Asymptotic behavior of the number of solutions for non-Archimedean Diophantine approximations with restricted denominators

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Abstract

We consider metric results for the asymptotic behavior of the number of solutions of Diophantine approximation inequalities with restricted denominators for Laurent formal power series with coefficients in a finite field. We consider, in particular, approximations by rational functions whose denominators are powers of irreducible polynomials, and we study the strong law of large numbers for solutions of the inequalities under consideration.

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1 Introduction

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The metric theory of Diophantine approximation and particularly, the asymptotic behavior of the number of solutions of Diophantine approximation inequalities has arisen an important literature in the real case, see e.g. the books [4,9]. Such results can also be naturally extended to the case of Laurent formal power series with coefficients in a finite field. Let us quote e.g. [5] and [7] which discuss the strong law of large numbers in metric theory of Diophantine approximation in positive characteristics. In the present paper, we consider specific inequalities with restricted denominators (powers of irreducible polynomials) with an approximation function which does not only depend on the degree of the denominator.

As usual, let $\mathbb{F}_q$ be a finite field of cardinality q, and we denote as

$$\mathbb{F}[X], \qquad \mathbb{F}(X), \qquad \mathbb{F}((X)^{-1}), \qquad \mathbb{L}$$

the set of polynomials (with $\mathbb{F}_q$-coefficients), the set of rational functions, the set of Laurent formal power series, and the set of Laurent formal power series of negative degree, respectively. Here we define the degree of

$$f = a_n X^n + a_{n-1} X^{n-1} + \cdots$$

by $\deg f = n$. We consider the topology on $\mathbb{L}$ induced by the metric $d(f, g) = |f - g|$ for $f, g \in \mathbb{L}$, and we denote by m the Haar probability measure on $\mathbb{L}$.

We consider for a given formal power series $f \in \mathbb{L}$ the solutions $\frac{P}{Q}$ with P, Q polynomials with coefficients in $\mathbb{F}_q$ of

$$\left| f - \frac{P}{Q} \right| < \frac{\Psi(Q)}{|Q|}, \qquad P,\ Q : \text{coprime},\ \ Q : \text{monic} \tag{1}$$

where $|Q| = q^{\deg Q}$.

In the case where the approximation function Ψ depends only on the degree of Q, that is, if Ψ has form $\Psi(Q) = \frac{1}{q^{n+l_n}}$ if $\deg Q = n$, with l_n being a nonnegative integer, then the strong law of large numbers holds whenever $\sum \frac{1}{q^{l_n}} = \infty$. Moreover, some limit theorems can be obtained under a mild condition on l_n, see [3,5,1,2]. Note that although it is not explicitly stated as such in [5], the proof of the Khintchine type theorem stated in [5] implies results on the asymptotic behavior of the number of solutions for non-Archimedean Diophantine approximations. It is furthermore proved in [7] that the strong law of large numbers also holds even if we do not assume coprimeness of P and Q.

However, it seems to be not so easy to get the strong law of large numbers for Ψ not depending only on degree of Q. Indeed, the only known result in the general case (Ψ not depending only on $\deg Q$) is a Duffin-Schaeffer type theorem, i.e., a generalized Khintchine type theorem (see [5]). In this paper, we thus consider some special cases of approximation function Ψ:

(i) Ψ is finite only when Q is irreducible, otherwise it takes value ∞,
(ii) Ψ is finite only when Q is the t-power, t being fixed, of a single monic irreducible polynomial, otherwise it takes value ∞,
(iii) Ψ is finite when Q is some power of a single monic irreducible polynomial, otherwise it takes value ∞.

More precisely, we consider the three following inequalities for coprime polynomials P and Q:

(i) $\left|f - \dfrac{P}{Q}\right| < \dfrac{1}{q^{2n+l_n}}, \qquad \deg Q = n, \quad Q$: monic and irreducible, where the sequence of nonnegative integers (l_n) is given (see Section 2);

(ii) $\left|f - \dfrac{P}{Q}\right| < \dfrac{1}{q^{(t+1)n+l_{Q_1}}}$, $\deg Q_1 = n$, Q_1 : monic and irreducible, $Q = Q_1^t$, where t is a fixed positive integer and the sequence of nonnegative integers (l_{Q_1}) is given (see Section 3);

(iii) $\left|f - \dfrac{P}{Q}\right| < \dfrac{1}{q^{(t+1)n+l_{Q_1}+l_t}}$, $\deg Q_1 = n, Q_1$: monic and irreducible, $Q = Q_1^t$, for some positive integer t, where both sequences of nonnegative integers (l_{Q_1}) and $(l_t : t \geq 1)$ are given and $\sum_{t\geq 1} \frac{1}{q^{l_t}}$ is assumed to be a convergent series (see Section 4).

Obviously, (ii) is a special case of (iii) and (i) is a special case of (ii). However, we estimate the asymptotic behavior of the number of solutions of these inequalities step by step, first, for the sake of clarity, and secondly, because (i) itself is interesting as a Diophantine approximation problem: this corresponds to the approximation of irrational numbers by rational numbers with prime denominators. Note that (ii) is somehow a natural generalization of (i). On the other hand, (iii) illustrates the difficulty of finding a sufficient condition for Ψ holding the strong law of large numbers: indeed, we have to add as an extra hypothesis that $\sum_{t\geq 1} \frac{1}{q^{l_t}}$ is assumed to be a convergent series.

The main tool of our proofs will be the following lemma, also used in [7]. We recall here that the notation $X \ll Y$ is equivalent to the notation $X = O(Y)$.

Lemma 1.1 (Sprindžuk, p. 45) *Let $(\xi_n(\omega) : n \geq 1)$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{B}, P)$. Moreover let $(\eta_n : n \geq 1)$ and $(\hat{\eta}_n : n \geq 1)$ be sequences of real numbers such that*

(i) $0 \leq \eta_n \leq \hat{\eta}_n \leq 1$, for all $n \geq 1$,
(ii) for any positive integers $N_1 < N_2$

$$\int_\Omega \Big(\sum_{n=N_1}^{N_2} \xi_n(\omega) - \eta_n\Big)^2 dP \ll \sum_{n=N_1}^{N_2} \hat{\eta}_n.$$

Then, one has

$$\sum_{n=1}^{N} \xi_n(\omega) = \sum_{n=1}^{N} \eta_n + O(\Psi(N)^{1/2} \log^{\frac{3+\varepsilon}{2}} \Psi(N)) \quad \text{for } P\text{-a.e.},$$

where $\varepsilon > 0$ *is arbitrary and* $\Psi(N) = \sum_{n=1}^{N} \hat{\eta}_n$.

This lemma can be considered as a refinement of Kintchine's theorem. The idea of this lemma (called Schmidt's method in [4]) was used in metric theory of classical Diophantine approximation in the 50's (see e.g. [8]). As application in the real case of Schmidt's method, one obtains asymptotic formulas for the number of solutions of Diophantine inequalities for restricted sets of denominators such as the set of prime numbers (see Theorem 18 in [9]) or sets of positive lower density (see e.g. Chapter 4 in [4]).

We should note that we may apply this lemma if the approximation function Ψ is large. However, in such a case, the error term might be larger than the main term. This is one of the reasons we have chosen an approximation fuction Ψ of type (ii) and (iii) in the right-hand side of Inequality (1).

In all that follows, the denominators that we consider are assumed to be monic.

2 Metric Diophantine approximation by irreducible polynomial denominators

We consider in this section an inequality of type (1) with restricted denominators that are supposed to be monic irreducible polynomials and a function $\Psi\colon \mathbb{F}_q[X] \to \mathbb{N}$ of the form $Q \mapsto |Q| + l_Q$, where l_Q takes nonnegative integer values for Q monic irreducible, and infinite value otherwise. We thus consider the following inequality over $\mathbb{L}$:

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n+l_Q}} \tag{2}$$

where P and Q are coprime, $\deg Q = n$, and l_Q takes infinite value whenever Q is not monic irreducible.

Theorem 2.1 *For almost all* $f \in \mathbb{L}$, *the number of solutions of (2) with* $\deg Q \leq N$ *satisfies*

$$\Psi(N) + O(\Psi^{1/2}(N) \log^{\frac{3+\varepsilon}{2}} \Psi(N))$$

with

$$\Psi(N) = \sum_{n=1}^{N} \sum_{Q:\deg Q=n} \frac{1}{q^{n+l_Q}}$$

for any $\varepsilon > 0$.

Remark 2.2 *If $\Psi(N)$ does not diverge to ∞ as $N \to \infty$, then Theorem 2.1 means that there exist at most finitely many solutions for a.e. $f \in \mathbb{L}$.*

We first need the following lemma:

Lemma 2.3 *We fix coprime monic polynomials Q and Q' such that $n = \deg Q$ and $m = \deg Q'$. Let l be a nonnegative integer. The number of pairs of polynomials (P, P') that satisfy*

$$\left| \frac{P}{Q} - \frac{P'}{Q'} \right| < \frac{1}{q^{m+l}}$$

is less than q^{n-l}.

Proof of Lemma 2.3 Since

$$\left| \frac{P}{Q} - \frac{P'}{Q'} \right| = \left| \frac{PQ' - P'Q}{Q\,Q'} \right|$$

we have

$$n + m - \deg(PQ' - P'Q) > m + l$$

and so

$$\deg(PQ' - P'Q) < n - l. \tag{3}$$

Note that the number of polynomials of degree less than $n-l$ is less than q^{n-l}. Now suppose that there exist two pairs of polynomials (P_1, P_1') and (P_2, P_2') that satisfy

$$P_1Q' - P_1'Q = P_2Q' - P_2'Q$$

with

$$\deg P_1,\, \deg P_2 < \deg Q \quad \text{and} \quad \deg P_1',\, \deg P_2' < \deg Q'.$$

Since $(Q, Q') = 1$, we deduce from

$$(P_1 - P_2)Q' = (P_1' - P_2')Q$$

that we have $P_1 = P_2$ and $P_1' = P_2'$. Thus the number of pairs (P, P') such that

$$\left| \frac{P}{Q} - \frac{P'}{Q'} \right| < \frac{1}{q^{m+l}}$$

is less that q^{n-l}. We note that, by (3), if $l \geq n$ then there exists no such pair (P, P'). ■

Proof of Theorem 2.1 For Q monic polynomial of $\deg Q = n$, we put

$$F_{\frac{P}{Q}} := \{f \in \mathbb{L} : \left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n+l_Q}}\},$$

$$F_Q := \cup_{P:0\leq \deg P<\deg Q,(P,Q)=1} \; F_{\frac{P}{Q}},$$

and

$$F_n := \cup_{Q:\text{irr.}, \deg Q=n} F_Q.$$

Let us note that the sets F_Q's are disjoint for a given n. Furthermore, one checks that

$$m(F_{\frac{P}{Q}}) = \frac{1}{q^{2n+l_Q}},\ m(F_Q) = \frac{q^n-1}{q^{2n+l_Q}},\ m(F_n) = \sum_{Q:\text{irr.}, \deg Q=n} \frac{q^n-1}{q^{2n+l_Q}}.$$

Hence

$$m(F_n) \sim \sum_{Q:\text{irr.}, \deg Q=n} \frac{1}{q^{n+l_Q}}.$$

Note that we can give a more precise estimate for $m(F_n)$ but that we do not need more in the present proof.

Let us apply Lemma 1.1 by setting for all n, $\xi_n := \chi_{F_n}$, the indicator function of the set F_n, and $\eta_n := \hat{\eta_n} := \int \xi_n dm$. Condition (i) of Lemma 1.1 is satisfied. Let us consider now Condition (ii). Then it is enough to show

$$\sum_{n=N_1}^{N_2} \sum_{m=N_1}^{N_2} m(F_n \cap F_m) - m(F_n)m(F_m) \ll \sum_{n=N_1}^{N_2} m(F_n)$$

for any positive integers $N_1 < N_2$. For this, it is sufficient to show that

$$\sum_{m=N_1}^{n-1} m(F_n \cap F_m) - m(F_n)m(F_m) \ll m(F_n).$$

Now we see that

$$\begin{aligned}
&\sum_{m=N_1}^{n-1} m(F_n \cap F_m) - m(F_n)m(F_m) \\
&= \sum_{Q:\deg Q=n} \sum_{m=N_1}^{n-1} \sum_{Q':\deg Q'=m} m(F_Q \cap F_{Q'}) - m(F_Q)m(F_{Q'}) \\
&= \sum_{Q} \sum_{m=N_1}^{n-1} \sum_{Q'} \sum_{P} \sum_{P'} m(F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}}) - m(F_{\frac{P}{Q}})m(F_{\frac{P'}{Q'}}).
\end{aligned}$$

Let us distinguish two cases according the value of $2n + l_Q$ with respect to that of $2m + l_{Q'}$.

- We first assume that $2n + l_Q \geq 2m + l_{Q'}$. Then, $m(F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}}) = \frac{1}{q^{2n+l_Q}}$ whenever $F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}} \neq \emptyset$. In this case, it follows that $\left|\frac{P}{Q} - \frac{P'}{Q'}\right| < \frac{1}{q^{2m+l_{Q'}}}$. So by Lemma 2.3, we see that the number of pairs (P, P') with $F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}} \neq \emptyset$ is less than $q^{n-m-l_{Q'}}$. We thus deduce that

$$m(F_n \cap F_m) < \frac{q^{n-m-l_{Q'}}}{q^{2n+l_Q}} = \frac{1}{q^{n+l_Q}} \frac{1}{q^{m+l_{Q'}}}.$$

- If $2n + l_Q < 2m + l_{Q'}$, then we have $m(F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}}) = \frac{1}{q^{2m+l_{Q'}}}$ whenever $F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}} \neq \emptyset$ and $\left|\frac{P}{Q} - \frac{P'}{Q'}\right| < \frac{1}{q^{2n+l_Q}}$ by the same way.

Either case, we get

$$\begin{aligned}\sum_{m=N_1}^{n-1} & m(F_n \cap F_m) - m(F_n)m(F_m) \\ & < \sum_Q \sum_{m=N_1}^{n-1} \sum_{Q'} \frac{1}{q^{n+l_Q}} \frac{1}{q^{m+l_{Q'}}} - \frac{(1-\frac{1}{q^n})}{q^{n+l_Q}} \frac{(1-\frac{1}{q^m})}{q^{m+l_{Q'}}} \\ & < \sum_Q \sum_{m=N_1}^{n-1} \sum_{Q'} \frac{1}{q^{n+l_Q}} \frac{1}{q^{m+l_{Q'}}} \left(\frac{1}{q^n} + \frac{1}{q^m}\right) \\ & < \sum_Q \sum_{m=N_1}^{n-1} \sum_{Q'} \frac{1}{q^{n+l_Q}} \cdot \frac{1}{q^{m+l_{Q'}}} \cdot \frac{2}{q^m}.\end{aligned}$$

Finally we estimate

$$\sum_Q \sum_{m=N_1}^{n-1} \sum_{Q'} \frac{1}{q^{n+l_Q}} \frac{1}{q^{m+l_{Q'}}} \frac{1}{q^m} < \sum_Q \frac{1}{q^{n+l_Q}} \sum_{m=N_1}^{n-1} \frac{1}{q^{m+l_{Q'}}}$$

by using the fact that there exist at most q^m polynomials Q', which yields

$$\sum_Q \sum_{m=N_1}^{n-1} \sum_{Q'} \frac{1}{q^{n+l_Q}} \frac{1}{q^{m+l_{Q'}}} \frac{1}{q^m} \ll \sum_Q \frac{1}{q^{n+l_Q}} \sim m(F_n).$$

Consequently, we get

$$\sum_{m=N_1}^{n-1} m(F_n \cap F_m) - m(F_n)m(F_m) \ll m(F_n),$$

which completes the proof. ■

As an application, we now consider the particular case where l_Q vanishes, i.e., l_Q takes zero value if Q is monic irreducible, and l_Q takes infinite value otherwise:

Corollary 2.4 *For almost all $f \in \mathbb{L}$, one has*

$$Card\{1 \leq n \leq N : \exists Q \, irr.\,, \deg Q = n, \exists P \; s.t. \left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n}}\}$$

$$= \log N + O(\log^{1/2} N \cdot \log^{(3+\varepsilon)/2} \log N), \quad \textit{for any } \varepsilon > 0.$$

Proof We first note that the number $l(n)$ of monic irreducible polynomials of degree n is equivalent to $\frac{q^n}{n}$. Indeed, it is well known (see e.g. [6]) that

$$l(n) = 1/n \sum_{d|n} \mu(n/d) q^d$$

where μ is the Möbius function. Let $r(n) = \sum_{d|n,\, d<n} \mu(n/d) q^d$. One has $l(n) = \frac{1}{n} q^n + \frac{1}{n} r(n)$. Furthermore, $|r_n| \leq \sum_{d=1}^{\lfloor n/2 \rfloor} q^d = q \frac{q^{\lfloor n/2 \rfloor} - 1}{q-1}$. Consequently,

$$|r_n| \leq \frac{q^{\lfloor n/2 \rfloor + 1}}{q-1}, \tag{4}$$

and thus $l(n) \sim q^n/n$.

We then deduce Corollary 2.4 from Theorem 2.1 by noticing that

$$\Psi(N) = \sum_{n=1}^{N} \sum_{Q \text{irr.}, \deg Q = n} \frac{1}{q^n} = \sum_{n=1}^{N} \frac{1}{n} + \sum_{n=1}^{N} \frac{1}{n} \frac{r_n}{q^n},$$

and then, by applying (4). ■

3 Fixed powers of irreducible denominators

The aim of this section is to extend the set of admissible denominators in (1). We thus fix a positive integer $t \geq 2$ and consider denominators Q of the form $Q = Q_1^t$ with monic irreducible Q_1. We first discuss what could be a reasonable inequality (1) with respect to the strong law of large numbers. We thus also fix $k \geq 1$ and consider

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{(t+k)n}}, \quad Q = Q_1^t, \quad \deg Q_1 = n, \quad Q_1 : \text{irreducible}. \tag{5}$$

Let

$$F_n^{(t,k)} := \{ f \in \mathbb{L} : \; f \text{ satisfies (5) for some } Q = Q_1^t, \, \deg Q_1 = n \}.$$

One has

$$m(F_n^{(t,k)}) = \sum_{Q:Q=Q_1^t,\, \deg Q_1=n,\, Q_1 \mathrm{irr.}} m(F_Q^{(t,k)}) = \sum_Q \sum_P m(F_{\frac{P}{Q}}^{(t,k)})$$

where

$$F_{\frac{P}{Q}}^{(t,k)} := \{f \in \mathbb{L} : \left| f - \frac{P}{Q} \right| < \frac{1}{q^{(t+k)n}} \}$$

$$F_Q^{(t,k)} := \cup_{P:0\leq \deg P<\deg Q,\, (P,Q)=1} \quad F_{\frac{P}{Q}}^{(t,k)}.$$

Since Q_1 is monic and $\deg Q_1 = n$, then the number of polynomials Q of the form Q_1^t is less than q^n, and we deduce

$$m(F_n^{(t,k)}) < \frac{q^n \cdot q^{tn}}{q^{(t+k)n}} = \frac{1}{q^{(k-1)n}}. \tag{6}$$

If $k \geq 2$, we see that $\sum m(F_n^{(t,k)}) < \infty$. Thus, by the Borel Cantelli lemma, there exist at most finitely many solutions of (5) for a.e. f. Note that the bound for $k = 0$ is to large for a Diophantine approximation inequality. On the other hand, if $k = 1$, we check that $\sum m(F_n^{(t,k)}) = \infty$. Hence the only case which is worth of interest with respect to the strong law of large numbers is the case $k = 1$ that we study below.

We thus consider the following inequality

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{(t+1)n+l_{Q_1}}}, \quad Q = Q_1^t, \quad Q_1 \,\text{irreducible}, \deg Q_1 = n \tag{7}$$

where P and Q are coprime, t is a fixed positive integer, and l_{Q_1} takes non-negative integer values if Q_1 is a monic irreducible polynomial, and infinite value otherwise.

Theorem 3.1 *For almost all $f \in \mathbb{L}$, the number of solutions of (7) with $\deg Q_1 \leq N$ is equal to*

$$\Psi(N) + O(\Psi^{1/2}(N) \log^{\frac{3+\varepsilon}{2}} \Psi(N)), \quad \textit{for any } \varepsilon > 0$$

with

$$\Psi(N) = \sum_{n=1}^{N} \sum_{Q_1:Q_1 \, irr., \deg Q_1=n} \frac{1}{q^{n+l_{Q_1}}}.$$

Proof As in the proof of Theorem 2.1, we define for a given t and for

$Q = Q_1^t$ with Q_1 monic irreducible of degree n

$$
\begin{aligned}
E_{\frac{P}{Q}} &= \{f \in \mathbb{L} \,:\, \exists P \,\text{s.t.}\; |f - \tfrac{P}{Q}| < \tfrac{1}{q^{(t+1)n+l_{Q_1}}}\},\\
E_Q &= \cup_{P:0\leq \deg P<\deg Q,\,(P,Q)=1}\; E_{\frac{P}{Q}},\\
E_n &= \{f \in \mathbb{L} \,:\, f \,\text{satisfies (7) for some}\, Q_1,\, \deg Q_1 = n\}.
\end{aligned}
$$

In the proof of Theorem 2.1, we used the fact that the F_Q's were disjoint. However, the sets E_Q's may not be disjoint anymore even if we fix the degree n (note that the sets $E_{\frac{P}{Q}}$'s remain disjoint for a given Q). Because of this reason, we have to define ξ_n at a different level. We will thus have to change the summation with respect to the index m which ranges now from $N_1 \to n$ instead of $N_1 \to n-1$. Indeed, the term corresponding to $m = n$ in the below summation will not vanish anymore.

Let

$$\xi_n := \sum_{Q=Q_1^t,\deg Q_1=n,Q_1\,\text{irr.}} \chi_{E_Q}$$

and $\eta_n = \hat{\eta_n} = \int \xi_n dm$. One has

$$m(E_{\frac{P}{Q}}) = \frac{1}{q^{(t+1)n+l_{Q_1}}}, \qquad m(E_Q) = \frac{q^{tn} - q^{(t-1)n}}{q^{(t+1)n+l_{Q_1}}},$$

by noticing that there are $q^{tn} - q^{(t-1)n}$ polynomials P coprime with $Q = Q_1^t$, and

$$
\begin{aligned}
\eta_n = \sum_{Q_1:\deg Q_1=n,\,Q_1\,\text{irr.}} m(E_Q) &= \sum_{Q_1:\deg Q_1=n,\,Q_1\,\text{irr.}} \frac{1}{q^{n+l_{Q_1}}}(1-\frac{1}{q^n})\\
&\sim \sum_{Q_1:\deg Q_1=n,\,Q_1\,\text{irr.}} \frac{1}{q^{n+l_{Q_1}}}.
\end{aligned}
$$

Note that Condition (i) in Lemma 1.1 is satisfied. To check Condition (ii), we need to estimate

$$
\begin{aligned}
&\int (\sum_{n=N_1}^{N_2} \xi_n - \eta_n)^2 dm =\\
&\qquad \sum_{n,m=N_1}^{N_2} \sum_{\substack{\deg Q_1 = n\\ Q = {Q_1}^t}} \sum_{\substack{\deg Q_1' = m\\ Q' = {Q_1'}^t}} m(E_Q \cap E_{Q'}) - m(E_Q)m(E_{Q'}).
\end{aligned}
$$

It will be sufficient to show that

$$\sum_{m=N_1}^{n} \sum_{\deg Q_1=n,\ Q={Q_1}^t} \sum_{\deg Q'_1=m,\ Q'={Q'_1}^t} m(E_Q \cap E_{Q'}) \ - \ m(E_Q)m(E_{Q'}) \ll \eta_n.$$

Let us note that we deduce from the fact that the sets $E_{\frac{P}{Q}}$'s are disjoint that

$$\sum_{\substack{\deg Q_1 = n \\ Q = {Q_1}^t}} \sum_{\substack{\deg Q'_1 = m \\ Q' = {Q'_1}^t}} m(E_Q \cap E_{Q'}) \ - \ m(E_Q)m(E_{Q'})$$
$$= \sum_{Q_1} \sum_{Q'_1} \sum_{P} \sum_{P'} m(E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}}) \ - \ (E_{\frac{P}{Q}})m(E_{\frac{P'}{Q'}}).$$

We again distinguish two cases.

- Let us assume that $(t+1)n + l_{Q_1} \geq (t+1)m + l_{Q'_1}$. Then, $m(E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}}) = \frac{1}{q^{(t+1)n+l_{Q_1}}}$, whenever $E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}} \neq \emptyset$, and the number of pairs (P, P') such that $E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}} \neq \emptyset$ is less than $q^{tn-m-l_{Q'_1}}$ by Lemma 2.3.
- If $(t+1)n + l_{Q_1} < (t+1)m + l_{Q'_1}$, then one has similarly $m(E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}}) = \frac{1}{q^{(t+1)m+l_{Q'_1}}}$ whenever $E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}} \neq \emptyset$ and the number of pairs (P, P') such that $E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}} \neq \emptyset$ is less than $q^{tm-n-l_{Q_1}}$.

Hence, we see that for a given (Q, Q')

$$\sum_{P} \sum_{P'} m(E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}}) - (E_{\frac{P}{Q}})m(E_{\frac{P'}{Q'}})$$
$$< \frac{1}{q^{n+l_{Q_1}}} \frac{1}{q^{tm+l_{Q'_1}}} \ - \ \frac{\left(q^{tn} \ - \ q^{(t-1)n}\right)}{q^{(t+1)n+l_{Q_1}}} \frac{\left(q^{m} \ - \ q^{(t-1)m}\right)}{q^{(t+1)m+l_{Q'_1}}}.$$

We thus deduce that

$$\sum_{\deg Q_1=n,\ Q={Q_1}^t} \sum_{\deg Q'_1=m,\ Q'={Q'_1}^t} m(E_Q \cap E_{Q'}) \ - \ m(E_Q)m(E_{Q'})$$
$$< \sum_{Q_1} \sum_{Q'_1} \frac{1}{q^{n+l_{Q_1}}} \frac{1}{q^{m+l_{Q'_1}}} \ - \ \frac{\left(1 \ - \ \frac{1}{q^n}\right)}{q^{n+l_{Q_1}}} \frac{\left(1 \ - \ \frac{1}{q^m}\right)}{q^{m+l_{Q'_1}}}$$
$$= \sum_{Q_1} \sum_{Q'_1} \frac{1}{q^{n+l_{Q_1}}} \frac{1}{q^{m+l_{Q'_1}}} \left(\frac{1}{q^n} \ + \ \frac{1}{q^m}\right).$$

Consequently,

$$\sum_{m=N_1}^{n} \sum_{\deg Q_1=n,\ Q={Q_1}^t} \sum_{\deg Q_1'=m,\ Q'={Q_1'}^t} m(E_Q \cap E_{Q'}) \ - \ m(E_Q)m(E_{Q'})$$

$$< \sum_{Q_1} \sum_{m=N_1}^{n} \sum_{Q_1'} \frac{1}{q^{n+l_{Q_1}}} \cdot \frac{1}{q^{m+l_{Q_1'}}} \cdot \frac{2}{q^m} \ll \sum_{Q_1} \frac{1}{q^{n+l_{Q_1}}} \ \sim \ \eta_n,$$

which concludes the proof of Theorem 3.1. ∎

We deduce similarly as for Corollary 2.4 the following application in the case where l_{Q_1} vanishes, i.e., l_{Q_1} takes infinite value if Q_1 is not monic irreducible, and zero value otherwise.

Corollary 3.2 *For almost all $f \in \mathbb{L}$,*

$$Card\{1 \ \le \ n \ \le N \ : \ \exists Q_1 \ irr., \deg Q_1 = n, \ \exists P \ s.t. \left| f \ - \ \frac{P}{{Q_1}^{t+1}} \right| \ < \ \frac{1}{q^{(t+1)n}} \}$$

$$= \log N \ + \ O(\log^{1/2} N \log^{(3+\varepsilon)/2} \log N), \quad \textit{for any } \varepsilon > 0.$$

4 Variable powers of irreducible denominators

In Section 3, we have considered denominators which are powers of an irreducible polynomial, that is, $Q = Q_1^t$, for a fixed $t \ge 2$. Let us generalize this situation to the case where the power t is variable. Thus we consiser the following inequality;

$$\left| f \ - \ \frac{P}{Q} \right| \ < \ \frac{1}{|Q_1|^{t+1} q^{l_{Q_1}+l_t}}, \qquad t \ge 1, \quad Q = Q_1^t, \qquad Q_1 \, \text{irreducible}, \tag{8}$$

where l_{Q_1} takes nonnegative integer values if Q_1 is a monic irreducible polynomial, and infinite value otherwise, and l_t takes nonnegative integer values. As before, we also assume that P and Q are coprime.

Theorem 4.1 *We assume that the series $\sum_{t\ge 1} \frac{1}{q^{l_t}}$ is convergent. Then, for almost all $f \in \mathbb{L}$, the number of solutions of (8) with* $\deg Q \le N$ *is*

$$\Psi(N) \ + \ O(\Psi^{1/2}(N) \log^{\frac{3+\varepsilon}{2}} \Psi(N)) \qquad \textit{for any } \varepsilon > 0$$

with

$$\Psi(N) = \sum_{n=1}^{N} \sum_{(k,t):kt=n} \sum_{\substack{Q = Q_1^t, \\ Q_1\, irr. \\ \deg Q_1 = k}} \frac{1}{q^{k+l_t+l_{Q_1}}} \left(1 - \frac{1}{q^k}\right).$$

Proof For $Q = Q_1^t$, $k = \deg Q_1$, and $n = \deg Q$, we put

$$\begin{cases} G_{\frac{P}{Q}} = \left\{ f \in \mathbb{L} : \left| f - \frac{P}{Q} \right| < \frac{1}{q^{(t+1)k+l_{Q_1}+l_t}} \right\} \\ G_Q = \cup_{P: 0 \leq \deg P < \deg Q, \, (P,Q)=1} \, G_{\frac{P}{Q}} \end{cases}$$

Then we define $\xi_n, \eta_n, \hat{\eta}_n$ for all n as

$$\xi_n := \sum_{\substack{(k,t) \\ kt=n}} \sum_{Q_1 : \text{ irr.}, \deg Q_1 = k} \chi_{G_{Q_1^t}} \quad \text{and} \quad \eta_n = \hat{\eta}_n = \int \xi_n dm.$$

One has

$$m(G_{\frac{P}{Q}}) = \frac{1}{q^{(t+1)k+l_{Q_1}+l_t}}, \qquad m(G_Q) = \frac{1}{q^{(t+1)k+l_{Q_1}+l_t}} \left(q^{tk} - q^{(t-1)k} \right).$$

We first estimate η_n:

$$\begin{aligned} \eta_n &= \sum_{(k,t):kt=n} \sum_{Q_1:\text{irr.},\deg Q_1=k} m(G_{Q_1^t}) \\ &= \sum_{(k,t):kt=n} \sum_{Q_1:\text{irr. },\deg Q_1=k} \frac{1}{q^{(t+1)k+l_{Q_1}+l_t}} \left(q^{tk} - q^{(t-1)k} \right) \\ &= \sum_{(k,t):kt=n} \sum_{Q_1:\text{irr.},\deg Q_1=k} \frac{1}{q^{k+l_{Q_1}+l_t}} \left(1 - \frac{1}{q^k} \right). \end{aligned}$$

Consequently, there exists $M > 0$ such that $\eta_n \leq M$ for any $n \geq 1$. Hence we can apply Lemma 1.1 to $(\frac{1}{M}\xi_n : n \geq 1)$ which satisfies Condition (i) of Lemma 1.1. Note that we have furthermore

$$\sum_{(k,t):kt=n} \sum_{Q_1:\text{irr.},\deg Q_1=k} \frac{1}{q^{k+l_{Q_1}+l_t}} \ll \eta_n.$$

It remains to show (and this will be sufficient) that the sequence (ξ_n) satisfies Condition (ii) of Lemma 1.1. One has

$$
\begin{aligned}
&\int \left(\sum_{n=N_1}^{N_2} (\xi_n - \eta_n) \right)^2 dm \\
&= \int \left(\sum_{n=N_1}^{N_2} \left(\sum_{(k,t),\ kt=n} \ \sum_{Q_1 \text{irr., deg } Q_1 = k} (\chi_{G_{Q_1^t}} - m(G_{Q_1^t})) \right) \right)^2 dm \\
&= \int \left[\sum_{n=N_1}^{N_2} \left(\sum_{(k_1,t_1),\ k_1 t_1 = n} \ \sum_{Q_1 \text{ irr., deg } Q_1 = k_1} (\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}})) \right) \right] \\
&\qquad \left[\sum_{m=N_1}^{N_2} \left(\sum_{(k_2,t_2),\ k_2 t_2 = m} \ \sum_{Q_2 \text{ irr., deg } Q_2 = k_2} (\chi_{G_{Q_2^{t_2}}} - m(G_{Q_2^{t_2}})) \right) \right] dm \\
&= \int \left[\sum_{k_1=1}^{N_2} \sum_{t_1 = \lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \left(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}}) \right) \right] \\
&\qquad \left[\sum_{k_2=1}^{N_2} \sum_{t_2 = \lceil \frac{N_1}{k_2} \rceil}^{\lfloor \frac{N_2}{k_2} \rfloor} \sum_{Q_2} \left(\chi_{G_{Q_2^{t_2}}} - m(G_{Q_2^{t_2}}) \right) \right] dm \\
&= 2 \int \sum_{k_1=1}^{N_2} \sum_{k_2=1}^{k_1} \sum_{t_1 = \lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \sum_{t_2 = \lceil \frac{N_1}{k_2} \rceil}^{\lfloor \frac{N_2}{k_2} \rfloor} \sum_{Q_2 \neq Q_1} \left(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}}) \right) \\
&\qquad \times \left(\chi_{G_{Q_2^{t_2}}} - m(G_{Q_2^{t_2}}) \right) dm \\
&\quad + \int \sum_{k_1=1}^{N_2} \sum_{t_1 = \lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \sum_{t_2 = \lceil \frac{N_1}{k_2} \rceil}^{\lfloor \frac{N_2}{k_2} \rfloor} \left(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}}) \right) \left(\chi_{G_{Q_1^{t_2}}} - m(G_{Q_1^{t_2}}) \right) dm \\
&=: 2[\mathrm{A}] + [\mathrm{B}] \\
&=: 2[\mathrm{A}] + [\mathrm{B1} : t_1 = t_2] + 2\,[\mathrm{B2} : t_1 > t_2],
\end{aligned}
$$

by setting

$$
\begin{aligned}
[A] &:= \int \sum_{k_1=1}^{N_2} \sum_{k_2=1}^{k_1} \sum_{t_1 = \lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \sum_{t_2 = \lceil \frac{N_1}{k_2} \rceil}^{\lfloor \frac{N_2}{k_2} \rfloor} \sum_{Q_2 \neq Q_1} \left(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}}) \right) \\
&\qquad \times \left(\chi_{G_{Q_2^{t_2}}} - m(G_{Q_2^{t_2}}) \right) dm \\
[B1] &= \int \sum_{k_1=1}^{N_2} \sum_{t_1 = \lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \left(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}}) \right)^2 dm,
\end{aligned}
$$

$$[B2] = \int \sum_{k_1=1}^{N_2} \sum_{t_1=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \sum_{t_2:t_2<t_1, t_2=\lceil \frac{N_1}{k_2} \rceil}^{\lfloor \frac{N_2}{k_2} \rfloor} \left(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}}) \right) \times \left(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_2}}) \right) dm.$$

Estimate for [A]

To estimate [A], we distinguish as in the proofs of Theorem 2.1 and 3.1 two cases. We first suppose that $(t_1+1)k_1 + l_{Q_1} + l_{t_1} \geq (t_2+1)k_2 + l_{Q_2} + l_{t_2}$, i.e., $m(G_{\frac{P}{Q_1^{t_1}}}) \leq m(G_{\frac{P}{Q_2^{t_2}}})$. We then decompose [A] by introducing a further summation over P and P'. If $G_{Q_1^{t_1}} \cap G_{Q_2^{t_2}} \neq \emptyset$, then there exist P and P' such that $G_{\frac{P}{Q_1^{t_1}}} \cap G_{\frac{P'}{Q_2^{t_2}}} \neq \emptyset$. One has $m(G_{\frac{P}{Q_1^{t_1}}} \cap G_{\frac{P'}{Q_2^{t_2}}}) = \frac{1}{q^{(t_1+1)k_1+l_{Q_1}+l_{t_1}}}$ and similarly as in the previous proofs, $\left|\frac{P}{Q} - \frac{P'}{Q'}\right| < \frac{1}{q^{(t_2+1)k_2+l_{Q_2}+l_{t_2}}}$. From Lemma 2.3, there exist at most $k_1t_1 - (k_2+l_{Q_2}+l_{t_2})$ such pairs of polynomials (P, P'). Thus we get

$$\begin{aligned}
& m(G_{Q_1^{t_1}} \cap G_{Q_2^{t_2}}) - m(G_{Q_1^{t_1}})m(G_{Q_2^{t_2}}) \\
\leq & \frac{q^{k_1t_1-(k_2+l_{Q_2}+l_{t_2})}}{q^{(t_1+1)k_1+l_{Q_1}+l_{t_1}}} - \frac{q^{k_1t_1} - q^{(t_1-1)k_1}}{q^{(t_1+1)k_1+l_{Q_1}+l_{t_1}}} \frac{q^{k_2t_2} - q^{(t_2-1)k_2}}{q^{(t_2+1)k_2+l_{Q_2}+l_{t_2}}} \\
\leq & \frac{1}{q^{k_1+l_{Q_1}+l_{t_1}}} \frac{1}{q^{k_2+l_{Q_2}+l_{t_2}}} \left(\frac{1}{q^{k_1}} + \frac{1}{q^{k_2}} \right) \\
\leq & \frac{1}{q^{k_1+l_{Q_1}+l_{t_1}}} \frac{1}{q^{k_2+l_{Q_2}+l_{t_2}}} \frac{2}{q^{k_2}}
\end{aligned}$$

since we are assuming $k_1 \geq k_2$. The same holds when assuming $(t_1+1)k_1 + l_{Q_1} + l_{t_1} < (t_2+1)k_2 + l_{Q_2} + l_{t_2}$. Now we have

$$\begin{aligned}
& \sum_{k_1=1}^{N_2} \sum_{k_2=1}^{k_1} \sum_{t_1=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \sum_{t_2=\lceil \frac{N_1}{k_2} \rceil}^{\lfloor \frac{N_2}{k_2} \rfloor} \sum_{Q_2 \neq Q_1} \frac{1}{q^{k_1+l_{Q_1}+l_{t_1}}} \frac{1}{q^{k_2+l_{Q_2}+l_{t_2}}} \frac{1}{q^{k_2}} \\
= & \sum_{k_1=1}^{N_2} \sum_{t_1=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \frac{1}{q^{k_1+l_{Q_1}+l_{t_1}}} \sum_{k_2=1}^{k_1} \sum_{t_2=\lceil \frac{N_1}{k_2} \rceil}^{\lfloor \frac{N_2}{k_2} \rfloor} \sum_{Q_2 \neq Q_1} \frac{1}{q^{k_2+l_{Q_2}+l_{t_2}}} \frac{1}{q^{k_2}}.
\end{aligned}$$

We then use the fact that $\sum \frac{1}{q^{l_t}} < \infty$ to deduce that

$$[A] \ll \sum_{k_1=1}^{N_2} \sum_{t_1=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \frac{1}{q^{k_1+l_{Q_1}+l_{t_1}}} \ll \sum_{n=N_1}^{N_2} \eta_n.$$

Estimates for [B1] and [B2]

We now estimate [B1]:

$$\int \sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} \left(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}})\right)^2 dm \leq \sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} m(G_{Q_1^{t_1}}) = \sum_{n=N_1}^{N_2} \eta_n.$$

Concerning [B2], we need to estimate

$$\sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} \sum_{t_2} m(G_{Q_1^{t_1}} \cap G_{Q_1^{t_2}}) - m(G_{Q_1^{t_1}}) m(G_{Q_1^{t_2}})$$

with $t_1 > t_2$. Now we decompose here again $[B2]$ by introducing a further summation over P and P' and by comparing $(t_1+1)k_1 + l_{Q_1} + l_{t_1}$ with $(t_2+1)k_1 + l_{Q_1} + l_{t_2}$. We thus assume that $(t_1+1)k_1+l_{Q_1}+l_{t_1} \geq (t_2+1)k_1+l_{Q_1}+l_{t_2}$, the other case being handled similarly. We have to extend Lemma 2.3 in the following way: we prove that there exist at most $q^{t_1k_1-(k_1+l_{Q_1}+l_{t_2})}$ pairs of polynomials (P, P') that satisfy

$$\left| \frac{P}{Q_1^{t_1}} - \frac{P'}{Q_1^{t_2}} \right| < \frac{1}{q^{(t_2+1)k_1+l_{Q_1}+l_{t_2}}}.$$

Consider indeed such a pair (P, P'). Then

$$k_1t_1 - \deg(P - P'Q_1^{t_1-t_2}) > (t_2+1)k_1 + l_{Q_1} + l_{t_2}.$$

Thus we see that

$$\deg(P - P'Q_1^{t_1-t_2}) < (t_1-t_2)k_1 - (k_1 + l_{Q_1} + l_{t_2}).$$

Hence there exist at most $q^{(t_1-t_2)k_1 - (k_1+l_{Q_1}+l_{t_2})}$ polynomials of the form $P - P'Q_1^{t_1-t_2}$. Let us fix now (P, P'). If

$$P_1 - P_1'Q_1^{t_1-t_2} = P_2 - P_2'Q_1^{t_1-t_2},$$

then

$$(P_1 - P_2) = Q_1^{t_1-t_2}(P_1' - P_2').$$

It is easy to see that if $P_1 \neq P_2$, then

$$(t_1 - t_2)k_1 \leq \deg(P_1 - P_2) < t_1 k_1.$$

Hence there exist at most $q^{t_2 k_1}$ pairs of polynomials (P_1, P_1') that satisfy $P_1 - P_1' Q_1^{t_1-t_2} = P - P' Q_1^{t_1-t_2}$. Consequently, the possible numbers of pairs (P, P') is at most $q^{t_1 k_1 - (k_1 + l_{Q_1} + l_{t_2})}$. We deduce that

$$m(G_Q \cap G_{Q'}) \leq \frac{1}{q^{(t_1+1)k_1 + l_{Q_1} + l_{t_2}}} q^{t_1 k_1 - (k_1 + l_{Q_1} + l_{t_2})} = \frac{1}{q^{k_1 + l_{Q_1} + l_{t_1}}} \frac{1}{q^{k_1 + l_{Q_1} + l_{t_2}}}.$$

Therefore we have, by using the fact that $\sum \frac{1}{q^{l_t}}$ converges

$$\begin{aligned}
\sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} \sum_{t_2} m(G_{Q_1^{t_1}} \cap G_{Q_1^{t_2}}) - m(G_{Q_1^{t_1}}) m(G_{Q_1^{t_2}}) \\
&\ll \sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} \frac{1}{q^{k_1 + l_{Q_1} + l_{t_1}}} \frac{1}{q^{k_1 + l_{Q_1}}} \\
&\ll \sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} \frac{1}{q^{k_1 + l_{Q_1} + l_{t_1}}} \ll \sum_{n=N_1}^{N_2} \eta_n,
\end{aligned}$$

which ends the proof of Theorem 4.1. ■

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