BRUN EXPANSIONS OF STEPPED SURFACES

V. BERTHÉ AND TH. FERNIQUE

Abstract. This paper presents a geometric version of Brun multidimensional continued fraction algorithm acting on so-called stepped surfaces (these are a generalization of arithmetic discrete planes). This geometric extension of Brun algorithm is motivated by the discrete plane recognition problem: given a set of points in $\mathbb{Z}^d$, does there exist an arithmetic discrete plane that contains it? We present here a strategy based on multidimensional continued fractions, inspired by the one-dimensional Sturmian case. We will see that the role played respectively by words, substitutions, and classical continued fractions will be played here by stepped surfaces [2], dual maps [3, 10], and Brun algorithm [9, 19].

Keywords: Brun algorithm, digital planarity, standard arithmetic discrete plane, flip, free group morphism, multidimensional continued fractions, stepped surface, substitution, discrete geometry.

Introduction

The discrete plane recognition problem can be stated as follows: given a set of points in $\mathbb{Z}^d$, does there exist an arithmetic discrete plane that contains it? By arithmetic discrete plane, we mean a set of points $\mathbf{x} \in \mathbb{Z}^d$ verifying $\langle \mathbf{x}, \alpha \rangle < \rho \leq \langle \mathbf{x}, \alpha \rangle + ||\alpha||_1$, where $\alpha \in \mathbb{R}^d_+ \backslash \{\mathbf{0}\}$ and $\rho \in \mathbb{R}$; this discretization scheme corresponds to the notion of standard arithmetic discrete plane originated in [18] and formalized in [1, 12]. We will call them in all that follows *stepped planes* according to the terminology of [3].

This question is classical and central in the field of discrete geometry for the segmentation of discrete surfaces and for polyhedrization issues for instance. Indeed, numerous applications can be derived in image analysis and synthesis, volume modeling, pattern recognition... There exist various strategies for the discrete plane recognition problem, such as described for instance in the surveys [7, 8]. These methods are based on linear programming, on computational geometry by performing separability tests, or on the so-called preimage technique which consists in determining the set of parameters (α, ρ) of the arithmetic discrete planes that contain the given set of points. In particular, geometric approaches based on two-dimensional Farey frations have been studied in [24] in order to determine the preimage, the underlying continued fraction algorithm being investigated in [13].

In the one-dimensional case, there exists a natural strategy for the recognition problem based on word combinatorics. We first code discrete lines thanks to the Freeman code. One checks that an arithmetic standard line is made of horizontal and vertical steps. One can code such a standard line by using the *Freeman code* over the two-letter alphabet $\{0, 1\}$ as follows: one codes horizontal steps by a 0, and vertical ones by a 1. One gets a so-called Stumian word. Sturmian words are widely studied, see e.g. Chap. 2 in [16] and Chap. 6 in [17]. Most of the combinatorial properties of Sturmian words can be described in terms of the continued fraction expansion of the slope of the discrete line that they code. Let us recall that a substitution is a morphism of the free monoid whereas an S-adic word is an infinite word generated as the limit of an infinite composition of a finite number of substitutions (for more details, see Chap. 12 in [17]). Sturmian words are proven to be S-adic words; the rules for the iteration of these substitutions follow the continued fraction of the slope of the line which is coded. We deduce from the combinatorial properties of Sturmian words the following two facts: first, the factors 00 and 11 cannot occur simultaneoulsy in a Sturmian word, that is, one of the two letters 0 and 1 occurs as an isolated letter. Hence, up to a prefix of length 1, any infinite Sturmian word can be written as $\sigma_0(v)$ or $\sigma_1(v)$, where v is an infinite word over $\{0,1\}$, and the substitutions σ_0 and σ_1 are defined as $\sigma_0 \colon 0 \mapsto 0,\ \sigma_0 \colon 1 \mapsto 10$ and $\sigma_1 \colon 0 \mapsto 01,\ \sigma_1 \colon 1 \mapsto 1$. Secondly, we use the fact that v is itself a Sturmian word. We thus can reiterate the process. Suppose now we are given a connected union of translates of horizontal and vertical segments with integer vertices and length 1. We apply the previous process to the finite word coding this union of segments by taking care of the boundaries. This corresponds to the method developed in discrete geometry terms in [26, 22].

The aim of this paper is to present a strategy based on multidimensional continued fractions inspired by the one-dimensional Sturmian case. For a more precise description of a recognition algorithm based on the present paper, see [11]. The role played respectively by words, substitutions, and classical continued fractions will be played here by stepped surfaces [2], dual maps [3, 10], and Brun algorithm [9, 19].

More precisely, a *stepped surface* is defined as a union of faces of integer translates of the unit cube such that the orthogonal projection onto the hyperplane with normal vector $(1, 1, \cdots, 1)$ induces an homeomorphism from the stepped surface onto the antidiagonal plane. See for instance Fig. 1.2 below. The main difference between a stepped surface and a stepped plane is that it is possible to recognize locally whether a set of points in $\mathbb{Z}^d$ is a subset of the set of vertices of a stepped surface (see [15]). Nevertheless, stepped surfaces do not behave so distinctly from stepped planes: indeed, they can be generated by performing flips on stepped planes [2].

Let us recall that a *flip* is a classical notion in the study of dimer tilings and lozenge tilings associated with the triangular lattice. It consists in a local reorganization of tiles that transforms a tiling into another one. Such a reorganization can also be seen in the 3-dimensional space on the stepped surfaces. Suppose indeed that a stepped surface contains 3 faces that form the lower faces of a unit cube with integer vertices. By replacing these three faces by the upper faces of this cube, one obtains another functional stepped surface. According to [2], any functional stepped surface can be obtained from a stepped plane plane by a sequence of flips, possibly infinite but locally finite, in the sense that, for any bounded neighborhood of the origin in the antidiagonal plane, there is only a finite number of flips whose domain has a projection which intersects this neighborhood. We thus can come back via the notion of flips from stepped surfaces to stepped planes.

We have chosen here to work with Brun's algorithm for the following reasons. We use the fact that it is a unimodular multidimensionl continued fraction in the sense of [5], and that it is (weakly) convergent. Our study also applies to other algorithms with analogous properties such as Jacobi-Perron's algorithm.

Let us note that we consider here standard arithmetic discrete planes. When replacing the norm $|| \, ||_1$ by the norm $|| \, ||_\infty$ in the definition of arithmetic discrete planes, one gets so-called naive arithmetic discrete planes. The latter are usually considered in the recognition problem. We focus here on standard arithmetic discrete planes for technical reasons. Both notions are strongly related as shown e.g. in [21], Theo. 1.

A further motivation to the present study is the question of an efficient generation by means of dual maps of a stepped plane. We also plan to extend our approach to discretizations of higher codimension. Indeed, we are here particularly interested in projection tilings defined following the so-called cut and project method (see for instance [20]). Roughly speaking, a $d \to k$ canonical projection tiling is a tiling of $\mathbb{R}^k$ obtained by projecting onto $\mathbb{R}^k$ the k-dimensional unit faces lying in a "slice" $V + [0,1)^d$ of $\mathbb{R}^d$, where $V \subset \mathbb{R}^d$ is a k-dimensional affine space. We consider here only $d \to d-1$ canonical projection tilings, that is, codimension one tilings.

Let us outline the contents of the present paper. Section 1 is devoted to the basic introductory material, namely stepped planes and stepped surfaces, weighted and geometric sums. The notion of flip is introduced in Section 1.2, as well as its extension to weighted sums as pseudo-flips. Substitutions, and more generally morphisms of the free group, have a natural generalization in higher dimension which is recalled in Section 2. Their action on stepped planes and stepped surfaces is detailed respectively in Section 2.2 and 2.3. Brun expansions are introduced in Section 3. In particular, Section 3.2 and 3.3 are devoted to the notion of Brun expansion of a stepped plane which prepares the ground for the introduction of Brun expansion of a stepped surface in Section 3.4, where the main "recognition" theorem of the present paper is proved:

Theorem 0.1. *Let $\mathcal{S}$ be a Brun expandable stepped surface with a finite Brun expansion of positive length N. Then, $\mathcal{S}$ is a stepped plane if and only if its image under the N-th iterate of the Brun map is a stepped plane of normal vector* $(1, \mathbf{0})$.

We conclude this paper by discussing properties of Brun expandable stepped surfaces in Sec. 4 and by proving the following "classification" theorem:

Theorem 0.2. *If a stepped surface has the same Brun expansion as a totally irrational vector, then it is a stepped plane.*

1. Stepped planes and stepped surfaces

We introduce in Sec. 1.1 the notions of stepped planes and stepped surfaces, weighted and geometric sums. Flips (as well as their extension to weighted sums as pseudo-flips) are defined in Section 1.2 where we prove the main theorem of this section, namely Th. 1.14.

1.1. **From tilings to weighted sums of faces.** Let $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ stand for the canonical basis of $\mathbb{R}^d$. For any $\mathbf{x} \in \mathbb{Z}^d$ and $i \in \{1, \dots, d\}$, we denote by $(\mathbf{x}, i^*)$ the face of the translate by $\mathbf{x}$ of the unit hypercube defined by $(\mathbf{x}, i^*) = \{\mathbf{e}_i + \sum_{j \neq i} \lambda_j \mathbf{e}_j \mid 0 \leq \lambda_j \leq 1\}$. Such a face is called *unit face*. Examples of faces are depicted in Fig. 1.1.

Notation 1.1. The set of unit faces is denoted by F.

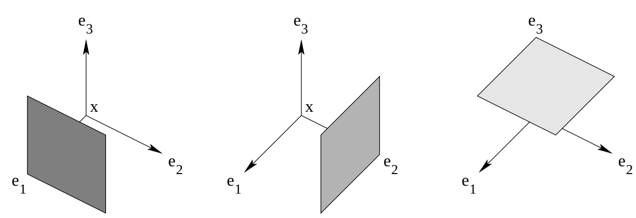

Figure 1.1. The faces $(\mathbf{x}, 1^*)$, $(\mathbf{x}, 2^*)$ and $(\mathbf{x}, 3^*)$ (from left to right), in the $d = 3$ case.

For any non-negative non-zero vector $\mathbf{u} \in \mathbb{R}^d_+ \backslash \{\mathbf{0}\}$ and $\rho \in \mathbb{R}$, we define the following set of faces:

$$P_{\alpha,\rho} = \{(\mathbf{x}, i^*) \mid \langle \mathbf{x}, \alpha \rangle < \rho \leq \langle \mathbf{x} + \mathbf{e}_i, \alpha \rangle\}, \tag{1.1}$$

where $\langle ., . \rangle$ denotes the canonical dot product. In fact, $P_{\alpha,\rho}$ consists in the intersection of the set F of faces with a "slice" of the form $V + (0, 1]^d$,where V is a translate of an hyperplane with normal vector α. Moreover, one checks that $\mathbf{x} \in \mathbb{Z}^d$ is a *vertex* of $P_{\alpha,\rho}$ (that is, $\mathbf{x}$ is a vertex of a face of $P_{\alpha,\rho}$) if and only if $\mathbf{x}$ satisfies:

$$\langle \mathbf{x}, \alpha \rangle < \rho \leq \sum_{i=1}^{d} \langle \mathbf{x} + \mathbf{e}_i, \alpha \rangle;$$

in other words, the set of vertices of $P_{\alpha,\rho}$ turns out to be a so-called *standard arithmetic discrete plane*, a classic object in discrete geometry introduced in [18].

Let π stand for the orthogonal projection onto $(\mathbf{e}_1 + \ldots + \mathbf{e}_d)^\perp$ where $(\mathbf{e}_1 + \ldots + \mathbf{e}_d)^\perp$ is the hyperplane $\{\mathbf{x} \in \mathbb{Z}^d \mid \langle \mathbf{x}, \mathbf{e}_1 + \ldots + \mathbf{e}_d \rangle = 0\}$. The projection π is a homeomorphism from $P_{\alpha,\rho}$ onto its image which is isomorphic to $\mathbb{R}^{d-1}$. We thus obtain a tiling of $\mathbb{R}^{d-1}$, whose tiles are being provided by the projected faces $\pi(\mathbf{x}, i^*)$'s, for $\mathbf{x} \in \mathbb{Z}^d$ and $i \in \{1, \cdots, d\}$. Fig. 1.2 (left) illustrates the $d = 3$ case.

It is convenient here to introduce a second viewpoint on stepped planes. We define a *weighted sum of faces* as a map $\mathcal{E} \colon F \to \mathbb{Z}$. We use the following notation:

$$\mathcal{E} = \sum_{(\mathbf{x},i^*)\in F,\ \mathcal{E}(\mathbf{x},i^*)\neq 0} \lambda_{(\mathbf{x},i^*)}(\mathbf{x}, i^*),$$

where $\lambda_{(\mathbf{x},i^*)} = \mathcal{E}(\mathbf{x}, i^*)$. The coefficient $\lambda := \lambda_{(\mathbf{x},i^*)}$ is called the *weight* of the face $(\mathbf{x}, i^*)$. The face $(\mathbf{x}, i^*)$ is said to have weight λ in $\mathcal{E}$.

Notation 1.2. We denote by $\mathfrak{F}$ the vector space of weighted sums.

Let us stress the fact that infinitely many coefficients may be non-zero in a weighted sum, that is, $\mathcal{F}$ differs from the vector space generated by the set of faces F.

Let $(\mathcal{E}_n)_{n\in\mathbb{N}}$ be a countable family of weighted sums. The weigthed sum $\sum_{n\in\mathbb{N}} \mathcal{E}_n$ is said to be well-defined if for every face $(\mathbf{x}, i^*)$, then $\sum_n \mathcal{E}_n(\mathbf{x}, i^*)$ is finite. In this latter case, $\sum_{n\in\mathbb{N}} \mathcal{E}_n$ is defined as the weighted sum that gives weight $\sum_n \mathcal{E}_n(\mathbf{x}, i^*)$ to the face $(\mathbf{x}, i^*)$.

Definition 1.3. The *stepped plane* of *normal vector* $\alpha \in \mathbb{R}^d_+\backslash\{\mathbf{0}\}$ and *intercept* $\rho \in \mathbb{R}$ is the weighted sum of faces with weights in $\{0, 1\}$ defined by:

$$\mathcal{P}_{\alpha,\rho} = \sum_{(\mathbf{x},i^*)\in P_{\alpha,\rho}} (\mathbf{x}, i^*)$$

where $P_{\alpha,\rho}$ is recalled to be equal to

$$P_{\alpha,\rho} = \{(\mathbf{x}, i^*) \mid \langle \mathbf{x}, \alpha \rangle < \rho \leq \langle \mathbf{x} + \mathbf{e}_i | \alpha \rangle\}.$$

Notation 1.4. We denote by $\mathfrak{P}$ the set of stepped planes.

Let us introduce now more general examples of weighted sums, namely *stepped surfaces*, introduced in [15].

Definition 1.5. A *stepped surface* is a weighted sum of faces with weights in $\{0, 1\}$ such that π is an homeomorphism from the union of faces with weight 1 onto $\mathbb{R}^{d-1}$.

Fig. 1.2 (right) illustrates the $d = 3$ case.

Notation 1.6. We denote by $\mathfrak{S}$ the set of stepped surfaces.

We first prove the following basic property of stepped surfaces that will be used hereafter.

Proposition 1.7. *If $\mathbf{x}$ and $\mathbf{y}$ are vertices of a stepped surface, then the coordinates of the vector $\mathbf{x} - \mathbf{y}$ are neither all positive nor all negative.*

Proof. Let $\mathbf{x} = (x_1, \cdots, x_d)$ and $\mathbf{y} = (y_1, \cdots, y_d)$ be two vertices of a stepped surface $\mathcal{S}$. We assume w.l.o.g. that $\mathbf{x} - \mathbf{y}$ has only positive entries (we write for short $\mathbf{x} - \mathbf{y} > \mathbf{0}$). Since the surface $\mathcal{S}$ is connected, there exists a finite sequence $\mathbf{z} = (\mathbf{z}_n)_{0\leq n\leq N}$ of vertices of $\mathcal{S}$ that satisfies $\mathbf{z}_0 = \mathbf{x}$, $\mathbf{z}_N = \mathbf{y}$, and for all $n \in \{0, \cdots, N-1\}$, $\mathbf{z}_{n+1} - \mathbf{z}_n = \mathbf{e}_{i_{n+1}}$, with $i_{n+1} \in \{1, \cdots, d\}$. We deduce from $\mathbf{x} - \mathbf{y} > \mathbf{0}$ that $\{i_1, \cdots, i_N\} = \{1, \cdots, d\}$.

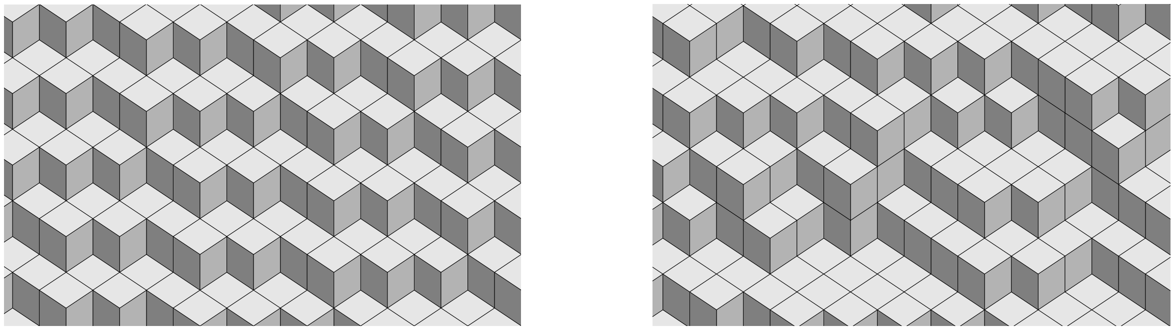

FIGURE 1.2. Left, a stepped plane (that is, a weighted sum of faces in $\mathbb{R}^3$), whose vertices form a standard arithmetic discrete plane. Right, a stepped surface. Both can be seen as tilings of $\mathbb{R}^2$ by projections of unit faces (that is, lozenges).

We now introduce the increasing sequence of integers $(r_{\mathbf{z}}(k))_{1\leq k\leq d}$ defined as

$$r_{\mathbf{z}}(k) := \min\{n \mid \ \mathrm{Card}(\{i_1, \cdots, i_n\}) = k\}.$$

One has $r_{\mathbf{z}}(k) \geq k$, for all k.

Let us prove by induction on k that for every k, we can define a new sequence $\mathbf{z}(k) = (\mathbf{z}(k)_n)_{0\leq n\leq N_k}$ of points of $\mathcal{S}$ that satisfies $r_{\mathbf{z}(k)}(k) = k$ with $\mathbf{z}(k)_0 - \mathbf{z}(k)_{N_k} > \mathbf{0}$, by reorganizing and sorting vectors from the sequence $(\mathbf{e}_{i_n})_{1\leq n\leq N}$.

We first note that $r_{\mathbf{z}}(1) = 1$ by definition. Let us assume $d = 2$. One has $\mathbf{y} - \mathbf{z}_{r_{\mathbf{z}}(2)-2} > \mathbf{0}$. Indeed, $i_{r_{\mathbf{z}}(k)-1} \neq i_{r_{\mathbf{z}}(k)}$ by definition of $r_{\mathbf{z}}(k)$. Hence, the sequence $\mathbf{z}(2) := (\mathbf{z}_{r_{\mathbf{z}}(2)-2}, \mathbf{z}_{r_{\mathbf{z}}(2)-1}, \mathbf{z}_{r_{\mathbf{z}}(2)-2})$ satisfies $r_{\mathbf{z}(2)}(2) = 2$.

We now assume $d \geq 3$. Let $2 \leq k \leq d-1$. We assume that the induction hypothesis holds true for $k-1$, that is, $r_{\mathbf{z}(k-1)} = k-1$ for some sequence $\mathbf{z}(k-1)$. We also assume that $r_{\mathbf{z}(k-1)} \geq k+1$. To simplify the notation, we write $r(k)$ for $r_{\mathbf{z}(k-1)}(k)$. We consider the three following vertices of $\mathcal{S}$

$$\mathbf{x}_{r(k)-2},\ \mathbf{x}_{r(k)-1} = \mathbf{x}_{r(k)-2} + \mathbf{e}_{i_{r(k)-1}},\ \mathbf{x}_{r(k)} = \mathbf{x}_{r(k)-2} + \mathbf{e}_{i_{r(k)-1}} + \mathbf{e}_{i(k)}.$$

One has again $i_{r(k)-1} \neq i_{r(k)}$ by definition of $r(k)$. The point $\mathbf{x}_{r(k)-2} + \mathbf{e}_{i_{r(k)}}$ belongs to the stepped surface $\mathcal{S}$ (we recall that $d \geq 3$). Indeed this point $\mathbf{x}_{r(k)-2} + \mathbf{e}_{i(k)}$ is the fourth vertex of a subface of $\mathcal{S}$, and thus belongs to $\mathcal{S}$. By replacing $\mathbf{x}_{r(k)-1}$ by $\mathbf{x}_{r(k)-2} + \mathbf{e}_{i_{r(k)}}$ in $\mathbf{z}(k-1)$, which amounts to exchange $\mathbf{e}_{i_{r(k)-1}}$ and $\mathbf{e}_{i_{r(k)}}$, and by iterating this process for the indices $i_{r(k)-1}, \cdots i_{r(k-1)+1}$, we obtain a new sequence $\mathbf{z}(k)$ of points of $\mathcal{S}$ such that $r_{\mathbf{z}(k)}(k) = k$.

We thus deduce that there exists a finite finite sequence $(\mathbf{z}(d))_{0\leq k\leq d}$ of vertices of $\mathcal{S}$ that satisfies $\mathbf{z}_d - \mathbf{z}_0 = \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_d} = \mathbf{e}_1 + \cdots + \mathbf{e}_d$, which contradicts the injectivity of the projection π on the stepped surface $\mathcal{S}$. $\square$

To conclude this section, let us introduce the notion of *geometric sums*.

Definition 1.8. A *geometric sum* (also called a *binary sum*) is a weighted sum of faces with weights in $\{0, 1\}$. We denote by $\mathfrak{B}$ (like binary) the set of geometric sums. If a face has weight 1 in a geometric sum, then it is said to *belong* to this geometric sum.

A geometric sum admits a geometrical representation by considering the union of its faces of weight 1. In particular, both stepped planes and stepped surfaces are geometric sums, so that one finally obtains (with all the inclusions being strict):

$$\mathfrak{P} \subset \mathfrak{S} \subset \mathfrak{B} \subset \mathfrak{F}.$$

1.2. **Flips and pseudo-flips.** We have considered sofar tilings of $\mathbb{R}^{d-1}$ by tiles which are projections of faces of unit hypercubes of $\mathbb{R}^d$. As mentioned in the introduction, such tilings are widely studied in mechanical physics and, in this context, a central notion is the *flip*. Roughly speaking, a flip is a local rearrangement of tiles, as depicted on Fig. 1.3 in the $d = 3$ case. Such a rearrangement

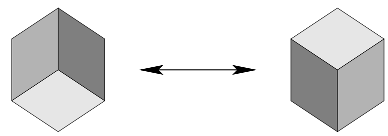

FIGURE 1.3. Flip

can be seen in $\mathbb{R}^d$. This is the viewpoint we take here. We thus consider flips as exchanges of faces in space $\mathbb{R}^d$. More formally, for any $\mathbf{x} \in \mathbb{Z}^d$, we define the following weighted sum $\mathcal{F}_{\mathbf{x}} \in \mathfrak{F}$:

$$\mathcal{F}_{\mathbf{x}} = \sum_{i=1}^{i=d} (\mathbf{x}, i^*) - \sum_{i=1}^{i=d} (\mathbf{x} - \mathbf{e}_i, i^*).$$

Since flips can be seen as a local exchange of faces, performing a flip on a stepped surface does not affect the fact that the projection π is an homeomorphims onto its image, and thus this operation preserves the property of being a stepped surface. More formally, we say that a stepped surface $\mathcal{S}' \in \mathfrak{S}$ is obtained *by performing a flip* $\mathcal{F}_{\mathbf{x}}$ over a stepped surface $\mathcal{S} \in \mathfrak{S}$ if one has: $\mathcal{S}' = \mathcal{S} \pm \mathcal{F}_{\mathbf{x}}$.

More generally, the notion of flip extends to weighted sums:

Definition 1.9 (Pseudo-flip)**.** A weighted sum $\mathcal{E}' \in \mathfrak{F}$ is obtained by performing a *pseudo-flip* $\mathcal{F}_{\mathbf{x}}$ on the weighted sum $\mathcal{E} \in \mathfrak{F}$ if one has: $\mathcal{E}' = \mathcal{E} \pm \mathcal{F}_{\mathbf{x}}$.

We use here the terminology pseudo-flip to stress the fact that the notion of pseudo-flip is not "geometric" contrarily to the notion of flip. This definition naturally leads to the notion of *pseudo-flip-accessibility* over weighted sums:

Definition 1.10 (Pseudo-flip-accessibility)**.** A weighted sum $\mathcal{E}'$ is said to be *pseudo-flip-accessible* from the weighted sum $\mathcal{E}$ if there exists a finite sequence $(\mathcal{E}_n)_{0 \leq n \leq N}$ of weighted sums such that $\mathcal{E} = \mathcal{E}_0$, $\mathcal{E}_n$ is obtained by performing a flip on $\mathcal{E}_{n-1}$, and $\mathcal{E}_N = \mathcal{E}'$. We thus get:

$$\mathcal{E}_N = \mathcal{E} + \sum_{n=0}^{N} \varepsilon_n \mathcal{F}_{\mathbf{x}_n}, \text{ with } \forall n,\ \varepsilon_n \in \{\pm 1\}.$$

Note that pseudo-flip-accessibility is a symmetric relation. Note also that faces of two pseudo-flip-accessible weighted sums have identical weights if they are far enough from the origin. Thus, weighted sums are not always pseudo-flip-accessible. For example, two stepped planes with different normal vectors are not pseudo-flip-accessible. This leads to extend the notion of pseudo-flip-accessibility by allowing not only finite but also infinite sequences of flips. We first define a distance $d_{\mathfrak{F}}$ over weighted sums of faces (the norm $|| \, ||$ stands for the Euclidean norm in $\mathbb{R}^d$):

Definition 1.11. Given two weighted sums $\mathcal{E}$ and $\mathcal{E}'$, we set $d(\mathcal{E}, \mathcal{E}') = 0$ if $\mathcal{E} = \mathcal{E}'$. Otherwise, $d(\mathcal{E}, \mathcal{E}')$ is equal to 2^{-R} where R is the supremum of the set of real numbers $r > 0$ such that $\mathcal{E}$ and $\mathcal{E}'$ take the same values on the faces $(\mathbf{x}, i^*) \in F$ with $||\mathbf{x}|| \leq r$.

Roughly speaking, the larger the balls $B(\mathbf{0}, r) = \{\mathbf{x} \in \mathbb{R}^d \mid ||\mathbf{x}|| < r\}$ the restriction of the weighted sums $\mathcal{E}$ and $\mathcal{E}'$ coincide on, the closer $\mathcal{E}$ and $\mathcal{E}'$ are.

Definition 1.12 (ω-pseudo-flip-accessibility)**.** A weigthed sum $\mathcal{E}'$ is said to be *ω-pseudo-flip-accessible* from the weighted sum $\mathcal{E}$ if there exists a sequence $(\mathcal{E}_k)_{k\in\mathbb{N}}$ of weighted sums that satisfies

(1) $\mathcal{E} = \mathcal{E}_0$,
(2) $\lim_{k\to\infty} d_{\mathfrak{F}}(\mathcal{E}_k, \mathcal{E}') = 0$,
(3) $\mathcal{E}_k$ is obtained by performing a flip on $\mathcal{E}_{k-1}$.

Proposition 1.13. *Any two stepped surfaces are ω-pseudo-flip-accessible.*

Proof. We say that $\mathbf{x} \in \mathbb{Z}^d$ is above (resp. below) the stepped surface $\mathcal{S}$ if there exists a point $\mathbf{y} \in \mathcal{S}$ such that $\sum_{i=1}^d x_i \geq \sum_{i=1}^d y_i$ (resp. $\sum_{i=1}^d x_i \leq \sum_{i=1}^d y_i$). We similarly say that $\mathbf{x} \in \mathbb{Z}^d$ is strictly above (resp. below) $\mathcal{S}$ if there exists $\mathbf{y} \in \mathcal{S}$ such that $\sum_{i=1}^d x_i > \sum_{i=1}^d y_i$ (resp. $\sum_{i=1}^d x_i < \sum_{i=1}^d y_i$). Let us note that this definition is consistent according to Proposition 1.7.

Let $\mathcal{S}$ and $\mathcal{S}'$ be two stepped surfaces. Let $\varepsilon_{\mathcal{S},\mathcal{S}'}$ be the function from $\mathbb{Z}^d$ into $\{0, \pm 1\}$ defined as follows (see also Fig. 1.4):

$$\varepsilon_{\mathcal{S},\mathcal{S}'} = \begin{cases} 1 & \text{if } \mathbf{x} \text{ is above} \mathcal{S} \text{ and strictly below} \mathcal{S}' \\ -1 & \text{if } \mathbf{x} \text{ is below} \mathcal{S} \text{ and strictly above} \mathcal{S}' \\ 0 & \text{otherwise} \end{cases}$$

We label as $(\mathbf{x})_{n\in\mathbb{N}}$ the set of $\mathbf{x}$ such that $\varepsilon_{\mathcal{S},\mathcal{S}'}(\mathbf{x}) \neq \mathbf{0}$ in such a way that if $n \leq m$, then $||\mathbf{x}_n|| \leq ||\mathbf{x}_m||$, and $||\mathbf{x}_n|| \neq ||\mathbf{x}_m||$, if $n \neq m$. We define for $N \in \mathbb{N}$ the weighted sum

$$\mathcal{S}_N := \mathcal{S} + \sum_{0\leq n\leq N} \varepsilon_{\mathcal{S},\mathcal{S}'}(\mathbf{x}_n)\mathcal{F}_{\mathbf{x}_n}.$$

One checks that the sequence $(\mathcal{S}_n)_n$ converges toward $\mathcal{S}$, which implies that $\mathcal{S}'$ is ω-pseudo-flip-accessible from $\mathcal{S}$. □

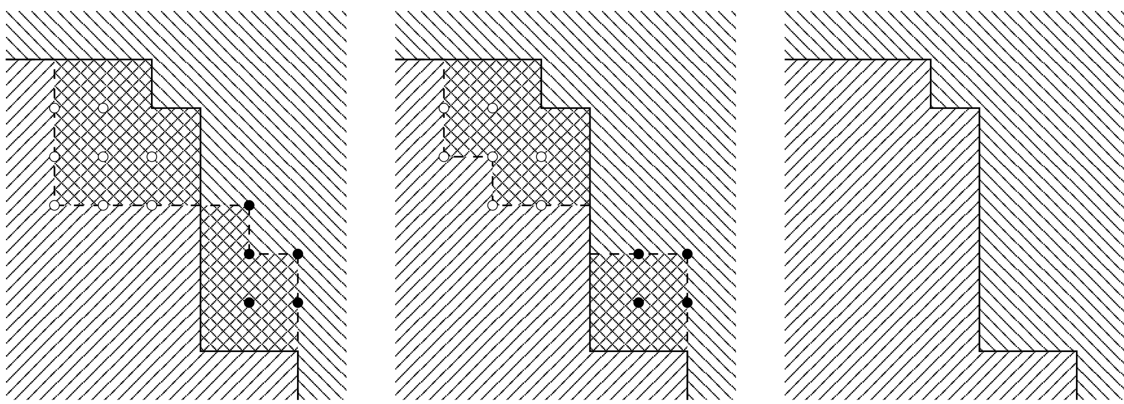

FIGURE 1.4. Two stepped surfaces $\mathcal{S}$ (dashed line) and $\mathcal{S}'$ (plain line) in the $d = 2$ case. The function $\varepsilon_{\mathcal{S},\mathcal{S}'}$ maps vertices above $\mathcal{S}$ and strictly below $\mathcal{S}'$ to $+1$ (white points), and vertices below $\mathcal{S}$ and strictly above $\mathcal{S}'$ to -1 (black points). Other vertices are all mapped on 0. By adding or subtracting flips according to this function, the stepped surface $\mathcal{S}$ is transformed into the stepped surface $\mathcal{S}'$ (from left to right).

Consequently, the notion of ω-pseudo-flip accessibility could seem to be not restrictive enough. Nevertheless, the following theorem illustrates the interest of this notion that will play a key role in the following sections (see in particular the proof of Th. 2.11):

Theorem 1.14. *If a geometric sum is ω-pseudo-flip-accessible from a stepped surface, then it is a stepped surface.*

Proof. We first start by considering two pseudo-flip-accessible geometric sums. Let us show that one can rearrange the finite sequence of pseudo-flips in order to obtain only geometric intermediate sums. Let the geometric sum $\mathcal{E}$ be pseudo-flip-accessible from the geometric sum $\mathcal{E}'$. There exist a finite sequence of flips $(\mathcal{F}_{\mathbf{x}_n})_{n=1,\ldots,N}$, and a finite sequence of weights $(\lambda_n)_{n=1,\ldots,N}$ such that:

$$\mathcal{E}' = \mathcal{E}_N = \mathcal{E} + \sum_{n=1}^{N} \lambda_n \mathcal{F}_{\mathbf{x}_n},$$

where, for every n, $\lambda_n \in \mathbb{Z}^*$, $\mathbf{x}_n \in \mathbb{Z}^d$. We furthermore assume w.l.o.g. that $n \neq n'$ yields $\mathbf{x}_n \neq \mathbf{x}_{n'}$. The intermediate sums $\mathcal{E}_k = \mathcal{E} + \lambda_1 \mathcal{F}_{\mathbf{x}_1} + \ldots + \lambda_k \mathcal{F}_{\mathbf{x}_n}$, for $k \in \{1, \ldots, N-1\}$, are not necessarily geometric.

Let us rearrange the sequence of flips as follows. Let $\preceq$ be the partial order over $\mathbb{Z}^d$ defined by:

$$(x_1, \ldots, x_d) \preceq (y_1, \ldots, y_d) \text{ if, for all } i \in \{1, \ldots, d\},\ x_i \leq y_i.$$

By sorting flips according to the partial order $\preceq$, respectively for the positive and negative λ_k's, we can assume that

$$\mathbf{x}i \preceq \mathbf{x}_j \ \Rightarrow \ \begin{cases} i \leq j & \text{if } \lambda_{\tau(i)} > 0 \text{ and } \lambda_{\tau(j)} > 0 \\ i \geq j & \text{if } \lambda_{\tau(i)} < 0 \text{ and } \lambda_{\tau(j)} < 0. \end{cases}$$

Let us prove by induction on $k \in \{0, \cdots, N\}$ that all the intermediate sums $\mathcal{E}_k$ are geometric. Note that $\mathcal{E}_0 = \mathcal{E}$ (as well as $\mathcal{E}_N = \mathcal{E}'$) are geometric.

We suppose that $\mathcal{E}_k$ is geometric for some k, with $0 \leq k \leq N-1$. Let us assume that $\lambda_{\tau(k+1)} > 0$ (the case $\lambda_{\tau(k+1)} < 0$ is handled in a symmetric way). Let us fix i. We first consider the weight of the face $(\mathbf{x}_{\tau(k+1)} - \mathbf{e}_i, i^*)$ in $\mathcal{E}_{k+1}$. One has $\mathcal{E}_{k+1}(\mathbf{x}_{\tau(k+1)} - \mathbf{e}_i, i^*) = \mathcal{E}_k(\mathbf{x}_{\tau(k+1)} - \mathbf{e}_i, i^*) - \lambda_{\tau(k+1)}$. Let ℓ such that $\ell > k+1$. By assumption, one has $\mathbf{x}_{\tau(\ell)} - \mathbf{e}_i \neq \mathbf{x}_{\tau(k+1)} - \mathbf{e}_i$; furthermore, if $\mathbf{x}_{\tau(\ell)} = \mathbf{x}_{\tau(k)} - \mathbf{e}_i$, then $\lambda_{\tau(\ell)} < 0$. Hence

$$\mathcal{E}_{k+1}(\mathbf{x}_{\tau(k+1)} - \mathbf{e}_i, i^*) = \mathcal{E}_k(\mathbf{x}_{\tau(k+1)} - \mathbf{e}_i, i^*) - \lambda_{\tau(k+1)} \geq \mathcal{E}_N(\mathbf{x}_{\tau(k+1)} - \mathbf{e}_i, i^*) \in \{0,1\}.$$

We deduce from $\mathcal{E}_k(\mathbf{x}_{\tau(k+1)} - \mathbf{e}_i, i^*) \in \{0,1\}$ that $\lambda_{\tau(k+1)} = 1$, and thus that $\mathcal{E}_{k+1}(\mathbf{x}_{\tau(k+1)} - \mathbf{e}_i, i^*) \in \{0,1\}$.

We consider now the weight of the face $(\mathbf{x}_{\tau(k+1)}, i^*)$. One has for $1 \leq \ell \leq k$, $\mathbf{x}_{\tau(\ell)} \neq \mathbf{x}_{\tau(k)}$; furthermore, if $\mathbf{x}_{\tau(\ell)} - \mathbf{e}_i = \mathbf{x}_{\tau(k)}$, then $\lambda_{\tau(\ell)} < 0$. Hence

$$\mathcal{E}_{k+1}(\mathbf{x}_{\tau(k+1)}, i^*) = \mathcal{E}_k(\mathbf{x}_{\tau(k+1)}, i^*) + \lambda_{\tau(k+1)} \leq \mathcal{E}_0(\mathbf{x}_{\tau(k+1)}, i^*) \in \{0,1\}.$$

We deduce that $\mathcal{E}_k(\mathbf{x}_{\tau(k+1)}, i^*) = 0$, and that $\mathcal{E}_{k+1}(\mathbf{x}_{\tau(k+1)}, i^*) = 1$, which ends the induction proof.

Let us end now the proof of Theorem 1.14. The idea of the proof is to use the fact that being a homeomorphism for π is a "local" property. Indeed, let $\mathcal{E}$ be a geometric sum which is ω-pseudo-flip-accessible from the stepped surface $\mathcal{S}$. Let $r > 0$. Let us prove that π is an homeomorphism from the union of faces $(\mathbf{x}, i^*)$ of weight 1 in $\mathcal{E}$ that satisfy $||\mathbf{x}|| \leq r$ onto its image. Let $(\mathcal{E})_r$ (resp. $(\mathcal{S})_r$) stand for the restriction of $\mathcal{E}$ (resp. $\mathcal{S}$). to this set. We have proved that the geometric sum $(\mathcal{E})_r$ is pseudo-flip-accessible from $(\mathcal{S})_r$ with intermediates sums being geometric sums, and thus stepped surfaces, since $\mathcal{S}$ is a stepped surface. Since π is a homeomorphism from the stepped surface $\mathcal{S}$ onto $\mathbb{R}^{d-1}$ and since this is invariant by performing the previous flips, this yields that π is a homeomorphism from the union of faces of $\mathcal{E}$ with weight 1 which are in $B(\mathbf{0}, r)$ onto its image in $\mathbb{R}^{d-1}$. Since this holds for any $r > 0$, this proves that π is a homeomorphism from $\mathcal{E}$ onto $\mathbb{R}^{d-1}$, that is, $\mathcal{E}$ is a stepped surface. □

The following stronger result is proven in [2]: let us fix α a real vector with positive entries and $\rho \in \mathbb{R}$; then, any stepped surface $\mathcal{S}$ is *ω-flip-accessible* from the stepped plane $\mathcal{P}_{\alpha,\rho}$, that is, there exists a sequence $(\mathcal{S}_k)_{k \in \mathbb{N}}$ of *stepped surfaces* that satisfies

(1) $\mathcal{P}_{\alpha,\rho} = \mathcal{S}_0$,
(2) $\lim_{k\to\infty} d_{\mathfrak{F}}(\mathcal{S}_k, \mathcal{S}) = 0$,
(3) $\mathcal{S}_k$ is obtained by performing a flip on $\mathcal{S}_{k-1}$.

For example, the right stepped surface of Fig. 1.2 can be obtained by performing (infinitely many) flips over the left stepped plane of the same figure. Note that there exist stepped surfaces which are not ω-flip-accessible (a trivial example consists in a stepped plane of normal vector $\mathbf{e}_i$, for a given $i \in \{1, 2, \cdots, d\}$: one easily checks that no flip can be performed on such a stepped plane). A complete characterization of ω-flip-accessibility in terms of *shadows* is provided in [2], see also [6].

2. Action of the linear group on stepped surfaces

We introduce in this section the notion of dual map associated with a unimodular morphism of the free group. Dual maps are shown to act on stepped planes in Sec. 2.2 and on stepped surfaces in Sec. 2.3.

2.1. Free group morphisms, dual maps and the linear group.

The *free group* over $\{1, \ldots, d\}$ is denoted by F_d. A *morphism* of F_d is thus a map $\sigma : F_d \to F_d$ such that, for any u, v in F, $\sigma(uv) = \sigma(u)\sigma(v)$.

The *incidence matrix* of a morphism σ of F_d, denoted by M_σ, is the $d \times d$ matrix whose entry at i-th row and j-th column is the number of occurrences of the letter i in $\sigma(j)$ minus the number of occurrences of the letter i^{-1} in $\sigma(j)$. Note that two different morphisms can have the same incidence matrix and that every matrix with entries in $\mathbb{Z}$ can be seen as the incidence matrix of a morphism. Morphisms whose incidence matrix has determinant ± 1 are said to be *unimodular*; they play an important role in what follows.

A morphism of the free group σ is an *automorphism* if there exists a morphism, denoted by σ^{-1}, such that $\sigma \circ \sigma^{-1} = Id$. Note that an automorphism is necessarily unimodular, but that a unimodular morphism is not always an automorphism.

Last, a morphism of F_d^* is said to be *non-negative* if it maps each letter of $\{1, \ldots, d\}$ to a word over $\{1, \ldots, d\}$, and it is said to be *non-erasing* if it does maps no letter to the empty word. Non-negative non-erasing morphisms are usually called *substitutions*.

Example 2.1. Let us illustrate the previous notions with the following morphism:

$$\sigma : \begin{cases} 1 & \mapsto & 12 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 1 \end{cases}$$

It acts over the free group over $\{1, 2, 3\}$. For example, one has

$$\sigma(1^{-1}312) = \sigma(1)^{-1}\sigma(3)\sigma(1)\sigma(2) = (12)^{-1}11213 = 2^{-1}1^{-1}11213 = 2^{-1}1213.$$

Its incidence matrix is:

$$M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since this matrix has determinant one, σ is a unimodular morphism. It is moreover invertible. Indeed one checks that

$$\sigma^{-1} : \begin{cases} 1 & \mapsto & 3 \\ 2 & \mapsto & 3^{-1}1 \\ 3 & \mapsto & 3^{-1}2 \end{cases}.$$

One has, for example:

$$\sigma^{-1} \circ \sigma(1) = \sigma^{-1}(12) = 33^{-1}1 = 1 = \sigma(3) = \sigma \circ \sigma^{-1}(1).$$

Its incidence matrix is

$$M_{\sigma^{-1}} = (M_\sigma)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

Now, let us briefly recall the notion of *dual maps* associated with unimodular morphisms, introduced in [3] for positive morphisms and extended to the general case in [10]. We first introduce the Parikh map $\mathbf{f} : \{1, \cdots, d\}^\star \to \mathbb{Z}^d$ defined by $\mathbf{f}(w) = (|w|_1 - |w|_{-1}, \cdots, |w|_d - |w|_{d^{-1}})$, where $|w|_i$ stands for the number of occurrences of the letter i in w.

Definition 2.2 (Dual map)**.** Let σ be a unimodular morphism of the free group F_d. The dual map $E_1^*(\sigma)\colon \mathfrak{F} \to \mathfrak{F}$ associated with σ is defined as follows. Let $\mathcal{E}$ be a weighted sum. The weighted sum $E_1^*(\sigma)(\mathcal{E})$ gives to a face $(\mathbf{y}, j^*)$ the weight

$$\sum_{(\mathbf{x},i^*),\ \sigma(j)=p\cdot i\cdot s} \mathcal{E}(M_\sigma(\mathbf{x}+\mathbf{f}(p)), i^*) - \sum_{(\mathbf{x},i^*),\ \sigma(j)=p\cdot i^{-1}\cdot s} \mathcal{E}(M_\sigma(\mathbf{x}+\mathbf{f}(p)-\mathbf{e}_i), i^*).$$

In particular, $E_1^*(\sigma)$ acts on the weighted sums restricted to one face as follows

$$E_1^*(\sigma)(\mathbf{x}, i^*) = \sum_{\sigma(j)=p\cdot i\cdot s} (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p)), j^*) - \sum_{\sigma(j)=p\cdot i^{-1}\cdot s} (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p)+\mathbf{e}_i), j^*). \tag{2.1}$$

One easily checks that $E_1^*(\sigma)$ satisfies $E_1^*(\sigma)(\mathcal{E}+\mathcal{E}') = E_1^*(\sigma)(\mathcal{E})+E_1^*(\sigma)(\mathcal{E}')$, for any two weighted sums $\mathcal{E}$ and $\mathcal{E}'$. Furthermore, one key property of dual maps is their behaviour with respect to composition:

Theorem 2.3 ([10])**.** *Let σ, σ' be two morphisms of the free group F_d. One has*

$$E_1^*(\sigma \circ \sigma') = E_1^*(\sigma') \circ E_1^*(\sigma).$$

Hence "desusbtituting" a weighted sum with respect to the morphism $E_1^*(\sigma)$ reduces to the application of $E_1^*(\sigma^{-1})$.

It is worth noticing that $E_1^*(\sigma)$ maps two faces $(\mathbf{y}, j^*)$ and $(\mathbf{z}, j^*)$, for some j that satisfies $1 \leq j \leq d$ onto weighted sums of faces which are equal up to a translation of vector $M_\sigma^{-1}(\mathbf{z}-\mathbf{y})$. In other words, $E_1^*(\sigma)$ is uniquely characterized by its action on the faces $(\mathbf{0}, j^*)$, for $1 \leq j \leq d$, and by the matrix M_σ^{-1}.

Example 2.4. Let us compute the dual map associated with the morphism introduced in Ex. 2.1. It suffices to compute its action over unit faces $(\mathbf{0}, j^*)$, $j = 1, 2, 3$:

$$E_1^*(\sigma) : \begin{cases} (\mathbf{0}, 1^*) & \mapsto & (\mathbf{0}, 1^*) + (\mathbf{0}, 2^*) + (\mathbf{0}, 3^*) \\ (\mathbf{0}, 2^*) & \mapsto & (-\mathbf{e}_3, 1^*) \\ (\mathbf{0}, 3^*) & \mapsto & (-\mathbf{e}_3, 2^*). \end{cases}$$

One can also compute the dual map associated with the inverse of σ:

$$E_1^*(\sigma^{-1}) \ : \ \left\{ \begin{array}{lcl} (\mathbf{0},1^*) & \mapsto & (\mathbf{e}_1,2^*) \\ (\mathbf{0},2^*) & \mapsto & (\mathbf{e}_1,3^*) \\ (\mathbf{0},3^*) & \mapsto & (\mathbf{0},1^*) - (\mathbf{e}_1,2^*) - (\mathbf{e}_1,3^*). \end{array} \right.$$

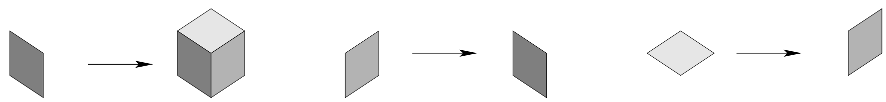

FIGURE 2.1. Action of the dual map $E_1^*(\sigma)$, where $\sigma : 1 \to 12, 2 \to 13, 3 \to 1$, over the unit faces $(\mathbf{0},1^*)$, $(\mathbf{0},2^*)$ and $(\mathbf{0},3^*)$ (from left to right).

2.2. **Action on stepped planes.** Dual maps associated with unimodular morphisms of the free group are defined over the set $\mathfrak{F}$ of weighted sums of faces. In this section, we are particularly interested in the way these maps act over the set $\mathfrak{P}$ of stepped planes.

The action of dual maps over stepped planes is well characterized by the following theorem, which extends Th. 14 in [14] stated in the case of dual maps of *non-negative* (unimodular) morphisms:

Proposition 2.5. *Let σ be a unimodular morphism. Let $\rho \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+^d \backslash \mathbf{0}$. If $M_\sigma^\top \alpha \in \mathbb{R}_+^d$, then $E_1^*(\sigma)$ maps the stepped plane $\mathcal{P}_{\alpha,\rho}$ onto the stepped plane $\mathcal{P}_{M_\sigma^\top \alpha,\rho}$. Conversely, if $M_\sigma^\top \alpha \notin \mathbb{R}_+^d$, then the image of $\mathcal{P}_{\alpha,\rho}$ is not a geometric sum.*

Proof. First, note that $\mathcal{E} = E_1^*(\sigma)(\mathcal{P}_{\alpha,\rho})$ is well-defined and is a weighted sum. We fix $j \in \{1, \cdots, d\}$. Let $(\mathbf{y}, j^*)$ be a face and let us compute the weight $\mathcal{E}(\mathbf{y}, j^*)$. We use the notation

$$\sigma(j) = w_1^{(j)} w_2^{(j)} \cdots w_{l_i}^{(j)}.$$

Let $(\mathbf{x}, i^*)$ such that $\mathcal{P}_{\alpha,\rho}(\mathbf{x}, i^*) = 1$. By definition,

$$\begin{aligned} \mathcal{E}(\mathbf{y}, j^*) &= \#\{k \mid w_k^{(j)} = i \text{ and } \mathbf{y} = M_\sigma^{-1}(\mathbf{x} - \mathbf{f}(w_1^{(j)} \ldots w_{k-1}^{(j)}))\} \\ &\quad -\#\{k \mid w_k^{(j)} = i^{-1} \text{ and } \mathbf{y} = M_\sigma^{-1}(\mathbf{x} - \mathbf{f}(w_1^{(j)} \ldots w_{k-1}^{(j)}) + \mathbf{e}_i)\}. \end{aligned}$$

Note that, with $r_k = M_\sigma \mathbf{y} + f(w_1^{(j)} \ldots w_k^{(j)})$ for $0 \le k \le l$, the first cardinality of the right-hand side of the equality yields:

$$\mathbf{x} = r_{k-1} \quad \text{and} \quad \mathbf{x} + \mathbf{e}_i = r_{k-1} + \mathbf{e}_i = r_{k-1} + \mathbf{f}(i) = r_{k-1} + \mathbf{f}(w_k^{(j)}) = r_k,$$

while the second yields:

$$\mathbf{x} = r_{k-1} - \mathbf{e}_i = r_{k-1} + \mathbf{f}(i^{-1}) = r_{k-1} + \mathbf{f}(w_k^{(j)}) = r_k \quad \text{and} \quad \mathbf{x} + \mathbf{e}_i = r_{k-1}.$$

Recall that $\mathcal{P}_{\alpha,\rho}(\mathbf{x}, i^*) = 1$ if and only if $\langle \mathbf{x}, \alpha \rangle < \rho \le \langle \mathbf{x} + \mathbf{e}_i, \alpha \rangle$. Therefore, one has:

$$\mathcal{E}(\mathbf{y}, j^*) = \#\{k \mid \langle r_{k-1}, \alpha \rangle < \rho \le \langle r_k, \alpha \rangle\} - \#\{k \mid \langle r_k, \alpha \rangle < \rho \le \langle r_{k-1}, \alpha \rangle\}.$$

It is not hard to see that, if $\langle r_0, \alpha \rangle \le \langle r_l, \alpha \rangle$, then this yields:

$$\mathcal{E}(\mathbf{y}, j^*) = \left\{ \begin{array}{ll} 1 & \text{if } \langle r_0, \alpha \rangle < \rho \le \langle r_l, \alpha \rangle, \\ 0 & \text{otherwise.} \end{array} \right.$$

Similarly, if $\langle r_0, \alpha \rangle > \langle r_l, \alpha \rangle$, then this yields:

$$\mathcal{E}(\mathbf{y}, j^*) = \left\{ \begin{array}{ll} -1 & \text{if } \langle r_0, \alpha \rangle > \rho \ge \langle r_l, \alpha \rangle, \\ 0 & \text{otherwise.} \end{array} \right.$$

Last, note that:

$$\langle r_0, \alpha \rangle = \langle M_\sigma \mathbf{y}, \alpha \rangle = \langle \mathbf{y}, M_\sigma^\top \alpha \rangle,$$

$$\langle r_l, \alpha \rangle = \langle M_\sigma \mathbf{y} + \mathbf{f}(\sigma(i)), \alpha \rangle = \langle M_\sigma (\mathbf{y} + \mathbf{e}_i), \alpha \rangle = \langle \mathbf{y} + \mathbf{e}_i, M_\sigma^\top \alpha \rangle.$$

Thus, $M_\sigma^\top \alpha \in \mathbb{R}_+^d$ yields $\langle r_0, \alpha \rangle \leq \langle r_l, \alpha \rangle$ and, according to what preceeds, $\mathcal{E}(\mathbf{y}, j^*) = 1$ if and only if $\langle \mathbf{y}, M_\sigma^\top \alpha \rangle < \rho \leq \langle \mathbf{y} + \mathbf{e}_i, M_\sigma^\top \alpha \rangle$. This proves that $\mathcal{E} = \mathcal{P}_{M_\sigma^\top \alpha, \rho}$.

Conversely, $M_\sigma^\top \alpha \notin \mathbb{R}_+^d$ yields that, for some i, $\langle r_0, \alpha \rangle > \langle r_l, \alpha \rangle$. Hence, there is a face $(\mathbf{y}, j^*)$ such that $\mathcal{E}(\mathbf{y}, j^*) = -1$. This shows that $\mathcal{E} \notin \mathfrak{B}$. □

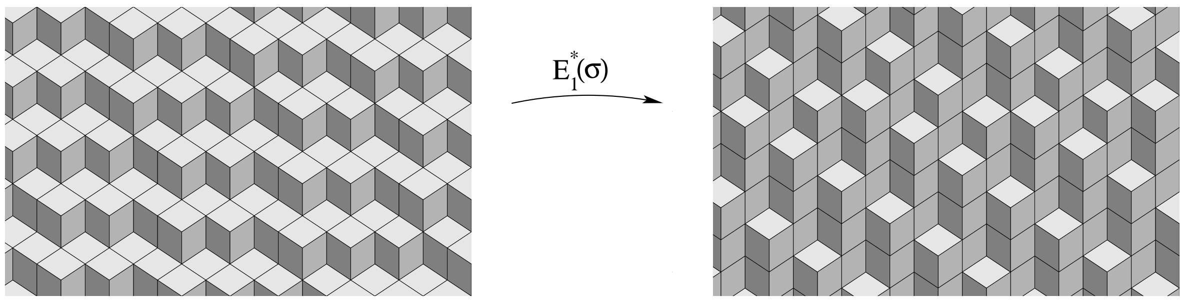

FIGURE 2.2. If the image of a stepped plane $\mathcal{P}_{\alpha,\rho}$ by a dual map $E_1^*(\sigma)$ is geometric, then it is the stepped plane $\mathcal{P}_{M_\sigma^\top \alpha, \rho}$.

In particular, Prop. 2.5 easily yields the following theorem:

Theorem 2.6. *Let σ be a unimodular morphism. If the image by $E_1^*(\sigma)$ of a stepped plane is geometric, then it is a stepped plane:*

$$E_1^*(\sigma)(\mathfrak{P}) \cap \mathfrak{B} \subset \mathfrak{P}.$$

Note that, in the particular but important following case (which includes, for example, unimodular substitutions), it is no more necessary to assume that the image is geometric:

Proposition 2.7. *If σ is a unimodular morphism whose incidence matrix has non-negative entries, then $E_1^*(\sigma)$ maps stepped planes onto stepped planes:*

$$M_\sigma \geq 0 \ \Rightarrow \ E_1^*(\sigma)(\mathfrak{P}) \subset \mathfrak{P}.$$

Proof. Since $M_\sigma^\top \alpha \in \mathbb{R}_+^d$ for any $\alpha \in \mathbb{R}_+^d \backslash \mathbf{0}$, the result follows from Prop. 2.5. □

It is also worth stressing that Prop. 2.5 yields that the image of a stepped plane by a dual map depends only on the incidence matrix of the underlying unimodular morphism:

Proposition 2.8. *Let σ and σ' be two unimodular morphisms with the same incidence matrix $M_\sigma = M_{\sigma'}$. If a stepped plane $\mathcal{P} \in \mathfrak{P}$ has a geometric image by $E_1^*(\sigma)$, then one has:*

$$E_1^*(\sigma)(\mathcal{P}) = E_1^*(\sigma')(\mathcal{P}).$$

2.3. **Action on stepped surfaces.** In this section, results concerning the action of dual maps over stepped planes are extended to stepped surfaces, by relying on the notion of flip introduced in Sec. 1. First, the following lemma shows that dual maps act very particularly over flips:

Lemma 2.9. *Let σ be a unimodular morphism. One has:*

$$\forall \mathbf{x} \in \mathbb{Z}^d, \quad E_1^*(\sigma)(\mathcal{F}_{\mathbf{x}}) = \mathcal{F}_{M_\sigma^{-1}\mathbf{x}}.$$

Proof. For $i = 1, \cdots, d$, let $l_i = |\sigma(i)|$ and $p_k^{(i)}$ stand for the prefix of length k of $\sigma(i)$ (in its reduced form). We recall the notation

$$\sigma(i) = w_1^{(i)} w_2^{(i)} \cdots w_{l_i}^{(i)}.$$

Let $\mathbf{x} \in \mathbb{Z}^d$. One has

$$\begin{array}{rcll} E_1^*(\sigma)(\mathcal{F}_x) & = & \sum_{i=1}^d \sum_{j \mid \sigma(j)=p\cdot i\cdot s} & (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p)), j^*) - (M_\sigma^{-1}(\mathbf{x}-\mathbf{e}_i-\mathbf{f}(p)), j^*) \\ & - & \sum_{i=1}^d \sum_{j \mid \sigma(j)=p\cdot i^{-1}\cdot s} & (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p)+\mathbf{e}_i), j^*) - (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p)), j^*). \end{array}$$

By using the previous notation, this implies that $E_1^*(\sigma)(\mathcal{F}_x)$ equals:

$$\sum_{j=1}^d \sum_{0\le k\le l_j-1,\ w_{k+1}^{(j)} \in \{1,\cdots,d\}} (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p_k^{(j)})), j^*) - (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(w_{k+1}^{(j)})-\mathbf{f}(p_k^{(j)})), j^*)$$

$$+\sum_{j=1}^d \sum_{0\le k\le l_j-1,\ w_{k+1}^{(j)} \in \{1,\cdots,d\}^{-1}} (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p_k^{(j)})), j^*) - (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(w_{k+1}^{(j)})-\mathbf{f}(p_k^{(j)})), j^*).$$

One thus gets

$$E_1^*(\sigma)(\mathcal{F}_x) = \sum_{j=1}^d \sum_{0\le k\le l_j-1,\ w_{k+1}^{(j)} \in \{1,\cdots,d\}} (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p_k^{(j)})), j^*) - (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p_{k+1}^{(j)})), j^*)$$

$$+\sum_{j=1}^d \sum_{0\le k\le l_j-1,\ w_{k+1}^{(j)} \in \{1,\cdots,d\}^{-1}} (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p_k^{(j)})), j^*) - (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p_{k+1}^{(j)})), j^*).$$

This implies:

$$\begin{array}{rcll} E_1^*(\sigma)(\mathcal{F}_x) & = & \sum_{1\le j\le d,\ w_1^{(j)}\in\{1,\cdots,d\}} & (M_\sigma^{-1}(\mathbf{x}), j^*) - (M_\sigma^{-1}(\mathbf{x}) - M_\sigma^{-1}(\mathbf{f}(\sigma(j))), j^*) \\ & + & \sum_{1\le j\le d,\ w_1^{(j)}\in\{1,\cdots,d\}^{-1}} & (M_\sigma^{-1}(\mathbf{x}), j^*) - (M_\sigma^{-1}(\mathbf{x}) - M_\sigma^{-1}(\mathbf{f}(\sigma(j))), j^*) \\ & = & \sum_{1\le j\le d,\ w_1^{(j)}\in\{1,\cdots,d\}} & (M_\sigma^{-1}(\mathbf{x}), j^*) - (M_\sigma^{-1}(\mathbf{x}) - \mathbf{e}_j, j^*) \\ & + & \sum_{1\le j\le d,\ w_1^{(j)}\in\{1,\cdots,d\}^{-1}} & (M_\sigma^{-1}(\mathbf{x}), j^*) - (M_\sigma^{-1}(\mathbf{x}) - \mathbf{e}_j, j^*) \\ & = & \mathcal{F}_{M_\sigma^{-1}\mathbf{x}}, & \end{array}$$

which ends the proof. □

Th. 2.6 and Lem. 2.9 characterize the action of dual maps over, respectively, stepped planes and flips. In order to extend this to stepped surfaces by relying on the results of Sec. 1, we also need the following lemma:

Lemma 2.10. *Let σ be a unimodular morphism. The map $E_1^*(\sigma)$ is uniformy continuous over $\mathfrak{F}$.*

Proof. Let σ be a unimodular morphism. Let $\mathcal{E} \in \mathfrak{F}$. Let $r > 0$. From the expression of $E_1^*(\sigma)$, we deduce that there exists $R > 0$ such that for all $(\mathbf{y}, j^*)$, with $||\mathbf{y}|| \le r$, which has a non-zero weight in $E_1^*(\sigma)(\mathcal{E})$, then all the faces $(\mathbf{x}, i^*) \in F$ for which $(\mathbf{y}, j^*)$ occurs with a non-zero weight in $E_1^*(\sigma)(\mathbf{x}, i^*)$ satisfy $||\mathbf{x}|| \le R$, hence the uniform continuity. □

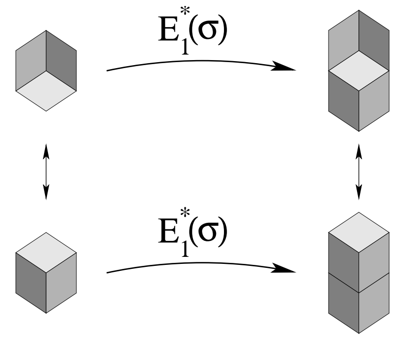

FIGURE 2.3. The weighted sums $\sum_i (\mathbf{x}, i^*)$ (top left) and $\sum_i (\mathbf{x}-\mathbf{e}_i, i^*)$ (bottom left) are respectively mapped by the dual map $E_1^*(\sigma)$ on $E_1^*(\sigma)(\sum_i (\mathbf{x}, i^*))$ (top right) and $E_1^*(\sigma)(\sum_i (\mathbf{x}-\mathbf{e}_i, i^*))$ (bottom right). The weighted sum defined by the difference of the two former weighted sums (left) is equal to the flip $\mathcal{F}_{\mathbf{x}}$, and it turns out that the difference of the two latter weighted sums (right) is equal to the flip $\mathcal{F}_{M\sigma^{-1}\mathbf{x}}$. In short, $E_1^*(\sigma)(\mathcal{F}_{\mathbf{x}}) = \mathcal{F}_{M\sigma^{-1}\mathbf{x}}$.

We are now in a position to extend Th. 2.6:

Theorem 2.11. *Let σ be a unimodular morphism with incidence matrix M_σ. If the image by $E_1^*(\sigma)$ of a stepped surface $\mathcal{S}$ is a geometric sum, then $\mathcal{S}$ is a stepped surface:*

$$E_1^*(\sigma)(\mathfrak{S}) \cap \mathfrak{B} \subset \mathfrak{S}.$$

Proof. Let $\mathcal{S}$ be a stepped surface and assume that $E_1^*(\sigma)(\mathcal{S})$ is a geometric sum. We first prove that there exists a nonzero vector $\alpha \in [0,1]^d$ such that $M_\sigma^\top \alpha$ has nonnegative entries. For all $n \in \mathbb{N}$, $(\mathcal{S})_n$ stands for the restriction of $\mathcal{S}$ to $\{\mathbf{x} \in \mathbb{Z}^d \mid ||\mathbf{x}|| \leq r\}$, and $|(\mathcal{S})_n|$ (resp. $|(\mathcal{S})_n|_i$) for the number of faces (resp. of type i) with positive weight in $(\mathcal{S})_n$. We use similar notation for $E_1^*(\sigma)((\mathcal{S})_n)$. For n large enough, one has $|E_1^*(\sigma)(\mathcal{S})_n| > 0$. Indeed, according to Lemma 2.10, the sequence $(E_1^*(\sigma)(\mathcal{S})_n)_n$ tends with n to $E_1^*(\sigma)(\mathcal{S})$ which is a geometric sum by assumption. We thus can define

$$\alpha_n = \frac{1}{|(\mathcal{S})_n|}(|(\mathcal{S})_n|_1, \cdots, |(\mathcal{S})_n|_d), \quad \text{and } \beta_n = \frac{1}{|E_1^*(\sigma)(\mathcal{S})_n|}(|E_1^*(\sigma)(\mathcal{S})_n|_1, \cdots, |E_1^*(\sigma)(\mathcal{S})_n|_d).$$

By (2.1), there exists for every n a nonnegative integer λ_n such that $\lambda_n \beta_n = \lambda_n M_\sigma^\top \alpha_n$. The vectors α_n and β_n have norm $|| \, ||_1$ equal to 1 and the nonnegative real numbers λ_n take bounded values. By compacity, we thus can extract from sequences $(\alpha_n)_n$, $(\beta_n)_n$ and $(\lambda_n)_n$ convergent sequences of respective limits α, β and λ which satisfy $\lambda\beta = M_\sigma^\top \alpha$ and $\alpha, \beta \in [0,1]^d \setminus \{\mathbf{0}\}$ and $\lambda > 0$.

According to Prop. 1.13, there exist a sequence $(\mathcal{E}_k)_{k\in\mathbb{N}}$ of weighted sums and a sequence of flips $(\mathcal{F}_{\mathbf{x}_k})_{k\in\mathbb{N}}$, such that $\mathcal{P}_{\alpha,0} = \mathcal{S}_0$, $\lim_{k\to\infty} d_{\mathfrak{F}}(\mathcal{E}_k, \mathcal{E}) = 0$, and, for any $k \geq 1$, $\mathcal{E}_k = \mathcal{E}_{k-1} \pm \mathcal{F}_{\mathbf{x}_k}$.

Let us fix k. One has:

$$E_1^*(\sigma)(\mathcal{E}_{k+1}) = E_1^*(\sigma)(\mathcal{E}_k) \pm E_1^*(\sigma)(\mathcal{F}_{\mathbf{x}_k}),$$

and according to Lem. 2.9, this implies:

$$E_1^*(\sigma)(\mathcal{E}_{k+1}) = E_1^*(\sigma)(\mathcal{E}_k) \pm \mathcal{F}_{M_\sigma^{-1}\mathbf{x}_k}.$$

Furthermore, according to Lem. 2.10, one has $\lim_{k\to\infty} d_{\mathfrak{F}}(E_1^*(\sigma)(\mathcal{E}_k), E_1^*(\sigma)(\mathcal{E})) = 0$.

We deduce from Th. 2.6 that the geometric sum $E_1^*(\sigma)(\mathcal{S})$ is ω-pseudo-flip-accessible from the stepped plane $\mathcal{P}_{M_\sigma^\top \alpha, 0}$, and Th. 1.14 finally yields that $E_1^*(\sigma)(\mathcal{S})$ is a stepped surface. □

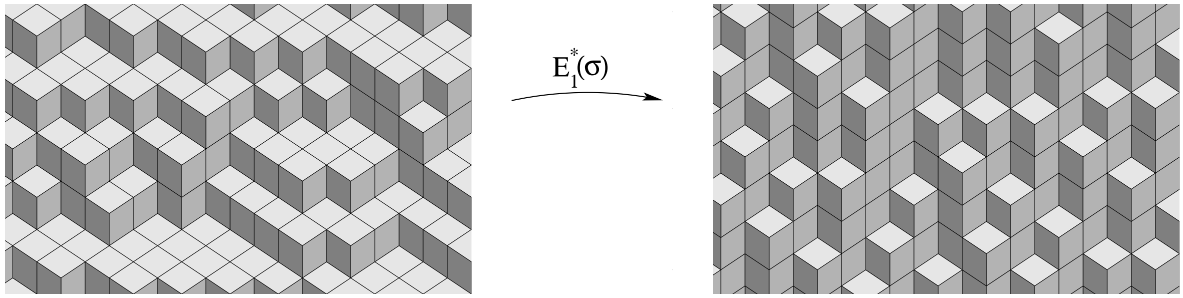

FIGURE 2.4. If the image of a stepped surface by a dual map is geometric, then it is a stepped surface.

As in the case of stepped planes, Prop. 2.5 implies that the image of a stepped plane by a dual map depends only on the incidence matrix of the underlying unimodular morphism (see Prop. 2.12 below). Note that Eq. (2.1) easily shows that, if $M_\sigma = M_{\sigma'}$ but $\sigma \neq \sigma'$, then $E_1^*(\sigma)$ and $E_1^*(\sigma')$ do not act in the same way over faces of a stepped surface.

Proposition 2.12. *Let σ and σ' be two unimodular morphisms with the same incidence matrix $M_\sigma = M_{\sigma'}$. If the image of a stepped surface $\mathcal{S} \in \mathfrak{S}$ by $E_1^*(\sigma)$ is a geometric sum, then one has:*

$$E_1^*(\sigma)(\mathcal{S}) = E_1^*(\sigma')(\mathcal{S}).$$

Proof. Let σ and σ' be two unimodular morphisms with the same incidence matrix $M_\sigma = M_{\sigma'}$. We assume that the image of a stepped surface $\mathcal{S} \in \mathfrak{S}$ by $E_1^*(\sigma)$ is a geometric sum. According to the proof of Theorem 2.11, there exists a nonzero vector $\alpha \in [0,1]^d$ such that $M_\sigma^\top \alpha$ has nonnegative entries. According to Prop. 1.13, there exist a sequence $(\mathcal{E}_k)_{k\in\mathbb{N}}$ of weighted sums, a sequence of flips $(\mathcal{F}_{\mathbf{x}_k})_{k\in\mathbb{N}}$ and a sequence of weights $(\varepsilon_k)_{k\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ such that $\mathcal{P}_{\alpha,0} = \mathcal{E}_0$, $\lim_{k\to\infty} d_{\mathfrak{F}}(\mathcal{E}_k, \mathcal{E}) = 0$, and, for any $k \geq 1$, $\mathcal{E}_k = \mathcal{S}_{k-1} + \varepsilon_k \mathcal{F}_{\mathbf{x}_k}$. According to Lem. 2.10, one has $\lim_{k\to\infty} d_{\mathfrak{F}}(E_1^*(\sigma)(\mathcal{E}_k), E_1^*(\sigma)(\mathcal{S})) = 0$ and similarly $\lim_{k\to\infty} d_{\mathfrak{F}}(E_1^*(\sigma)(\mathcal{E}_k), E_1^*(\sigma')(\mathcal{S})) = 0$. We conclude by noticing that for every k, one has $E_1^*(\sigma)(\mathcal{E}_k) = E_1^*(\sigma')(\mathcal{E}_k)(= \mathcal{P}_{M_\sigma^\top \alpha,0} + \sum_{n=0}^k \varepsilon_n \mathcal{F}_{M_\sigma^{-1}\mathbf{x}_n})$. □

We have, similarly as in the case of stepped planes (see Prop. 2.7):

Proposition 2.13. *Let σ is a unimodular morphism whose incidence matrix has non-negative entries. Then, $E_1^*(\sigma)$ maps stepped surfaces onto stepped surfaces:*

$$M_\sigma \geq 0 \;\Rightarrow\; E_1^*(\sigma)(\mathfrak{S}) \subset \mathfrak{S}.$$

Proof. According to Th. 2.11, it suffices to prove that the image by $E_1^*(\sigma)$ of any stepped surface is a geometric sum. Suppose that there exists a stepped surface $\mathcal{S} \in \mathfrak{S}$ such that $E_1^*(\sigma)(\mathcal{S}) \notin \mathfrak{B}$. The image of $\mathcal{S}$ by $E_1^*(\sigma)$ depends only on M_σ according to Prop. 2.12. Since M_σ has non-negative entries, we can assume that σ is a substitution. Thus, the faces of $E_1^*(\sigma)(\mathcal{S})$ have non-negative weights, according to Eq. (2.1). Hence, $E_1^*(\sigma)(\mathcal{S}) \notin \mathfrak{B}$ yields that there exist two distinct faces $(\mathbf{x}_1, i_1^*)$ and $(\mathbf{x}_2, i_2^*)$ of weight 1 in $\mathcal{S}$ and $j \in \{1, \ldots, d\}$ such that:

$$\sigma(j) = p_1 i_1 s_1 = p_2 i_2 s_2 \quad \text{and} \quad M_\sigma^{-1}(\mathbf{x}_1 - \mathbf{f}(p_1)) = M_\sigma^{-1}(\mathbf{x}_2 - \mathbf{f}(p_2)).$$

This yields $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{f}(p_1) - \mathbf{f}(p_2)$. Moreover, $(\mathbf{x}_1, i_1^*) \neq (\mathbf{x}_2, i_2^*)$ yields $p_1 \neq p_2$. We can assume w.l.o.g. that p_2 is a prefix of p_1. This implies the vector $\mathbf{f}(p_1) - \mathbf{f}(p_2) - \mathbf{f}(i_2)$ has non-negative entries. Let us now consider the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ defined by $\mathbf{y}_1 = \mathbf{x}_1 + \sum_{i=1}^d \mathbf{e}_i$ and $\mathbf{y}_2 = \mathbf{x}_2 + \mathbf{e}_2$. Both are vertices of, respectively, the faces $(\mathbf{x}_1, i_1^*)$ and $(\mathbf{x}_2, i_2^*)$, hence of $\mathcal{S}$. They satisfy:

$$\mathbf{y}_1 - \mathbf{y}_2 = (\mathbf{x}_1 + \sum_{i=1}^d \mathbf{e}_i) - (\mathbf{x}_2 + \mathbf{e}_2) = (\underbrace{\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{e}_2}_{\geq \mathbf{0}}) + \sum_{i=1}^d \mathbf{e}_i.$$

Therefore $\mathbf{y}_1$ and $\mathbf{y}_2$ are two vertices of $\mathcal{S}$ such that $\mathbf{y}_1 - \mathbf{y}_2$ has only non-negative entries: this contradicts Prop. 1.7, which ends the proof. □

3. Brun expansions

3.1. **Brun expansion of a real vector.** In this section, we define *Brun expansions* of d-dimensional real vectors. For more details on multidimensional continued fractions and Brun's algorithm, the reader is referred to [5, 19]. We first introduce the so-called *Brun map* ($\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$):

Definition 3.1. The d-dimensional *Brun map* T is defined over $[0,1]^d \backslash \{0\}$ by:

$$T(\alpha_1, \ldots, \alpha_d) = \left(\frac{\alpha_1}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i} - \left\lfloor \frac{1}{\alpha_i} \right\rfloor, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_d}{\alpha_i} \right),$$

where $i = \min\{j \mid \alpha_j = ||\alpha||_\infty\}$. Furthermore, we set $T(\mathbf{0}) = \mathbf{0}$.

It is worth noticing that, in the $d = 1$ case, the map T is nothing but the classic Gauss map. For $d \geq 2$, the Brun map consists in applying the Gauss map on the first largest entry, the other entries being divided by this largest entry.

Recall that the Gauss map is used to define the continued fraction expansion of a real number. Similarly, the Brun map is used to define a d-dimensional extension of continued fraction expansions, called *Brun expansion*:

Definition 3.2. The *Brun expansion* of a real vector $\alpha \in [0,1]^d \backslash \{0\}$ is the sequence $(a_n, i_n)_{n\geq 1}$ with values in $\mathbb{N}^* \times \{1, \ldots, d\}$ defined, while $T^n(\alpha) \neq \mathbf{0}$, by:

$$a_n = \lfloor ||T^n(\alpha)||_\infty^{-1} \rfloor \quad \text{and} \quad i_n = \min\{j \mid \langle T^n(\alpha), \mathbf{e}_j \rangle = ||T^n(\alpha)||_\infty\}.$$

One writes: $\alpha = [(a_1, i_1), (a_2, i_2), \ldots]$. If $T^n(\alpha) \neq \mathbf{0}$, for some n, then the algorithm is said to terminate and the expansion of α is finite.

It is convenient to provide a matrix viewpoint on Brun expansions by considering a projective version of it. Brun's algorithm can be described in projective terms as follows: one subtracts as much as possible the second largest coordinate to the largest one. For more details, see e.g. [4]. In order to simplify the notation, we choose here to normalize our projectivized vectors by setting their first coordinate equal to 1 (see Eq. (3.1) below). For $a \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$, we thus introduce the following $(d+1) \times (d+1)$ matrix:

$$B_{a,i} = \begin{pmatrix} a & & 1 & \\ & I_{i-1} & & \\ 1 & & 0 & \\ & & & I_{d-i} \end{pmatrix},$$

where I_p stands for the $p \times p$ identity matrix and all the unspecified coefficients are equal to zero. Note that $B_{a,i}$ has integer entries and determinant -1, and thus belongs to the linear group $GL(d+1, \mathbb{Z})$.

Consider now $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0,1]^d \backslash \{\mathbf{0}\}$. For $i = \min\{j \mid \alpha_j = ||\alpha||_\infty\}$ and $a = \lfloor \alpha_i^{-1} \rfloor$, an easy computation shows that

$$(1, \alpha) = ||\alpha||_\infty B_{a,i}(1, T(\alpha)), \tag{3.1}$$

where $(1, \mathbf{u}) = (1, u_1, \ldots, u_d)$ for $\mathbf{u} = (u_1, \ldots, u_d)$. In particular, if α has Brun expansion $(a_n, i_n)_{n\geq 1}$, this yields, for every suitable n:

$$(1, \alpha) = \mu_n M_n(1, T^{n+1}(\alpha)), \tag{3.2}$$

where $\mu_n = ||T^0(\alpha)||_\infty \times \ldots \times ||T^n(\alpha)||_\infty$ and $M_n = B_{a_1,i_1} \ldots B_{a_{n+1},i_{n+1}}$.

We rely on this matrix viewpoint to recall the property of *(uniform) weak convergence* of Brun's algorithm:

Theorem 3.3. *Let $\alpha \in [0,1]^d \backslash \{0\}$ with Brun expansion $(a_n, i_n)_{n\geq 1}$. One has:*

$$\forall \varepsilon > 0,\ \exists N \ s.t. \ \left(n \geq N,\ \mathbf{u} \in [0,1]^d\right) \ \Rightarrow \ ||(1, \alpha) - \mu_n M_n(1, \mathbf{u})|| \leq \varepsilon,$$

where $\mu_n = ||T^0(\alpha)||_\infty \times \ldots \times ||T^n(\alpha)||_\infty$ and $M_n = B_{a_1,i_1} \ldots B_{a_{n+1},i_{n+1}}$. One says that the Brun algorithm is weakly convergent*.*

The property of weak convergence allows one to approximate real vectors by rational vectors. Indeed, if $\alpha \in [0,1]^d \backslash \{0\}$ has Brun expansion $(a_n, i_n)_n$, we define its *n-th convergent* as the integer vector:

$$(q_n, \mathbf{p}_n) = M_n(1, \mathbf{0}) \in \mathbb{Z}^{d+1},$$

where $M_n = B_{a_1,i_1} \ldots B_{a_{n+1},i_{n+1}}$. According to Th. 3.3, one has:

$$\lim_{n\to\infty} \mu_n(q_n, \mathbf{p}_n) = (1, \alpha).$$

By dividing by $\mu_n q_n$ the d last entries, this yields:

$$\lim_{n\to\infty} \frac{\mathbf{p}_n}{q_n} = \alpha.$$

In other words, the Brun expansion of a vector α produces a sequence $(\frac{\mathbf{p}_n}{q_n})_{n\geq 0}$ of rational vectors which tends towards α.

3.2. **Brun morphisms.** Let us see now how to extend the Brun map to stepped planes. One first idea consists in defining the Brun expansion of a stepped plane as the Brun expansion of its normal vector. This yields identical Brun expansions for stepped planes which have colinear normal vectors. According to our convention concerning the projectivization of Brun's algorithm, we restrict ourselves to the following set of stepped planes (see Prop. 3.7 below for the notation $\mathfrak{P}_\exists$):

$$\mathfrak{P}_\exists = \{\mathcal{P}_{(1,\alpha),\rho} \mid \alpha \in [0,1]^d,\ \rho \in \mathbb{R}\}.$$

The Brun expansion of $\mathcal{P}_{(1,\alpha),\rho} \in \mathfrak{P}_\exists$ is then simply defined as the Brun expansion of $\alpha \in [0,1]^d$. Fig. 3.1 and 3.2 illustrate the $d = 3$ case.

However, this way of defining Brun expansions of stepped planes is disapointing: we just rely on normal vectors, with stepped planes only representing these vectors. Let us now use results of Sec. 2 in order to act directly over stepped planes. The idea is to use Eq. (3.1) and Prop. 2.5 by introducing morphisms whose incidence matrices are the $B_{a,i}$'s.

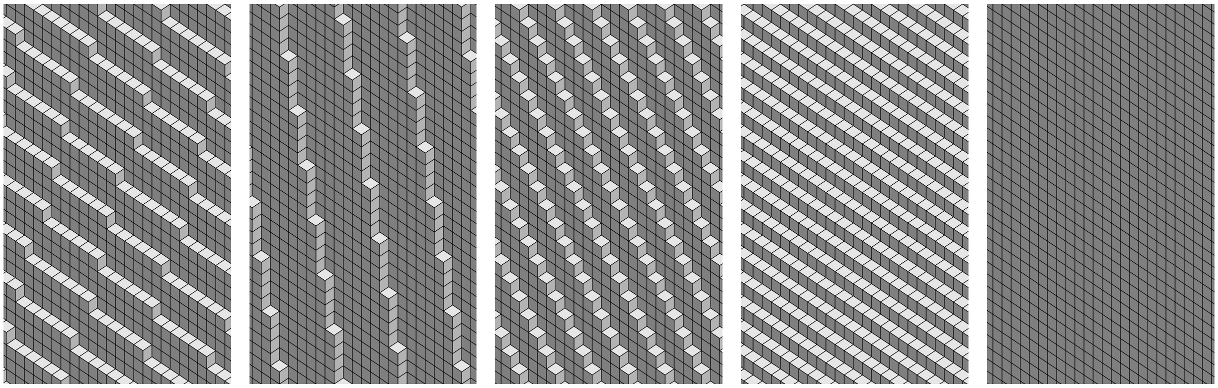

FIGURE 3.1. Stepped planes $\mathcal{P}_{(1,T^n(\alpha)),\rho}$ for $0 \leq n \leq 4$, where $\alpha = (\frac{1}{19}, \frac{25}{76})$ has the finite Brun expansion $[(3,2),(6,1),(4,1),(1,2)]$.

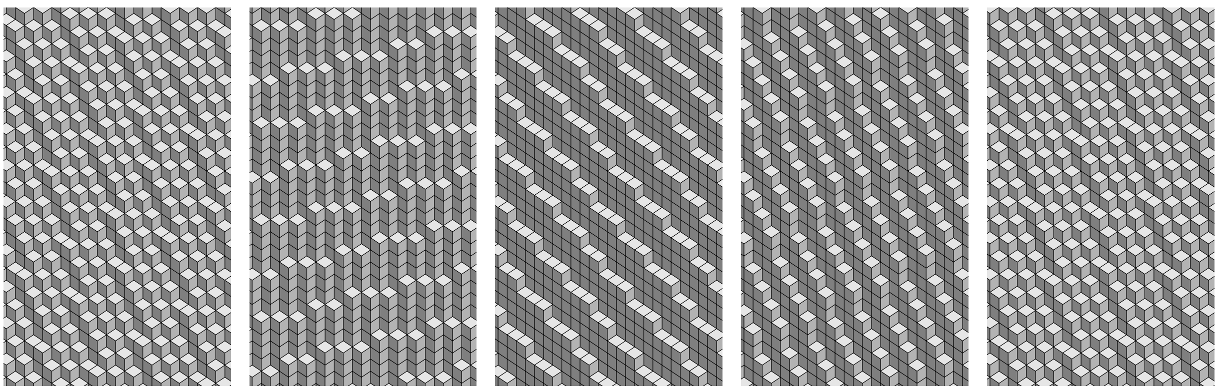

FIGURE 3.2. Stepped planes $\mathcal{P}_{(1,T^n(\alpha)),\rho}$ for $0 \leq n \leq 4$, where $\alpha = (\frac{\sqrt{2}}{2}, \frac{\pi}{4})$ has an infinite Brun expansion $[(1,2),(1,1),(3,2),(2,1),\ldots]$.

Definition 3.4 (Brun morphism)**.** Let $\beta_{a,i}$ be the morphism of free group over $\{1,\ldots,d+1\}$ defined, for $a \in \mathbb{N}$ and $i \in \{1,\ldots,d\}$, by:

$$\beta_{a,i} \ : \ \begin{cases} 1 & \mapsto & 1^a(i+1) \\ (i+1) & \mapsto & 1 \\ j+1 & \mapsto & j+1 \ \text{ for } j \neq i \text{ and } j \in \{1,\cdots,d\}. \end{cases}$$

It is easily checked that $\beta_{a,i}$ has incidence matrix $B_{a,i}$. In particular, $\beta_{a,i}$ is unimodular since $B_{a,i} \in GL(d+1,\mathbb{Z})$. Note that $\beta_{a,i}$ is a substitution, that is, a positive and non-erasing morphism. Let us note furthermore that the $B_{a,i}$'s are symmetric matrices. The substitution $\beta_{a,i}$ is an automorphism:

$$\beta_{a,i}^{-1} \ : \ \begin{cases} 1 & \mapsto & (i+1) \\ (i+1) & \mapsto & (i+1)^{-a}1 \\ j+1 & \mapsto & j+1 \end{cases} \quad \text{and} \quad M_{\beta_{a,i}^{-1}} = B_{a,i}^{-1} = \begin{pmatrix} 0 & & 1 & \\ & I_{i-1} & & \\ 1 & & -a & \\ & & & I_{d-i} \end{pmatrix}.$$

According to Eq. (2.1), one computes (see also Fig. 3.3):

$$E_1^*(\beta_{a,i}) \;:\; \begin{cases} (\mathbf{0},1^*) & \mapsto \quad (\mathbf{0},(i+1)^*) + \sum_{k=0}^{a-1}(-k\mathbf{e}_{i+1},1^*) \\ (\mathbf{0},(i+1)^*) & \mapsto \quad (-a\mathbf{e}_{i+1},1^*) \\ (\mathbf{0},(j+1)^*) & \mapsto \quad (\mathbf{0},(j+1)^*) \text{ for } j \neq i,\ 1 \leq j \leq d \end{cases}$$

$$E_1^*(\beta_{a,i}^{-1}) \;:\; \begin{cases} (\mathbf{0},1^*) & \mapsto \quad (a\mathbf{e}_1,(i+1)^*) \\ (\mathbf{0},(i+1)^*) & \mapsto \quad (\mathbf{0},1^*) - \sum_{k=1}^{a}(k\mathbf{e}_1,(i+1)^*) \\ (\mathbf{0},(j+1)^*) & \mapsto \quad (\mathbf{0},(j+1)^*) \text{ for } j \neq i,\ 1 \leq j \leq d. \end{cases}$$

Then, Eq. (3.1) and Prop. 2.5 yield:

Proposition 3.5. *For any* $\alpha \in [0,1]^d \backslash \{\mathbf{0}\}$ *and* $\rho \in \mathbb{R}$,

$$\mathcal{P}_{(1,\alpha),\rho} = E_1^*(\beta_{a,i})(\mathcal{P}_{||\alpha||_\infty(1,T(\alpha)),\rho}),$$

or, equivalently (according to Th. 2.3):

$$E_1^*(\beta_{a,i}^{-1})(\mathcal{P}_{(1,\alpha),\rho}) = \mathcal{P}_{||\alpha||_\infty(1,T(\alpha)),\rho},$$

where $i = \min\{j \mid \alpha_j = ||\alpha||_\infty\}$ *and* $a = \lfloor \alpha_i^{-1} \rfloor$.

Therefore, we can relate the action of the Brun map T on a vector α to the action of a dual map $E_1^*(\beta_{a,i}^{-1})$ on the stepped plane $\mathcal{P}_{(1,\alpha),\rho}$. Note that there is a simple "geometric" way to recognize whether a stepped plane (and even a stepped surface) is the image of another stepped plane (resp. a stepped surface) under the action of a substitution $\beta_{a,i}$: the condition given below means that each occurrence of a face of type $(i+1)^*$ is included in the image of a face of type 1^* by $E_1^*(\beta_{a,i})$.

Lemma 3.6. *Let us fix* $a \in \mathbb{N}^*$ *and* $i \in \{1, \cdots, d\}$. *Let* $\mathcal{S}$ *be a stepped surface. The weighted sum* $E_1^*(\beta_{a,i}^{-1})(\mathcal{S})$ *is a stepped surface if and only for all face* $(\mathbf{x},(i+1)^*)$ *such that* $\mathcal{S}(\mathbf{x},(i+1)^*) = 1$, *then* $\mathcal{S}(\mathbf{x} - k\mathbf{e}_{i+1}, 1^*) = 1$ *for* $0 \leq k \leq a-1$.

Proof. Let us assume that $E_1^*(\beta_{a,i}^{-1})(\mathcal{S})$ is a stepped surface, say $\mathcal{S}'$. One has $\mathcal{S} = E_1^*(\beta_{a,i})(\mathcal{S}')$, according to Theorem 2.3. Let $(\mathbf{x},(i+1)^*)$ such that $\mathcal{S}(\mathbf{x},(i+1)^*) = 1$. There exists $\mathbf{y} \in \mathbb{Z}^{d+1}$ such that $(\mathbf{x},(i+1)^*)$ occurs in the image of $(\mathbf{y},1^*)$ under $E_1^*(\beta_{a,i})$. According to the expression of $E_1^*(\beta_{a,i})$, and since there are no cancellations of faces, then all the faces that occur in $E_1^*(\beta_{a,i})(\mathbf{y},1^*)$ have weight 1 in $\mathcal{S}$.

Conversely, we assume that for all face $(\mathbf{x},(i+1)^*)$ such that $\mathcal{S}(\mathbf{x},(i+1)^*) = 1$, then $\mathcal{S}(\mathbf{x} - k\mathbf{e}_{i+1}, 1^*) = 1$ for $0 \leq k \leq a-1$. We now consider the following partition of the set of faces that have weight 1 in $\mathcal{S}$. Let

$$I_1 := \{(\mathbf{z},(j+1)^*) \mid \mathcal{S}(\mathbf{z},(j+1)^*) = 1, \text{ for } j \neq i+1\},$$

$$I_2 := \{(\mathbf{z},(i+1)^*),\ (\mathbf{z} - k\mathbf{e}_{i+1},1^*), \text{ for } 0 \leq k \leq a-1 \mid \mathcal{S}(\mathbf{z},(i+1)^*) = 1\},$$

and

$$I_3 := \{(\mathbf{z},1^*) \mid \mathcal{S}(\mathbf{z},1^*) = 1 \text{ and } (\mathbf{z},1^*) \notin I_2\}.$$

One has

$$E_1(\beta_{a,i}^{-1})^*(\mathcal{S}) = \sum_{(\mathbf{z},(j+1)^*)\in I_1} E_1(\beta_{a,i}^{-1})^*((\mathbf{z},(j+1)^*)+$$

$$\sum_{(\mathbf{z},(i+1)^*)\in I_2} E_1(\beta_{a,i}^{-1})^*[(\mathbf{z},(i+1)^*) + \sum_{0\leq k\leq a-1} (\mathbf{z} - k\mathbf{e}_{i+1},1^*)] + \sum_{(\mathbf{z},1^*)\in I_3} E_1(\beta_{a,i}^{-1})^*(\mathbf{z},1^*).$$

One checks that the images of the faces of each of the three types are faces respectively of type $(j+1)^*$, 1^*, and $(i+1)^*$, which implies that $E_1^*(\beta_{a,i}^{-1})(\mathcal{S})$ is a geometric sum. We now apply Th. 2.11 to conclude, by noticing that the inverses of Brun morphisms satisfy the conditions of the theorem. □

It is also worth noticing that every stepped plane in $\mathfrak{P}_\exists$ can be desubstituted by at least one substitution $\beta_{a,i}$ (hence the notation), whereas a stepped plane in $\mathfrak{P}_\exists$ that can be desubstituted with respect to all substitutions $\beta_{a,i}$ has a normal vector of the form $(1, \mathbf{0})$:

Proposition 3.7. *One has*

$$\mathfrak{P}_\exists = \{\mathcal{P}_{(1,\alpha),\rho} \mid \rho \in \mathbb{R},\ \alpha \in [0,1]^d\} \subset \bigcup_{\substack{a\in\mathbb{N}^* \\ 1\le i\le d}} E_1^*(\beta_{a,i})(\mathfrak{P}_\exists)$$

$$\{\mathcal{P}_{(1,\mathbf{0}),\rho} \mid \rho \in \mathbb{R}\} = \bigcap_{\substack{a\in\mathbb{N}^* \\ 1\le i\le d}} E_1^*(\beta_{a,i})(\mathfrak{P}_\exists).$$

Proof. The first inclusion is a direct consequence of Prop. 3.5. Indeed, consider a stepped plane $\mathcal{P}_{(1,\alpha),\rho} \in \mathfrak{P}_\exists$. If $\alpha = \mathbf{0}$, one uses the fact that for any a, i, then $E_1^*(\beta_{a,i})(\mathcal{P}_{(1,0,\cdots,0,1/a,0,\cdots,0),\rho}) = \mathcal{P}_{(1,\mathbf{0}),\rho}$. If $\alpha \neq \mathbf{0}$, set $i = \min\{j \mid \alpha_j = ||\alpha||_\infty\}$ and $a = \lfloor \alpha_i^{-1} \rfloor$.

We prove now the second equality. If a weighted sum belongs to the intersection $\bigcap_{\substack{a\in\mathbb{N}^* \\ 1\le i\le d}} E_1^*(\beta_{a,i})(\mathfrak{P}_\exists)$, then it is a stepped plane with normal vector, say $\mathbf{u}$, by applying Prop. 2.7. We deduce from Prop. 2.5 that for all $a \in \mathbb{N}^*$ and for all i, one has $B_{a,i}^{-1}\mathbf{u} \geq 0$, which implies $\mathbf{u} = (1, \mathbf{0})$. □

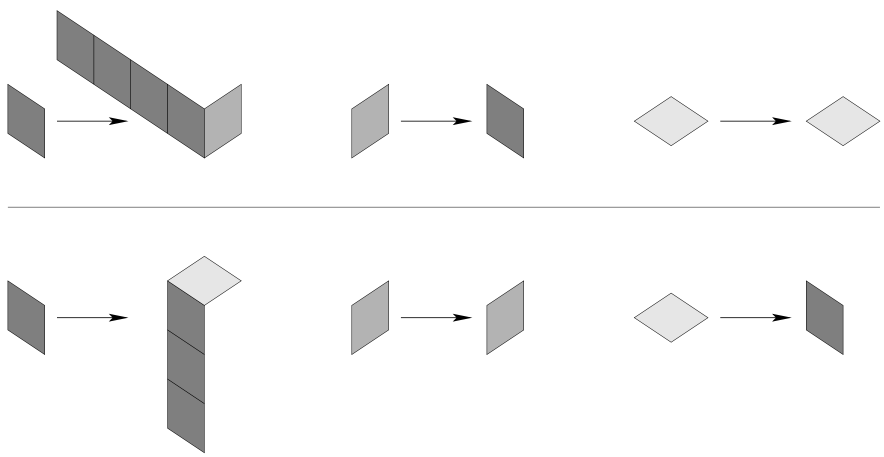

FIGURE 3.3. Action of the dual maps $E_1^*(\beta_{4,1})$ (top) and $E_1^*(\beta_{3,2})$ (bottom) over the unit faces $(\mathbf{0}, 1^*)$, $(\mathbf{0}, 2^*)$ and $(\mathbf{0}, 3^*)$ (from left to right).

3.3. **Brun expansion of a stepped plane.** We need a priori to know the normal vector of a given stepped plane for choosing the dual map $E_1^*(\beta_{a,i}^{-1})$ which shall be applied onto. Let us show that convenient a and i can be directly obtained from the geometry of the stepped plane in the flavour of Lem. 3.6, that is, roughly speaking, by "reading" them on faces. In other words, we will be able first, to order the coordinates α_i of a normal vector α, and thus, to deduce the smallest index i such that $\alpha_i = ||\alpha||_\infty$; this is the object of Lem. 3.8; secondly, we will be able to determine the partial quotient $\lfloor \frac{1}{\alpha_i} \rfloor$; this is the object of Lem. 3.11.

Lemma 3.8. *Let $\mathcal{P}_{(1,\alpha),\rho}$ be a stepped plane, with $\alpha = (\alpha_1, \cdots, \alpha_d) \in [0,1]^d$ and $\rho \in \mathbb{R}$. There exists $\mathbf{x} \in \mathbb{Z}^{d+1}$ such that the faces $(\mathbf{x}, (i+1)^*)$ and $(\mathbf{x} - \mathbf{e}_{j+1}, (i+1)^*)$, with $1 \leq i, j \leq d$, both belong to $\mathcal{P}_{(1,\alpha),\rho}$ if and only if $\alpha_i > \alpha_j$.*

Proof. We first assume that there exists $\mathbf{x} \in \mathbb{Z}^{d+1}$ such that the faces $(\mathbf{x}, (i+1)^*)$ and $(\mathbf{x}-\mathbf{e}_{j+1}, (i+1)^*)$ both belong to $\mathcal{P}_{(1,\alpha),\rho}$. One has

$$\langle \mathbf{x}, (1,\alpha)\rangle < \rho \le \langle \mathbf{x}+\mathbf{e}_{i+1}, (1,\alpha)\rangle$$

and

$$\langle \mathbf{x}-\mathbf{e}_{j+1}, (1,\alpha)\rangle < \rho \le \langle \mathbf{x}-\mathbf{e}_{j+1}+\mathbf{e}_{i+1}, (1,\alpha)\rangle.$$

We thus deduce that

$$\langle \mathbf{x}, (1,\alpha)\rangle < \langle \mathbf{x}-\mathbf{e}_{j+1}+\mathbf{e}_{i+1}, (1,\alpha)\rangle, \text{ i.e., } \langle \mathbf{e}_{j+1}, (1,\alpha)\rangle < \langle \mathbf{e}_{i+1}, (1,\alpha)\rangle,$$

which gives $\alpha_i > \alpha_j$.

Conversely, let assume that $\alpha_i > \alpha_j$. It is sufficient to prove that there exists $\mathbf{x} \in \mathbb{Z}^{d+1}$ such that

$$\langle \mathbf{x}, (1,\alpha)\rangle < \rho \le \langle \mathbf{x}-\mathbf{e}_{j+1}+\mathbf{e}_{i+1}, (1,\alpha)\rangle = \langle \mathbf{x}, (1,\alpha)\rangle + \alpha_i - \alpha_j.$$

If α has at least one irrational entry, the existence of such a vector $\mathbf{x} \in \mathbb{Z}^{d+1}$ follows by density of $(\langle \mathbf{y}, (1,\alpha)\rangle)_{\mathbf{y}\in\mathbb{Z}^{d+1}}$. If α has only rational entries, then we can assume by multiplying ρ by a suitable rational number that $(1,\alpha)$ has coprime non-negative integer entries. The existence of $\mathbf{x}$ follows now by applying Bezout's lemma. □

Remark 3.9. In other words, one has $\alpha_i \ge \alpha_j$, for $1 \le i,j \le d$, if and only if for all $\mathbf{x} \in \mathbb{Z}^{d+1}$ such that the face $(\mathbf{x}, (j+1)^*)$ belongs to $\mathcal{P}_{(1,\alpha),\rho}$, then $(\mathbf{x}-\mathbf{e}_{i+1}, (j+1)^*)$ does not belong to $\mathcal{P}_{(1,\alpha),\rho}$. Hence, $||\alpha||_\infty$ is defined as the coordinate of index given by the smallest $i \in \{1,\cdots,d\}$ that satisfies the following: for every $j \in \{1,\cdots,d\}$ and $j \ne i$, for every face of type $(\mathbf{x}, (j+1)^*)$ that belongs to $\mathcal{P}$, then $(\mathbf{x}-\mathbf{e}_{i+1}, (j+1)^*)$ does not belong to $\mathcal{P}$.

□

Definition 3.10. Let $\mathcal{P} = \mathcal{P}_{(1,\alpha),\rho}$ be a stepped plane. Let $\alpha \in [0,1]^d$ and let $i = ||\alpha||_\infty$.We set

$$a(\mathcal{P}) := \max\{a \in \mathbb{N}^* \mid \forall\, (\mathbf{x}, (i+1)^*) \text{ such that } \mathcal{P}(\mathbf{x}, (i+1)^*) = 1,$$
$$\text{then } \mathcal{P}(\mathbf{x}-k\mathbf{e}_{i+1}, 1^*) = 1 \text{ for } 0 \le k \le a-1\}.$$

According to Lem. 3.6 and Prop. 3.7, the parameter $a(\mathcal{P})$ takes finite values if and only if $\alpha \ne \mathbf{0}$.

Let us prove that the parameter $a(\mathcal{P})$ provides some information about the normal vector of the stepped plane $\mathcal{P}$:

Lemma 3.11. *Let $\mathcal{P} = \mathcal{P}_{(1,\alpha),\rho} \in \mathfrak{P}_\exists$, with $\alpha \ne \mathbf{0}$. Then, one has:*

$$a(\mathcal{P}_{(1,\alpha),\rho}) = \left\lfloor \frac{1}{\alpha_i} \right\rfloor, \text{ where } \alpha_i = ||\alpha||_\infty.$$

Proof. One has:

$$B_{a,i}^{-1}(1,\alpha) = (1, \alpha_1, \ldots, \alpha_{i-1}, 1 - a\alpha_i, \alpha_{i+1}, \ldots, \alpha_d).$$

According to Prop. 2.5 , $E_1^*(\beta_{a,i}^{-1})(\mathcal{P})$ is a stepped plane if and only if $B_{a,i}^{-1}(1,\alpha) \in \mathbb{R}_+^{d+1}$, i.e., $a \le \frac{1}{\alpha_i}$. The result follows by noticing that $\max\{a \in \mathbb{N} \mid \ a \le \frac{1}{\alpha_i}\} = \lfloor \alpha_i^{-1} \rfloor$, and from Lem. 3.6. □

We now are able to define a transformation $\tilde{T}_{\mathfrak{P}}$ acting directly on the set $\mathfrak{P}_\exists$:

Definition 3.12. Let $\tilde{T}_{\mathfrak{P}} : \mathfrak{P}_\exists \to \mathfrak{P}_\exists$ be the map defined for $\mathcal{P} = \mathcal{P}_{(1,\alpha),\rho} \in \mathfrak{P}_\exists$ by

$$\tilde{T}_{\mathfrak{P}}(\mathcal{P}) = E_1^*(\beta_{a(\mathcal{P}),i}^{-1})(\mathcal{P}), \text{ if } \alpha \ne \mathbf{0},$$

where $i \in \{1,\cdots,d\}$ is the smallest index that satisfies: for every $j \ne i$ and $j \in \{1,\cdots,d\}$, for every face of type $(\mathbf{x}, (j+1)^*)$ that belongs to $\mathcal{P}$, then $(\mathbf{x}-\mathbf{e}_{i+1}, (j+1)^*)$ does not belong to $\mathcal{P}$, and

$$\tilde{T}_{\mathfrak{P}}(\mathcal{P}_{(1,\mathbf{0}),\rho)}) = \mathcal{P}(1,\mathbf{0}),\rho.$$

We deduce from Prop. 3.5 that the map $\tilde{T}$ is related to the Brun map T by the following equality:

$$\forall \alpha \in [0,1]^d \backslash \{0\},\ \forall \rho \in \mathbb{R}, \quad \tilde{T}(\mathcal{P}_{(1,\alpha),\rho}) = \mathcal{P}_{||\alpha||_\infty (1,T(\alpha)),\rho}.$$

In short, we have defined a map $\tilde{T}_{\mathfrak{P}}$ which acts over a stepped plane $\mathcal{P}_{(1,\alpha),\rho}$ as the Brun map T acts on the vector $\alpha \in [0,1]^d \backslash \{\mathbf{0}\}$. Morever, given a stepped plane $\mathcal{P}_{(1,\alpha),\rho)}$, with $\alpha \in [0,1]^d$, this provides a geometric process for computing $\tilde{T}_{\mathfrak{P}}(\mathcal{P})$, without assuming that the normal vector of $\mathcal{P}$ is known. The next section shows how to extend the definition of $\tilde{T}_{\mathfrak{P}}$ to stepped surfaces.

3.4. **Brun expansion of a stepped surface.** We now can parallel what has been previously made on stepped planes for defining the Brun expansion of a stepped surface. According to our choice of projectivisation for Brun algorithm, we restrict ourselves to the following set of stepped surfaces:

Definition 3.13. We define $\mathfrak{S}_\exists$ as the set of stepped surfaces that satisfy:

(1) for every $j \in \{1, \cdots, d\}$, for every face of type $(\mathbf{x}, (j+1)^*)$ that belongs to $\mathcal{S}$, then $(\mathbf{x} - \mathbf{e}_1, (j+1)^*)$ does not belong to $\mathcal{S}$;
(2) there exists $i \in \{1, \cdots, d\}$ such that for all $j \in \{1, \cdots, d\}$, then for every face of type $(\mathbf{x}, (j+1)^*)$ that belongs to $\mathcal{S}$, the face $(\mathbf{x} - \mathbf{e}_{i+1}, (j+1)^*)$ does not belong to $\mathcal{S}$;
(3) let i be the smallest integer that satisfies (2); there exist $a \in \mathbb{N}^*$ such that for all face $(\mathbf{x}, (i+1)^*)$ such that $\mathcal{S}(\mathbf{x}, (i+1)^*) = 1$, then $\mathcal{S}(\mathbf{x} - k\mathbf{e}_{i+1}, 1^*) = 1$ for $0 \le k \le a-1$,

A stepped surface that belongs to $\mathfrak{S}_\exists$ is called *Brun expandable.*

The first condition comes from the fact that we have considered so far planes whose normal vector admits as largest coordinate the first one (see Rem. 3.9). The second condition is a necessary condition for a stepped surface to be a stepped plane: this condition means that $\alpha_i \ge \alpha_j$, once again according to Rem. 3.9. The third condition implies that $E_1^*(\beta_{a,i}^{-1})(\mathcal{S})$ is a stepped surface, according to Lem. 3.6. Note that one has

$$\mathfrak{S}_\exists \cap \mathfrak{P} = \mathfrak{P}_\exists.$$

Examples of stepped surfaces in $\mathfrak{S}_\exists$ are depicted in Fig. 4.1 and 4.2.

We can define a coefficient $a(\mathcal{S})$ analogous to the coefficient introduced in Def. 3.10. Note that a statement analog to Lem. 3.11 is no longer valid for stepped surfaces.

Definition 3.14. Let $\mathcal{S}$ be a Brun expandable stepped surface. Let I be the set of indices i in $\{1, \cdots, d\}$ that satisfy Condition (3) in Def. 3.13. We define $i(\mathcal{S})$ as the smallest element in I. We set

$$a(\mathcal{S}) := \max\{a \in \mathbb{N}^* \mid \forall\, (\mathbf{x}, (i(\mathcal{S})+1)^*) \text{ such that } \mathcal{S}(\mathbf{x}, (i(\mathcal{S})+1)^*) = 1,$$
$$\text{then } \mathcal{S}(\mathbf{x} - k\mathbf{e}_{i(\mathcal{S})+1}, 1^*) = 1, \text{ for } 0 \le k \le a-1\}.$$

Let us note that if $\mathcal{S}$ is a stepped plane in $\mathfrak{P}_\exists$, then the coefficient $a(\mathcal{S})$ coincides with the coefficient introduced in Def. 3.10. According to Lem. 3.6, if a stepped surface $\mathcal{S}$ is Brun expandable and $a(\mathcal{S}) < \infty$, then $E_1^*(\beta^{-1}_{a(\mathcal{S}),i(\mathcal{S})})(\mathcal{S})$ is itself a stepped surface.

Definition 3.15. Let $\tilde{T}_{\mathfrak{S}}\colon \mathfrak{S}_\exists \to \mathfrak{S}_\exists$ be the map defined for $\mathcal{S} \in \mathfrak{S}_\exists$ by:

$$\tilde{T}_{\mathfrak{S}}(\mathcal{S}) = E_1^*(\beta^{-1}_{a(\mathcal{S}),i(\mathcal{S})})(\mathcal{S}),$$

and

$$\tilde{T}_{\mathfrak{S}}(\mathcal{S}) = \mathcal{S}, \text{ if } a(\mathcal{S}) = +\infty.$$

One checks that the restriction of $\tilde{T}_{\mathfrak{S}}$ to $\mathfrak{P}$ coincides with the map $\tilde{T}_{\mathfrak{P}}$.

This allows to naturally extend the definition of Brun expansions from stepped planes to stepped surfaces:

Definition 3.16. The *Brun expansion* of a Brun expandable stepped surface $\mathcal{S} \in \mathfrak{S}_\exists$ is defined as the sequence $(a_n, i_n)_{n\geq 1}$ with values in $\mathbb{N}^* \times \{1, \ldots, d\}$ defined, while $\tilde{T}^n(\mathcal{S}) \in \mathfrak{S}_\exists$, by

- $i_n = i(\tilde{T}^{n-1}(\mathcal{S})$
- $a_n = a(\tilde{T}^{n-1}(\mathcal{S}))$.

4. Brun expandable stepped surfaces

Note that if a stepped plane has the same Brun expansion as the vector $\alpha \in [0,1]^d$, then it is a stepped plane of the form $\mathcal{P}_{(1,\alpha),\rho}$ for some $\rho \in \mathbb{R}$: indeed α is characterized by its Brun expansion. More generally, a natural question is to characterize the stepped surfaces that have the same Brun expansion as a given vector $\alpha \in [0,1]^d$. Note that there are many stepped surfaces associated with a given Brun expansion: consider e.g. the stepped planes having the same normal vector.

4.1. Finite Brun expansions.

Let us first consider the case of finite Brun expansions.

Definition 4.1. A stepped surface $\mathcal{S}$ is said to have a finite Brun expansion of length N if

- either $\tilde{T}^N_{\mathfrak{S}}(\mathcal{S})$ is not Brun expandable
- or $a(\tilde{T}^N_{\mathfrak{S}}(\mathcal{S})) = +\infty$.

Fig. 4.1 and 4.2 provide examples of stepped surfaces with the same finite Brun expansion as the stepped plane of Fig. 3.1.

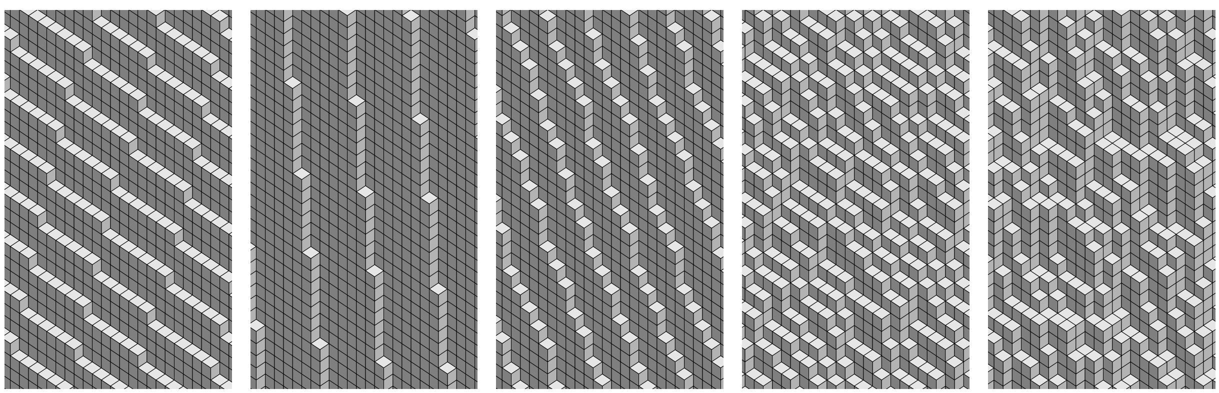

FIGURE 4.1. Stepped surfaces $\tilde{T}^n_{\mathfrak{S}}(\mathcal{S})$ for $0 \leq n \leq 4$ (from left to right), where $\mathcal{S}$ has the same Brun expansion as the stepped plane of Fig. 3.1. The rightmost stepped surface is not Brun expandable: none of its images by a dual map $E_1^*(\beta_{a,i}^{-1})$ is a stepped surface.

It is worth noticing that, in the case of finite Brun expansions, one has the following result:

Theorem 4.2. *Let $\mathcal{S}$ be a Brun expandable stepped surface with a finite Brun expansion of positive length N. Then, $\mathcal{S}$ is a stepped plane if and only if $\tilde{T}^N_{\mathfrak{S}}(\mathcal{S})$ is a stepped plane of normal vector $(1, \mathbf{0})$.*

Proof. We first assume that $\mathcal{S}$ is a stepped plane in $\mathfrak{P}_\exists$. One has $\tilde{T}^N_{\mathfrak{S}}(\mathcal{S}) \in \mathfrak{P}_\exists$, and thus $a(\tilde{T}^{N+1}_{\mathfrak{S}}(\mathcal{S})) = +\infty$ since $\mathcal{S}$ has finite expansion of length N. This implies that $\mathcal{S}$ has normal vector $(1, \mathbf{0})$. Conversely, note that:

$$\mathcal{S} = E_1^*(\beta_{a_1,i_1}) \circ \ldots \circ E_1^*(\beta_{a_N,i_N})(\tilde{T}^N_{\mathfrak{S}}(\mathcal{S})).$$

If $\tilde{T}^{N+1}_{\mathfrak{S}}(\mathcal{S})$ is a stepped plane of normal vector $(1, \mathbf{0})$, then $\mathcal{S} \in \mathfrak{P}_\exists$, according to Prop. 2.5. $\square$

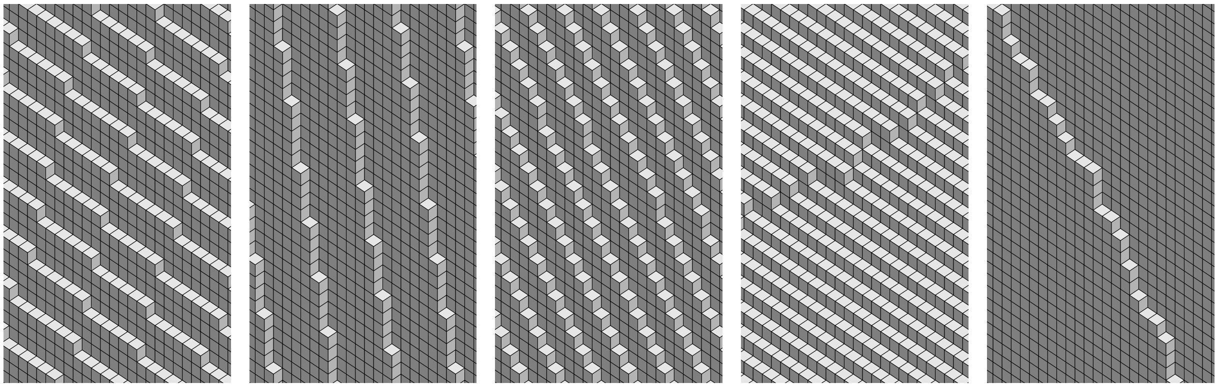

FIGURE 4.2. Stepped surfaces $\tilde{T}^n_{\mathfrak{G}}(\mathcal{S}$ for $0 \leq n \leq 4$ (from left to right), where $\mathcal{S}$ has the same Brun expansion as the stepped plane of Fig. 3.1. The image by any dual map $E_1^*(\beta_{a,i}^{-1})$ of the stepped surface $\tilde{T}^4_{\mathfrak{G}}(\mathcal{S})$ is a stepped surface.

This yields an algorithm to check wether a given Brun expandable stepped surface $\mathcal{S}$ with a finite Brun expansion is a stepped plane. Indeed, it suffices to compute the sequence $\tilde{T}^n_{\mathfrak{G}}(\mathcal{S})$ while it is possible. If it stops at step N, then the previous theorem yields that $\mathcal{S}$ is a stepped plane if and only if $\tilde{T}^N_{\mathfrak{G}}(\mathcal{S})$ is a stepped plane of normal vector $(1, \mathbf{0})$, which is easily recognizable. Moreover, we also obtain the normal vector of $\mathcal{S}$: according to Prop. 2.5, it is $M_N(1, \mathbf{0})$, where $M_N = B_{a_1,i_1} \ldots B_{a_N,i_N}$ and $(a_n, i_n)_{1 \leq n \leq N}$ is the Brun expansion associated with the sequence $\tilde{T}^n_{\mathfrak{G}}(\mathcal{S})$. For a recognition algorithm based on this idea, see [11].

4.2. **Infinite Brun expansions.** Let us now turn back to the case of infinite Brun expansions. According to the definition of $\tilde{T}_{\mathfrak{G}}$, a stepped surface $\mathcal{S}$ has an infinite Brun expansion if and only if, for any $n \geq 0$, $\tilde{T}^n_{\mathfrak{G}}(\mathcal{S})$ is Brun expandable and $a(\tilde{T}^n_{\mathfrak{G}}(\mathcal{S}))$ is finite. Here again, it is not clear to characterize the stepped surfaces have a given infinite Brun expansion. According to Sec. 3.3, there are at least the stepped planes whose normal vector have this infinite Brun expansion. Fig. 4.3 illustrates the fact that there might also exist other stepped surfaces. Note that the provided example of stepped surface is intuitively very close to a stepped plane (but not for the distance $d_{\mathfrak{F}}$), and that the infinite Brun expansion of this example is associated with a real vector whose entries are not linearly independent over $\mathbb{Q}$. The aim of the present section is to characterize Brun expandable stepped surfaces that have the same Brun expansion as a totally irrational vector. We recall that a vector $\alpha = (\alpha_1, \cdots, \alpha_d)$ is said to be totally irrational if $\dim_{\mathbb{Q}}(1, \alpha_1, \cdots, \alpha_d) = d + 1$.

Theorem 4.3. *If a stepped surface has the same Brun expansion as a totally irrational vector α, then it is a stepped plane of normal vector $(1, \alpha)$.*

Theorem 4.3 is a direct consequence of the following lemma. Indeed, if α is totally irrational, then $\langle \mathbf{y}, (1, \alpha) \rangle = \rho$ implies that $\mathbf{y} = \mathbf{0}$.

Lemma 4.4. *Let $\mathcal{S}$ be a Brun expandable stepped surface that has the same infinite Brun expansion as $\alpha \in [0,1]^d$. Then, there exist $\rho \in \mathbb{R}$ and a sequence $(\varepsilon_{\mathbf{y}})_{\mathbf{y} \in \mathbb{Z}^{d+1}}$ with values in $\{0, 1\}$ such that:*

$$\mathcal{S} = \mathcal{P}_{(1,\alpha),\rho} + \sum_{\mathbf{y} \in Y_\rho} \varepsilon_{\mathbf{y}} \mathcal{F}_{\mathbf{y}}.$$

where

$$Y_\rho := \{\mathbf{y} \in \mathbb{Z}^{d+1} | \, \langle \mathbf{y}, (1, \alpha) \rangle = \rho\}.$$

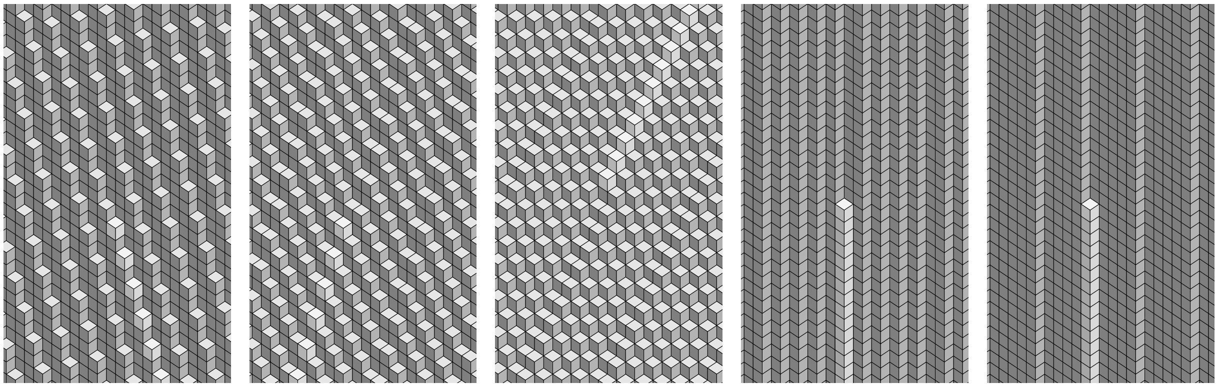

FIGURE 4.3. Stepped surfaces $(\tilde{T}^n(\mathcal{S}))_{0\leq n\leq 4}$ (from left to right), where $\mathcal{S}$ has the same (infinite) Brun expansion as $\alpha = (\sqrt{2}-1, \frac{3}{2}-\sqrt{2})$. Thus, $\mathcal{S}$ is an example of a stepped surface which has the same infinite Brun expansion as a stepped plane of normal vector $(1,\alpha)$, although both are not equal. More precisely, the faces where $\tilde{T}^n(\mathcal{S})$ and $\tilde{T}^n(\mathcal{P}_{(1,\alpha),0})$ are not equal are the brighter faces (which form sort of half-lines).

We will use in the proof of Lem. 4.4 the following following expanding property for the product of matrices $M_n = B_{a_1,i_1}\dots B_{a_{n+1},i_{n+1}}$ that will be deduced from the weak convergence of Brun algorithm. We first give a proof of Lem. 4.4 and then a proof of Lem. 4.5.

Lemma 4.5. *Let $\alpha \in [0,1]^d$ having an infinite Brun expansion $(a_n, i_n)_n$. Then, one has:*

$$\forall r \geq 0,\ \exists k_r \in \mathbb{N} \text{ s.t. } (n \geq k_r,\ \mathbf{x} \in C \text{ and } \langle M_n^{-1}\mathbf{x}, (1,\alpha)\rangle \neq 0) \ \Rightarrow\ ||M_n^{-1}\mathbf{x}|| \geq r,$$

where $M_n = B_{a_1,i_1}\dots B_{a_{n+1},i_{n+1}}$, for all n, and C stands for the set of integer vectors whose coordinates are neither all positive nor all negative.

Proof. We first prove Lem. 4.4. Let $\mathcal{S}$ be a Brun expandable stepped surface and $\alpha \in [0,1]^d$ having both the infinite Brun expansion $(a_n, i_n)_n$. There exists $\mathbf{t} \in \mathbb{Z}^d$ such that the translate $\mathcal{S}'$ of the stepped surface $\mathcal{S}$ by the vector $\mathbf{t}$ contains a face of the form $(\mathbf{0}, i^*)$, for some $i \in \{1, \cdots, d+1\}$. More precisely, $\mathcal{S}'$ is given by

$$\mathcal{S}' = \sum_{(\mathbf{x},i^*),\ \mathcal{S}(\mathbf{x},i^*)=1} (\mathbf{x}+\mathbf{t}, i^*).$$

The stepped surfaces $\mathcal{S}'$ and $\mathcal{S}$ have the same Brun expansion, since they are equal up to a translation vector. Let us prove that there exists a sequence $(\varepsilon_{\mathbf{y}})_{\mathbf{y}\in\mathbb{Z}^{d+1}}$ with values in $\{0, \pm 1\}$ such that:

$$\mathcal{S}' = \mathcal{P}_{(1,\alpha),0} + \sum_{\mathbf{y}\in Y_0} \varepsilon_{\mathbf{y}} \mathcal{F}_{\mathbf{y}}.$$

The corresponding result for $\mathcal{S}$ follows by setting $\rho = \langle \mathbf{t}, (1,\alpha)\rangle$.

Let $(\mathbf{x}, i^*)$ be a face that satisfies $\mathcal{S}'(\mathbf{x}, i^*) \neq \mathcal{P}_{(1,\alpha),0}(\mathbf{x}, i^*)$. Let us fix $r > 0$ such that $\mathbf{x} \in B(\mathbf{0}, r)$ and $\mathbf{x} + \mathbf{e}_j \in B(\mathbf{0}, r)$, for all $j \in \{1, \cdots, d+1\}$.

The Brun expansion of $\mathcal{S}'$ yields stepped surfaces $(\mathcal{S}'_k)_{k\geq 0}$ such that $\mathcal{S}'_0 = \mathcal{S}'$ and, for any k, $\mathcal{S}'_{k+1} = E_1^*(\beta_{a_k,i_k}^{-1})(\mathcal{S}'_k)$. Thus:

$$\forall k \geq 1,\ \mathcal{S}' = E_1^*(\beta_{a_1,i_1}) \circ E_1^*(\beta_{a_2,i_2}) \circ \dots \circ E_1^*(\beta_{a_k,i_k})(\mathcal{S}'_k) = E_1^*(\beta_k)(\mathcal{S}'_k),$$

where $\beta_k = \beta_{a_k,i_k} \circ \dots \circ \beta_{a_1,i_1}$ has incidence matrix $M_k = B_{a_k,i_k}\dots B_{a_1,i_1}$. For any $R \in \mathbb{R}$ and for any stepped surface $\mathcal{T}$, we denote by $(\mathcal{T})_R$ the restriction of the stepped surface $\mathcal{T}$ to the set of faces $(\mathbf{z}, j^*)$ such that $\mathbf{z} \in B(\mathbf{0}, R)$. According to the definition of dual maps, for every k, there exists r_k such that $(\mathcal{S}')_r = E_1^*(\beta_k)((\mathcal{S}'_k)_{r_k})$. According to Prop. 1.13, for every k, there exist a finite sequence of flips $(\mathcal{F}_{\mathbf{x}_n^{(k)}})_{0 \le n \le N_k}$ and a finite sequence of weights $(\varepsilon_n^{(k)})_{0 \le n \le N_k}$ with values in $\{\pm 1\}$ such that

$$(\mathcal{S}'_k)_{r_k} = \mathcal{P}_{(1, T^k(\alpha)), 0} + \sum_{n=0}^{N_k} \varepsilon_n^{(k)} \mathcal{F}_{\mathbf{x}_n^{(k)}}.$$

One deduces from Prop. 2.5 and Lem. 2.9:

$$\forall k \in \mathbb{N},\ (\mathcal{S}')_r = \mathcal{P}_{(1,\alpha),0} + \sum_{n=0}^{N_k} \varepsilon_n^{(k)} \mathcal{F}_{M_k^{-1} \mathbf{x}_n^{(k)}}.$$

Let $k = k_r$ with the notation of Lem. 4.5. One checks that $M_k^{-1} \mathbf{x}_n^{(k)} \in C$ (with the notation of Lem. 4.5), for all $n \le N_k$. Indeed $M_k^{-1} \mathbf{x}_n^{(k)}$ belongs to $\mathcal{P}$ or to $\mathcal{S}'$, which implies that $M_k^{-1} \mathbf{x}_n^{(k)}$ belongs to C by applying Prop. 1.7 (we use the fact that we have translated $\mathcal{S}$ so that $\mathcal{S}'$ contains a face located at the origin). Consequently, by multiplying by M_k, one deduces that $\mathbf{x}_n^{(k)} \in C$, for all $n \le N_k$. We thus deduce from Lem. 4.5 applied to $\mathbf{x}_n^{(k)}$ that $\langle M_k^{-1} \mathbf{x}_n^{(k)}, (1, \alpha) \rangle = 0$. We thus have proved that $M_k^{-1} \mathbf{x}_n^{(k)} \in Y_0$, for all $n \le N_k$, which implies that $\mathbf{x} \in Y_0$. One concludes by noticing that necessarily $\varepsilon_{\mathbf{x}} = 1$, otherwise $\mathcal{S}$ would not be a geometric sum. ☐

Proof. We now prove Lem. 4.5. First, let us define, for any $r > 0$, the following *positive* number:

$$a(r) = \min\{d(\mathbf{y}, (1, \alpha)^\perp) \mid \mathbf{y} \in C,\ \langle \mathbf{y}, (1, \alpha) \rangle \ne 0,\ ||\mathbf{y}|| \le r\}.$$

Let us then fix $r > 0$. The (uniform) weak convergence of Brun algorithm yields (see Th. 3.3):

$$\exists k_r \in \mathbb{N} \text{ s.t. } (n \ge k_r, \mathbf{u} \in [\mathbf{e_1}, \ldots, \mathbf{e_{d+1}}]) \Rightarrow$$

$$d(M_n^\top \mathbf{u}, \mathbb{R}(1, \alpha)) < \frac{a(r)}{r} d(M_n^\top \mathbf{u}, (1, \alpha)^\perp), \tag{4.1}$$

where $[\mathbf{e_1}, \ldots, \mathbf{e_{d+1}}]$ denotes the convex hull of the $\mathbf{e}_i$'s.

Let us fix $\mathbf{x} = (x_1, \cdots, x_{d+1}) \in C$ and $n \ge k_r$. We assume furthermore that $\langle M_n^{-1} \mathbf{x}, (1, \alpha) \rangle \ne 0$. We can associate with $\mathbf{x}$ a vector $\mathbf{u}$ in the convex hull of the $\mathbf{e}_i$'s such that $\langle \mathbf{x}, \mathbf{u} \rangle = 0$. Indeed, if there exists i such that $x_i = 0$, then $\mathbf{u} = \mathbf{e}_i$ suits. Otherwise, let i and j be such that $x_i > 0$ and $x_j < 0$. One easily checks that $\mathbf{u} = \frac{x_i}{|x_i - x_j|} \mathbf{e}_j - \frac{x_j}{|x_i - x_j|} \mathbf{e}_i$ belongs to the convex hull of the $\mathbf{e}_i$'s and satisfies $\langle \mathbf{x}, \mathbf{u} \rangle = 0$.

We decompose $M_n^{-1} \mathbf{x}$ and $M_n^\top \mathbf{u}$ as

$$M_n^{-1} \mathbf{x} = (M_n^{-1} \mathbf{x})_{(1,\alpha)} + (M_n^{-1} \mathbf{x})_{(1,\alpha)^\perp}, \text{ and } M_n^\top \mathbf{u} = (M_n^\top \mathbf{u})_{(1,\alpha)} + (M_n^\top \mathbf{u})_{(1,\alpha)^\perp},$$

where $(M_n^{-1} \mathbf{x})_{(1,\alpha)}$ and $(M_n^\top \mathbf{u})_{(1,\alpha)}$ belong to the line generated by $(1, \alpha)$, and $(M_n^{-1} \mathbf{x})_{(1,\alpha)^\perp}$, $(M_n^\top \mathbf{u})_{(1,\alpha)^\perp}$ to its orthogonal. We deduce from $\langle M_n^{-1} \mathbf{x}, M_n^\top \mathbf{u} \rangle = \langle \mathbf{x}, \mathbf{u} \rangle = 0$ that

$$\langle (M_n^{-1} \mathbf{x})_{(1,\alpha)}, (M_n^\top \mathbf{u})_{(1,\alpha)} \rangle = -\langle (M_n^{-1} \mathbf{x})_{(1,\alpha)^\perp}, (M_n^\top \mathbf{u})_{(1,\alpha)^\perp} \rangle.$$

One has $d(M_n^{-1} \mathbf{x}, (1, \alpha)^\perp) = ||(M_n^{-1} \mathbf{x})_{(1,\alpha)}||$, $d(M_n^{-1} \mathbf{x}, \mathbb{R}(1, \alpha) = ||(M_n^{-1} \mathbf{x})_{(1,\alpha)^\perp}||$, and similarly for $M_n^\top \mathbf{u}$. We thus get

$$|\langle (M_n^{-1} \mathbf{x})_{(1,\alpha)}, (M_n^\top \mathbf{u})_{(1,\alpha)} \rangle| = d(M_n^{-1} \mathbf{x}, (1, \alpha)^\perp)\, d(M_n^\top \mathbf{u}, (1, \alpha)^\perp),$$

and

$$|\langle (M_n^{-1} \mathbf{x})_{(1,\alpha)^\perp}, (M_n^\top \mathbf{u})_{(1,\alpha)^\perp} \rangle| \le d(M_n^{-1} \mathbf{x}, \mathbb{R}(1, \alpha))\, d(M_n^\top \mathbf{u}, \mathbb{R}(1, \alpha)).$$

Note that $\langle M_n^{-1}\mathbf{x}|(1,\alpha)\rangle \neq 0$ implies that $d(M_n^\top \mathbf{u}, \mathbb{R}(1,\alpha)) \neq 0$. This yields

$$\frac{d(M_n^{-1}\mathbf{x}, \mathbb{R}(1,\alpha))}{d(M_n^{-1}\mathbf{x}, (1,\alpha)^\perp)} \geq \frac{d(M_n^\top \mathbf{u}, (1,\alpha)^\perp)}{d(M_n^\top \mathbf{u}, \mathbb{R}(1,\alpha))}.$$

Therefore, one deduces from Eq. (4.1)

$$\frac{d(M_n^{-1}\mathbf{x}, \mathbb{R}(1,\alpha))}{d(M_n^{-1}\mathbf{x}, (1,\alpha)^\perp)} \geq \frac{r}{a(r)}. \tag{4.2}$$

Let us assume that $||M_n^{-1}\mathbf{x}|| \leq r$. Then one has $d(M_n^{-1}\mathbf{x}, \mathbb{R}(1,\alpha)) \leq ||M_n^{-1}\mathbf{x}|| \leq r$. Furthermore, by definition of $a(r)$, one has $d(M_n^{-1}\mathbf{x}, (1,\alpha)^\perp) \geq a(r)$. Indeed, one checks that $M_n^{-1}\mathbf{x} \in C$. This yields a contradiction with (4.2), which ends the proof. □

References

[1] E. Andrès, R. Acharya, C. Sibata, *Discrete analytical hyperplanes*, Graphical Models and Image Processing **59** (1997), 302–309.

[2] P. Arnoux, V. Berthé, Th. Fernique, D. Jamet, *Generalized substitutions, functional stepped surfaces and flips*, Theoret. Comput. Sci. **380** (2007), 251–267.

[3] P. Arnoux, S. Ito, *Pisot substitutions and Rauzy fractals*, Bull. Bel. Math. Soc. Simon Stevin **8** (2001), 181–207.

[4] P. Arnoux, A. Nogueira, *Mesures de Gauss pour des algorithmes de fractions continues multidimensionnelles*, Ann. Sci. École Norm. Sup. **26** (1993), 645–664.

[5] A. J. Brentjes *Multi-dimensional continued fraction algorithms*, Mathematical Centre Tracts 145, Matematisch Centrum, Amsterdam, 1981.

[6] O. Bodini, Th. Fernique, E. Remila, *A characterization of flip-accessibility for rhombus tilings of the whole plane*, in Proc. of *LATA'07* (2007), 139–150.

[7] V. Brimkov, D. Coeurjolly, R. Klette, *Digital Planarity - A Review*, Discr. Appl. Math. **155** (2007), 468–495.

[8] V. Brimkov, D. Coeurjolly, *Computational aspects of Digital Plane and Hyperplane Recognition*, in Proc. of *IWCIA'06*, LNCS **4040** (2006), 291–304.

[9] V. Brun, *Algorithmes euclidiens pour trois et quatre nombres*, 13th Congr. Math. Scand. Helsinki (1957), 45–64.

[10] H. Ei, *Some properties of invertible substitutions of rank d, and higher dimensional substitutions*, Osaka Journal of Mathematics **40** (2003), 543–562.

[11] Th. Fernique, *Generation and recognition of digital planes using multi-dimensional continued fractions*, in Proc. of *DGCI 14th International Conference* (2008), to appear.

[12] J. Françon, *Sur la topologie d'un plan arithmétique*, Theoret. Comput. Sci. **156** (1996), 159–176.

[13] D. J. Grabiner, *Farey nets and multidimensional continued fractions*, Monatsh. Math. **114** (1992), 35–61.

[14] Th. Fernique, *Multi-dimensional Sequences and Generalized Substitutions*, Int. J. Fond. Comput. Sci. **17** (2006), 575–600.

[15] D. Jamet, *On the language of standard discrete planes and surfaces*, in Proc. of *IWCIA'04*, LNCS **3322** (2004), 232–247.

[16] N. Lothaire, *Algebraic combinatorics on words*, Cambridge University Press, 2002.

[17] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics, and Combiantorics*, Lecture Notes in Mathematics **1794**, Springer Verlag. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel, 2002.

[18] J.-P. Reveillès, "Calcul en nombres entiers et algorithmique," Ph. D Thesis, Univ. Louis Pasteur, Strasbourg, France, 1991.

[19] F. Schweiger, *Multidimensional continued fractions*, Oxford Science Publications, Oxford University Press, Oxford (2000).

[20] M. Senechal, *Quasicrystals and geometry*, Cambridge University Press, Cambridge (1995).

[21] I. Sivignon, F. Dupont, J.-M. Chassery, *Discrete surfaces segmentation into discrete planes*, in Proc. of *IWCIA'04* , LNCS **3322** (2004), 458–473.

[22] A. Troesch, *Interprétation géométrique de l'algorithme d'Euclide et reconnaissance de segments*, Theoret. Comput. Sci. **115** (1993), 291–320.

[23] W. P. Thurston, *Groups, tilings and finite state automata*, Lectures notes distributed in conjunction with the Colloquium Series, in AMS Colloquium lectures, 1989.

[24] J. Vittone and J.-M. Chassery. *(n,m)-cubes and Farey Nets for Naive Planes Understanding*, in Proc. of *DGCI, 8th International Conference*, LNCS **1568** (1999), 76–87.

[25] J. Vittone and J.-M. Chassery, in Proc. of *DGCI, 9th International Conference*, LNCS **1953** (2001), 296–307.

[26] L.-D. Wu, *On the chain code of a line*, IEEE Transactions on Pattern Analysis and Machine Intelligence **4** (1982), 359–368.

LIRMM–Univ. Montpellier 2–CNRS–161 rue Ada 34392 Montpellier - France
E-mail address: berthe@lirmm.fr

LIRMM–Univ. Montpellier 2–CNRS–161 rue Ada 34392 Montpellier - France
E-mail address: fernique@lirmm.fr