we denote by $\mathbf{v} \mapsto \{\mathbf{v}\}_\alpha$ the canonical projection on $\mathbb{T}^2_\alpha$, $\mathbb{T}^2_\alpha$ being in one-to-correspondence with F_{D_α}.

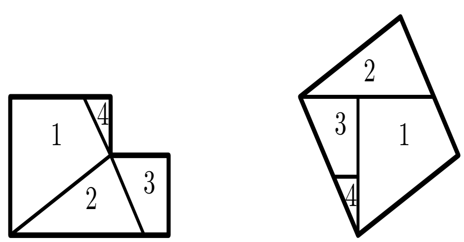


Figure 4. An exchange of pieces between F_{D_α} and the canonical representation of $\mathbb{R}^2/L_\alpha$, obtained by performing translations in L_α.

Theorem 7 *Let Let $\alpha \in [0, \pi/4]$. Let C'_α be the configuration associated with the discrete rotation $[r_\alpha]$. There exists a partition $P'_\alpha = \{J_{p'},\ p' \in Q'_\alpha\}$ of F_{D_α}*

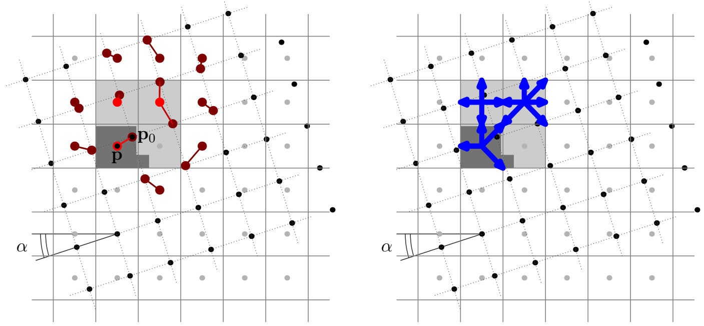


Figure 5. From a point $\mathbf{p}_0 \in \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ contained in the domain $F_{D_\alpha}(2\mathbf{v})$ (in dark gray), we can recover all the symbols that contribute to the block of size $[2,2]$ at position $2\mathbf{v}$ in C'_α; $F_B(2\mathbf{v})$ is depicted in light gray.

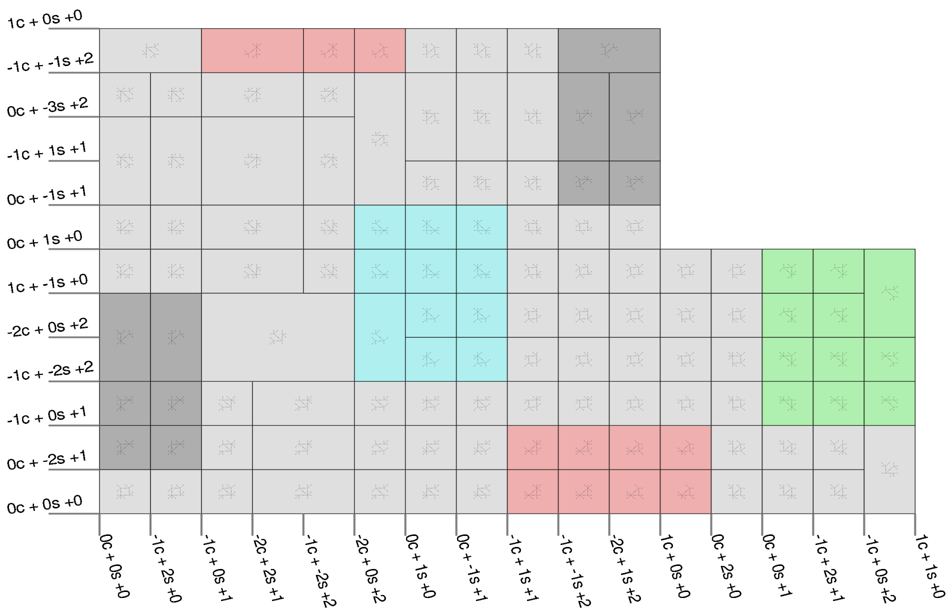


Figure 6. A partition of the domain F_{D_α}, for $\alpha \approx 0.464705$ rad. This partition gives the pattern of size $[2,2]$ that appears in $(C'_\alpha)^{[2,2]}(\mathbf{v})$, according to the position of $-\{\!\!\{2\mathbf{v}\}\!\!\}_\alpha$ inside that domain. On the axis the positions are labeled by expressions of the form $kc + k's + k''$, meaning that the corresponding line is located at $k\cos(\alpha) + k'\sin(\alpha) + k'' - \frac{1}{2}$ in F_{D_α}. For readability reasons, the scale is monotone but not linear.

into a finite number of frames such that

$$\forall \mathbf{v} \in \mathbb{Z}^2,\ (C'_\alpha)^{[2,2]}(\mathbf{v}) = p' \iff -\{\!\!\{2\mathbf{v}\}\!\!\}_\alpha \in J_{p'}.$$

We define by $l_{(C'_\alpha)^{[2,2]}} : \mathbb{T}^2_\alpha \to (Q'_\alpha)^{[2,2]}$ *the labelling function that associates with elements of the frame* $J_{p'} \in P'_\alpha$ *of* F_{D_α} *the corresponding pattern* p' *of* *size* $[2,2]$, *i.e.*,

$$\forall \mathbf{v} \in \mathbb{Z}^2, (C'_\alpha)^{[2,2]}(\mathbf{v}) = l_{(C'_\alpha)^{[2,2]}}(-\{2\mathbf{v}\}_\alpha).$$

The configuration $(C'_\alpha)^{[2,2]}$ *is thus a coding of the orbit of* 0 *under the* $\mathbb{Z}^2$*-action* $(\mathbb{T}^2_\alpha, \mathbf{v} \mapsto \mathbf{v} + \{-2\mathbf{i}\}_\alpha, \mathbf{v} \mapsto \mathbf{v} + \{-2\mathbf{j}\}_\alpha)$ *with respect to the labelling function* $l_{(C'_\alpha)^{[2,2]}}$.

PROOF. The proof is based on the following idea: for any $\mathbf{v} \in \mathbb{Z}^2$, there exists a unique $\gamma \in L_\alpha = r_\alpha(\mathbb{Z})$ such that $-\mathbf{v} \in \gamma + F_{D_\alpha}$, i.e., for any $\mathbf{v} \in \mathbb{Z}^2$, there exists a unique $\mathbf{w} \in \mathbb{Z}^2$ such that $-2\mathbf{v} \in -r_\alpha(\mathbf{w}) + F_{D_\alpha}$. One thus has $r_\alpha(\mathbf{w}) - 2\mathbf{v} = \{-2\mathbf{v}\}_\alpha = -\{2\mathbf{v}\}_\alpha$. Let us prove that it is possible to deduce the value of $(C'_\alpha)^{[2,2]}(\mathbf{v})$ from the location of $\{-2\mathbf{v}\}_\alpha$ in F_{D_α}.

For that purpose, we first check that for all points $\mathbf{w}$ of $\mathbb{Z}^2$ that have their image by r_α in $F_B(2\mathbf{v})$ we can compute $C_\alpha(\mathbf{w})$, according to Theorem 1 and Remark 6. Indeed, let $\mathbf{w}$ be the unique element $\mathbb{Z}^2$ such that $r_\alpha(\mathbf{w}) \in \mathbf{v} + F_{D_\alpha}$; if $x_{r_\alpha(\mathbf{w})-2\mathbf{v}} < \frac{1}{2}$, $[r_\alpha(\mathbf{w}) - 2\mathbf{v}] = 0$, else $[r_\alpha(\mathbf{w}) - \mathbf{v}] = 1$; we thus deduce the value of $C_\alpha(\mathbf{w})$, according to Theorem 1. ence, we get a first partition of F_{D_α} into a finite number of frames yielding the value of $C_\alpha(\mathbf{w})$.

The same argument applies for all points $\mathbf{w}' = r_\alpha(\mathbf{w})$ of $\mathbb{Z}_{\mathbf{i}_\alpha} + \mathbb{Z}_{\mathbf{j}_\alpha}$ that are inside $F_B(2\mathbf{v})$. We thus refine our first partition by intersecting it by translates by vectors of L_α, which ends the proof. □

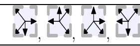
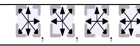
5.2 Application

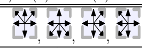
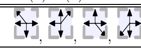
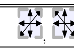
We can perform the same combinatorial study as in Section 4. In particular, Lemma 3 extends in a natural way. We do not detail here the corresponding results but focus on the following application to density of symbols. We assume in particular that α is not a Pythagorean angle. Similarly as in the study of C_α, the orbit of 0 under the $\mathbb{Z}^2$-action is dense and uniformly distributed in $\mathbb{T}^2_\alpha$. We thus deduce that

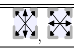
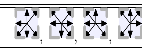
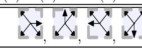
$$\forall p \in Q'_\alpha,\ \eta_{C'_\alpha}(p) = \sum_{p' \in (Q'_\alpha)^{[2,2]}} n(p', p)\, \mu(f_{p'}),$$

where $n(p', p)$ is the function that returns the number of occurrences of p in the pattern p' of size $[2,2]$, and $\mu(J_{p'})$ denotes the area of frame $J_{p'}$ associated with the symbol p' according to Theorem 7.

However practically, the computations for these symbolic maps are quite tedious. For each symbol p, there exist 40 patterns p' of size $[2,2]$ to compute. This leads to approximatively 360 inequations... and there are approximatively 25 symbols p to consider! The results describing the densities of the symbols in C'_α have been summarized in Figure 7. In the Pythagorean case, the study is also similar to the one developed for C_α.

α			
$[0..\arctan(\sqrt{2}/4)]$	$2\cos(\alpha)sin(\alpha) - 2\cos(\alpha) - 2\sin(\alpha) + 2$	$2\cos(\alpha)sin(\alpha) - 2\cos(\alpha) - 2\sin(\alpha) + 2$	$-(\cos(\alpha))^2 - \cos(\alpha)sin(\alpha) + 2\cos(\alpha) + \sin(\alpha) - 1$
$[\arctan(\sqrt{2}/4)..\arctan(1/2)]$	$2\cos(\alpha)sin(\alpha) - 2\cos(\alpha) - 2\sin(\alpha) + 2$	$2\cos(\alpha)sin(\alpha) - 2\cos(\alpha) - 2\sin(\alpha) + 2$	$-(\cos(\alpha))^2 + 2\cos(\alpha)sin(\alpha) + \cos(\alpha) - 2\sin(\alpha)$
$[\arctan(1/2)..\pi/6]$	$2\cos(\alpha)sin(\alpha) - 2\cos(\alpha) - 2\sin(\alpha) + 2$	$2\cos(\alpha)sin(\alpha) - 2\cos(\alpha) - 2\sin(\alpha) + 2$	0
$[\pi/6..\arctan(3/4)]$	$2\cos(\alpha)sin(\alpha) - 2\cos(\alpha) - 2\sin(\alpha) + 2$	$-2\cos(\alpha)sin(\alpha) + 1$	0
$[\arctan(3/4)..\pi/4]$	$2\cos(\alpha)sin(\alpha) - 2\cos(\alpha) - 2\sin(\alpha) + 2$	$-2\cos(\alpha)sin(\alpha) + 1$	0

α			
$[0..\arctan(\sqrt{2}/4)]$	$(\cos(\alpha))^2 - 2\cos(\alpha) + 1$	$-2(\sin(\alpha))^2 - 2\cos(\alpha)sin(\alpha) + \cos(\alpha) + 3\sin(\alpha) - 1$	$3\cos(\alpha)sin(\alpha) - \cos(\alpha) - 3\sin(\alpha) + 1$
$[\arctan(\sqrt{2}/4)..\arctan(1/2)]$	$(\cos(\alpha))^2 - 2\cos(\alpha) + 1$	$-2(\sin(\alpha))^2 - 2\cos(\alpha)sin(\alpha) + \cos(\alpha) + 3\sin(\alpha) - 1$	0
$[\arctan(1/2)..\pi/6]$	$2\cos(\alpha)sin(\alpha) - \cos(\alpha) - 2\sin(\alpha) + 1$	$-2(\sin(\alpha))^2 - 2\cos(\alpha)sin(\alpha) + \cos(\alpha) + 3\sin(\alpha) - 1$	0
$[\pi/6..\arctan(3/4)]$	0	0	0
$[\arctan(3/4)..\pi/4]$	0	0	$2(\cos(\alpha))^2 - \cos(\alpha)sin(\alpha) - 3\cos(\alpha) + \sin(\alpha) + 1$

α			
$[0..\arctan(\sqrt{2}/4)]$	0	0	0
$[\arctan(\sqrt{2}/4)..\arctan(1/2)]$	$-3\cos(\alpha)sin(\alpha) + \cos(\alpha) + 3\sin(\alpha) - 1$	0	0
$[\arctan(1/2)..\pi/6]$	$-2(\cos(\alpha))^2 + \cos(\alpha)sin(\alpha) + 3\cos(\alpha) - \sin(\alpha) - 1$	$(\cos(\alpha))^2 - 2\cos(\alpha)sin(\alpha) - \cos(\alpha) + 2\sin(\alpha)$	0
$[\pi/6..\arctan(3/4)]$	$-2(\cos(\alpha))^2 + \cos(\alpha)sin(\alpha) + 3\cos(\alpha) - \sin(\alpha) - 1$	$(\cos(\alpha))^2 - 2\cos(\alpha) + 1$	$-2(\sin(\alpha))^2 + 2\cos(\alpha)sin(\alpha) - \cos(\alpha) + \sin(\alpha)$
$[\arctan(3/4)..\pi/4]$	0	$-(\cos(\alpha))^2 + \cos(\alpha)sin(\alpha) + \cos(\alpha) - \sin(\alpha)$	$-2(\sin(\alpha))^2 + 2\cos(\alpha)sin(\alpha) - \cos(\alpha) + \sin(\alpha)$

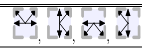
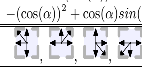
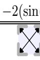
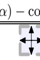
α				
$[0..\arctan(\sqrt{2}/4)]$	0	$-(\cos(\alpha))^2 - \cos(\alpha)sin(\alpha) + 2\cos(\alpha) + \sin(\alpha) - 1$	0	$4(\sin(\alpha))^2 - 4\sin(\alpha) + 1$
$[\arctan(\sqrt{2}/4)..\arctan(1/2)]$	0	$-(\cos(\alpha))^2 - \cos(\alpha)sin(\alpha) + 2\cos(\alpha) + \sin(\alpha) - 1$	0	$4(\sin(\alpha))^2 - 4\sin(\alpha) + 1$
$[\arctan(1/2)..\pi/6]$	0	$-(\cos(\alpha))^2 - \cos(\alpha)sin(\alpha) + 2\cos(\alpha) + \sin(\alpha) - 1$	0	$4(\sin(\alpha))^2 - 4\sin(\alpha) + 1$
$[\pi/6..\arctan(3/4)]$	$-2\cos(\alpha)sin(\alpha) + \cos(\alpha) + 2\sin(\alpha) - 1$	$-(\cos(\alpha))^2 + \cos(\alpha)sin(\alpha) + \cos(\alpha) - \sin(\alpha)$	$4(\sin(\alpha))^2 - 4\sin(\alpha) + 1$	0
$[\arctan(3/4)..\pi/4]$	$-2\cos(\alpha)sin(\alpha) + \cos(\alpha) + 2\sin(\alpha) - 1$	$-(\cos(\alpha))^2 + \cos(\alpha)sin(\alpha) + \cos(\alpha) - \sin(\alpha)$	$4(\sin(\alpha))^2 - 4\sin(\alpha) + 1$	0

Figure 7. Table describing $\eta_{C'_\alpha}(p)$ for each symbol p that appears in C'_α, with respect to the value of α.

References

[AI01] P. Arnoux and S. Ito. Pisot substitutions and Rauzy fractals. *Bull. Belg. Math. Soc. Simon Stevin*, 8(2):181–207, 2001.

[And92] E. Andrès. *Discrete circles and discrete rotations*. PhD thesis, Université Louis Pasteur, Strasbourg, France, 1992.

[AR91] P. Arnoux and G. Rauzy. Représentation géométrique de suites de complexité $2n+1$. *Bull. Soc. Math. France*, 119(2):199–215, 1991.

[BEIR] V. Berthé, H. Ei, S. Ito, and H. Rao. Invertible substitutions and sturmian words: an application of rauzy fractals. Theoret. Informatics Appl., to appear.

[BFJP] V. Berthé, C. Fiorio, D. Jamet, and F. Philippe. On some applications of generalized functionality for arithmetic discrete plane. To appear.

[BFZ05] V. Berthé, S. Ferenczi, and L.Q. Zamboni. Interactions between dynamics, arithmetics, and combinatorics: the good, the bad, and the ugly. In S. Kolyada, Y. Manin, and T. Ward, editors, *Algebraic and Topological Dynamics*, volume 385 of *Contemporary mathematics*, pages 333–364. American Mathematical Society, 2005.

[BN05] V. Berthé and B. Nouvel. Density of symbols in discretized rotation configurations. In S. Brlek and Reutenauer, editors, *Words 2005, 5th International Conference on Words,*, volume 36 of *Publications du LACIM*, pages 163–173, 2005.

[BV00] V. Berthé and L. Vuillon. Tilings and rotations on the torus: a two-dimensional generalization of Sturmian sequences. *Discrete Math.*, 223:27–53, 2000.

[Fog02] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[Lot02] M. Lothaire. *Algebraic combinatorics on words*. Cambridge University Press, 2002.

[Lot05] M. Lothaire. *Applied combinatorics on words*. Cambridge University Press, 2005.

[Nou] B. Nouvel. Substitutions, low-complexity configurations and Rauzy graphs. Preprint 2006.

[NR03] B. Nouvel and E. Rémila. On colorations induced by discrete rotations. In *DGCI*, number 2886 in LNCS, pages 174–183, 2003.

[NR04] B. Nouvel and E. Rémila. Characterization of bijective discretized rotations. In *International Workshop on Combinatorial Images Analysis, 10th International Conference, IWCIA 2004, Auckland, New Zealand, December 1-4, 2004*, number 3322 in LNCS, pages 248–259, 2004.

[NR05] B. Nouvel and E. Rémila. Configurations induced by discrete rotations: Periodicity and quasiperiodicity properties. *Discrete Applied Mathematics*, 2-3(147):325–343, 2005.

[Rau82] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.

[Rev91] J.-P. Reveillès. *Calcul en nombres entiers et algorithmique*. Thèse d'état, Université Louis Pasteur, Strasbourg, France, 1991.

[Sla67] N. B. Slater. Gaps and steps for the sequence $n\theta$ mod 1. *Proc. Cambridge Philos. Soc.*, 63:1115–1123, 1967.

[SW] C. Steineder and R. Winkler. Complexity of hartman sequences. J. Théor. Nombres Bordeaux, to appear.

[Van00] D. Vandeth. Sturmian words and words with a critical exponent. *Theoret. Comput. Science*, 242:283–300, 2000.