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# Construction of sports schedules with multiple venues

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**Abstract**

A graph theoretical model is presented for constructing calendars for sports leagues where balancing requirement have to be considered with respect to the different venues where competitions are to be located. An inductive construction is given for leagues having a number of teams $2n$ which is of the form $2^p$ in particular.


*Keywords:* Sports scheduling; Complete graph; Factorization; Edge coloring

## 1. Introduction

Many types of scheduling problems arise in the domain of sports (see for example [1,7–9,4]). We shall consider here the problem of constructing a season schedule for a sports league consisting of $2n$ teams. All these teams have to play against each other at least once.

In addition there are $n$ stadiums (not related to the various teams) in which the games have to be played. One desires that each team plays the same number of games in each stadium. Furthermore, all teams have to be involved in a game in each day where games are scheduled. Finally two teams should not play against each other twice in the same stadium.

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This question has been considered by various authors (see for instance Urban and Russell [11]) in different contexts. It occurs among other situations when one has to schedule intra-squad competitions on various drill stations for spring training; a case with $2n = 8$ teams is described in [11] and recently a solution for $2n = 16$ was obtained in [12]. There are four stations and the question was to find if a schedule satisfying all requirements does exist. An integer programming model was designed to construct solutions which would satisfy these requirements or at least violate the requirements as little as possible. Such a model has been refined by other authors but the approach is essentially the same.

A related problem (where each team plays exactly once against every other team) has been solved with different techniques based on groups, orthogonal latin squares, room squares (see [3,6,10] for a sample).

Here we intend to use a graph theoretical formulation and to develop an interactive procedure which is based on the existence of two disjoint semi-leagues in a league of $2n$ teams. This will give us a special type of factorization of the complete graph $K_{2n}$ (see [4] for the use of similar factorizations) and will provide the basis for a simple inductive construction.

By designing solutions that have a regular structure; we hope to get more insight into the problem and to be able to adapt the procedure to arbitrary values of $2n$.

We insist that our problem differs slightly from the classical case handled in [3,6,10] for instance: we introduce a new round so that each one of the $2n$ teams has to play $2n$ games (instead of $2n - 1$ in the usual case) and there is an additional condition on the round introduced.

We will use the terminology of Berge [2] for all graph theoretical concepts not defined here.

We will not consider the case where stadiums are associated to the various teams and regular patterns of alternating home-games and away-games have to be constructed. The reader is referred to [1,7–9,4,5] for results related to the construction of such patterns and for other basic models where teams may have to travel between several home cities of other teams.

## 2. The basic graph-theoretical model

Since each team competes every other team at least once, we may represent the games by edges in a graph as follows: each team $u$ corresponds to a node $u$ and each game between teams $u$ and $v$ to an edge $[u, v]$. So we will have a graph on $2n$ nodes where each edge $[u, v]$ occurs at least once. Now all teams have to be involved in a game every day where competitions are scheduled; it is known that a schedule in $2n - 1$ days can be constructed with $n$ games (involving the $2n$ teams) scheduled in each day, when we assume that each team meets every other team exactly once (see [5]). It is also required that every team plays the same number of games in each one of the $n$ stadiums. Since each team has to play $2n - 1$ games on $n$ stadiums, there will be in the best case one stadium in which only one game is played by this team and there will be two games involving this team in all remaining $n - 1$ stadiums.

So in order to have the same number (i.e., two) of games of each team in each stadium, each team should play $2n$ games (or a multiple of $n$). So let us consider a complete graph $K_{2n}$ on $2n$ nodes (each edge occurs exactly once); there we choose a 1-factor, say $[1, 2], [3, 4], \ldots, [2n-1, 2n]$ and we double all these edges. We obtain a graph $K^*_{2n}$ in which all nodes have degree $2n$. A schedule for these games in exactly $2n$ days is known to exist (it is a usual edge coloring of $K^*_{2n}$) if we do not consider the balancing requirements on the stadiums.

Now considering an edge coloring $F_1, F_2, \ldots, F_{2n}$ of $K^*_{2n}$, i.e., a partition of the edge set $E(K^*_{2n})$ into 1-factors (collections of $n$ node disjoint edges) $F_j$, we have to consider the stadiums $1, \ldots, n$ where the competitions occur. In other words in each $F_j$ we have to assign labels $1, 2, \ldots, n$ to the $n$ edges of $F_j$ in order to indicate in which stadium each competition occurs. This assignment has to be done in such a way that

(a) for any $i$ $(1 \leqslant i \leqslant n)$ each node of $K^*_{2n}$ is adjacent to exactly two edges with label $i$.
(b) for every pair of parallel edges $[2u-1, 2u]$ (introduced to transform $K_{2n}$ into $K^*_{2n}$) the labels are different.

Requirement (b) expresses the fact that no two teams can meet twice in the same stadium.

The question is now to determine whether there exists an edge $2n$-coloring $(F_1, \ldots, F_{2n})$ of $K^*_{2n}$ associated to an appropriate labeling (satisfying (a) and (b)). If it exists, we shall say that $K^*_{2n}$ has a *feasible* schedule.

We will examine this in the next sections and we will start with the cases where $2n = 4$ or 8. Then a general construction procedure will be sketched for $2n = 2^p$ where $p$ is integer and satisfies $p \geqslant 3$.

## 3. Some special cases

Let us consider first a league of $2n = 4$ teams; we construct a graph $K^*_{2n}$ by duplicating edges $[1, 2]$ and $[3, 4]$ (see Fig. 1a).

It is easy to see that no feasible solution exists for $2n = 4$ teams and $n = 2$ stadiums. Fig. 1a shows a "best possible" schedule.

Let us now consider the case $2n = 8$. For this we shall use the solution of the case $2n = 4$. Let $a, b, c, d, a', b', c', d'$ be the teams (nodes of $K^*_8$). We start by constructing a schedule for the games internal to $\{a, b, c, d\}$ and to $\{a', b', c', d'\}$; this gives the partial schedule represented in Fig. 2. Notice that nodes $a, a', b, b'$ have some imbalance of labels which will have to be compensated in the second part of the schedule.

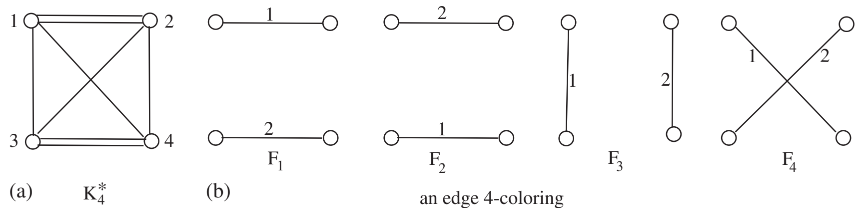


Fig. 1. The case of four teams.

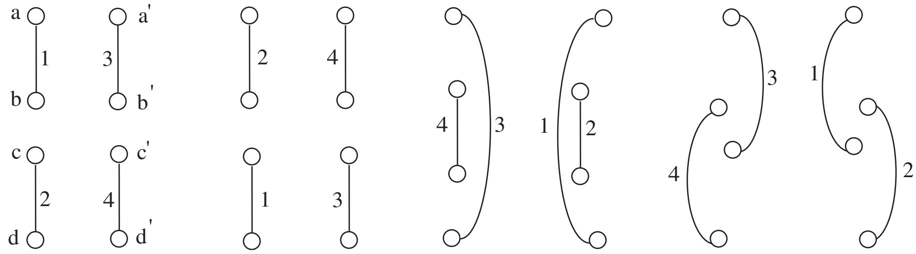


Fig. 2. A partial schedule for $K_8^*$.

| | $a'$ | $b'$ | $c'$ | $d'$ |
|---|---|---|---|---|
| $a$ | 4 | 1 | 4 | 2 |
| $b$ | 2 | 3 | 1 | 3 |
| $c$ | 3 | 1 | 2 | 4 |
| $d$ | 2 | 4 | 3 | 1 |

Fig. 3. The labels of the edges of $K_{4,4}$.

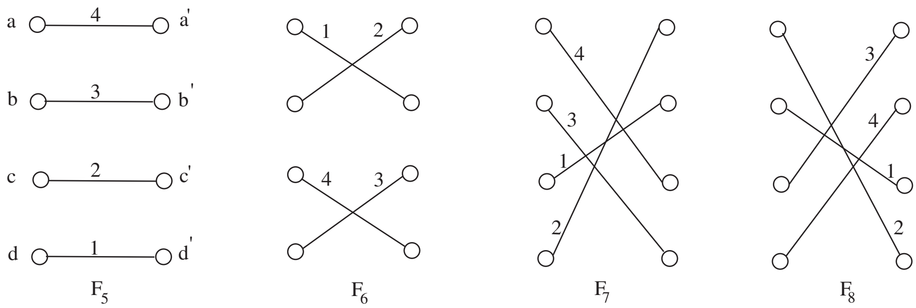


Fig. 4. The second part of the schedule for $K_8^*$.

We must now consider all games involving a team in $\{a, b, c, d\}$ and a team in $\{a', b', c', d'\}$. This can be represented by a $4 \times 4$ array $A$ giving the label (stadium) associated to each game.

For the case of $K_8^*$, the array $A$ is given in Fig. 3; $A(u, v) = i$ means that the game between teams $u$ and $v$ is played in stadium $i$. Considering the stadiums of the partial schedule of Fig. 2, we notice that each team plays exactly twice in each stadium.

It suffices now to give the second part of the schedule.

It is given in Fig. 4 and we observe that in each $F_j$ the labels of the edges are different. So we have constructed a feasible schedule for $2n = 8$ teams.

In the next section we shall give the general construction for the case where $2n = 2^p$ ($p$ integer, $p \geqslant 4$).

**Remark 1.** One may verify that no solution can be found for $2n = 6$ teams although there is a solution for the classical case (without additional round) where each team plays at most twice in each stadium (see [4]). This will be shown in the appendix.

## 4. The case of $2^p$ teams ($p \geqslant 4$)

We have seen that for $p = 3$, there exists a feasible schedule, while for $p = 2$, no such schedule could be constructed. We shall now describe the general construction which can be used for obtaining schedules in a league of $2^p$ teams. In such a situation we have $2^{p-1}$ simultaneous games and $2^{p-1}$ stadiums.

We may assume that there exists a feasible schedule for a league of $n = 2^{p-1}$ teams and a collection of $2^{p-2}$ stadiums.

In fact we will show a stronger result.

**Proposition 2.** *For $K_{2n}^*$ where $2n = 2^p (p \geqslant 3)$, there exists a feasible schedule such that for any $i (1 \leqslant i \leqslant 2^{p-1})$ the set $E_i$ of edges with label $i$ (games played on stadium $i$) is the union of two perfect matchings.*

This means in particular that $E_i$ consists of a collection of node disjoint even cycles covering all nodes. Notice that since from (b) parallel edges must have different labels, all these cycles will have length at least 4.

As an illustration one may verify that in the construction given for $K_8^*$ in Section 3, each $E_i$ $(1 \leqslant i \leqslant 4)$ is a cycle of length 8. So the construction proves the case $p = 3$. The general case will be established by giving a construction for $K_{2n}^*$.

For preparing the formulation of an inductive procedure we need some preliminaries.

Let us consider for the moment that the games to be played by the $2n = 2^p$ teams of a league are represented by the edges of a complete bipartite graph $K_{n,n}$ (where $n = 2^{p-1}$). This amounts to considering that we have two subleagues $A$, $B$ with $2^{p-1}$ teams each; all games must involve a team in $A$ and a team in $B$. Assume that we have $2^{p-1}$ stadiums. Then we can state

**Proposition 3.** *If the games are represented by the edges of $K_{n,n}$ (where $n = 2^{p-1}$ with $p \geqslant 3$), there exists a schedule in $n = 2^{p-1}$ days such that each team plays exactly one game in each stadium.*

**Proof.** For $K_{4,4}$ we construct the schedule given in Fig. 5. We have teams 1,2,3,4 on the left and $1', 2', 3', 4'$ on the right

Such a schedule can also be represented by the matrix $A$ in Fig. 6.

In Fig. 6, $F_1$ is represented by entries of $A$ with circled figures, $F_2$ by the entries with bold figures.

In order to obtain a matrix $\overline{A}$ (corresponding to a schedule) for $K_{n,n}$ from a matrix $A$ associated to $K_{n/2,n/2}$ we proceed as follows: we consider the symbols $1, 2, \ldots, n$ as forming a cyclic order.

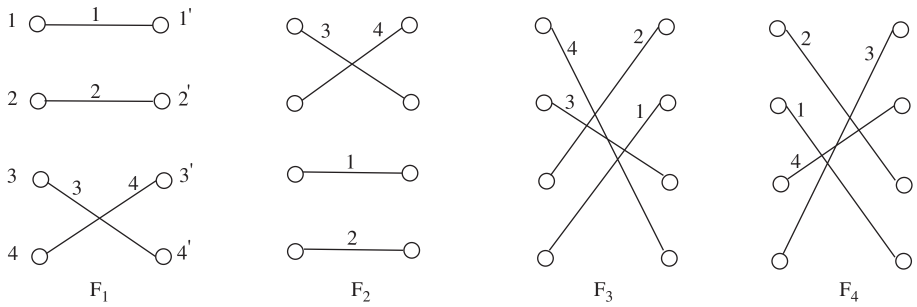


Fig. 5. A schedule for $K_{4,4}$.

| | 1’ | 2’ | 3’ | 4’ |
|---|---|---|---|---|
| 1 | 1 | **3** | 2 | 4 |
| 2 | **4** | 2 | 3 | 1 |
| 3 | 2 | 4 | **1** | 3 |
| 4 | 3 | 1 | 4 | **2** |

Fig. 6. The matrix $A$ corresponding to the schedule in Fig. 5.

| | 1’ | 2’ | 3’ | 4’ | 5’ | 6’ | 7’ | 8’ |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | **6** | **7** | 5 | 4 | 1 | 8 |
| 2 | 4 | 1 | **8** | **5** | 3 | 6 | 7 | 2 |
| 3 | **1** | **8** | 5 | 4 | 6 | 7 | 2 | 3 |
| 4 | **7** | **2** | 3 | 6 | 8 | 5 | 4 | 1 |
| 5 | 5 | 4 | 1 | 8 | **2** | **3** | 6 | 7 |
| 6 | 3 | 6 | 7 | 2 | **4** | **1** | 8 | 5 |
| 7 | 6 | 7 | 2 | 3 | 1 | 8 | **5** | **4** |
| 8 | 8 | 5 | 4 | 1 | 7 | 2 | **3** | **6** |

Fig. 7. The matrix $\overline{A}$ associated to a schedule for $K_{8,8}$.

Each entry of $A$ containing an $i$ is replaced by a square of four entries with values $\begin{array}{|c|c|}\hline 2i & 2i{+}1 \\ \hline 2i{+}2 & 2i\text{-}1 \\ \hline\end{array}$ if $i$ is odd or values $\begin{array}{|c|c|}\hline 2i{+}1 & 2i \\ \hline 2i\text{-}1 & 2i{+}2 \\ \hline\end{array}$ else.

Here the integers are taken modulo $n$ between 1 and $n$. For example, we would get the matrix $\overline{A}$ of Fig. 7 from the $A$ of Fig. 6.

One observes that all labels $1, \ldots, 8$ occur exactly once in each row and in each column. Now to every matching $F_j$ of $K_{4,4}$ correspond two matchings of $K_{8,8}$; for instance, consider $F_2$ in $K_{4,4}$ which corresponds to the entries in $A$ with bold figures.

The corresponding entries with bold figures in $\overline{A}$ define two matchings by taking first the first diagonal with the bold figures in the $2 \times 2$ squares corresponding to an odd $i$ in $A$ and the second diagonal for those corresponding to an even $i$ in $A$.

The second matching is obtained by taking the remaining teams with the bold figures in the $(2 \times 2)$ squares.

One sees that in each one of these matchings all eight edges have different labels.

So this construction will give the required schedule for $K_{n,n}$. This ends the proof of Proposition 3. □

**Proof** (*of Proposition 2*). Assume that the result is true for $K_n^*$ where $n = 2^{p-1}$ (with $p - 1 \geqslant 3$). We will show that it holds for $K_{2n}^*$.

Our league consists of two subleagues of $n$ teams each, the games inside these subleagues are represented by two graphs $K_n^*$ and $(K_n^*)'$.

So we have for $K_n^*$ a feasible schedule with $n/2 = 2^{p-2}$ stadiums; each team plays two games in each one of the stadiums. By assumption the set $E_i$ of edges with label $i$ is the union of two perfect matchings having $n/2$ edges each (for $i = 1, \ldots, n/2$). For each $i$ we change the labels on one of these two matchings from $i$ to $n/2 + i$. This gives a schedule for $K_n^*$ where each team plays exactly once in each one of $n$ stadiums. Now there is a one-to-one correspondence between the edges $e$ of $K_n^*$ and the edges $e'$ of $(K_n^*)'$. We take the same coloring for $(K_n^*)'$ as for $K_n^*$; but the labels are defined as follows: if $e$ has label $i$, then $e'$ will have label $i + n/2$ where all these values are taken modulo $n$ between 1 and $n$.

Now we form the graph $K_{2n}^*$ by taking $K_n^*$, $(K_n^*)'$ and the edges of $K_{n,n}$ colored and labeled as in the construction of Proposition 3.

Then it follows from the construction that each node of $K_{2n}^*$ is adjacent to exactly two edges with label $i$ for $i = 1, \ldots, n$ (in fact if $v$ is a node in $K_n^*$, one edge labeled $i$ is inside $K_n^*$; it belongs to a perfect matching in $K_n^*$. The other edge is between $v$ and a node in $(K_n^*)'$).

As a consequence the edges labeled $i$ in $K_{2n}^*$ form a collection of even cycles covering all nodes (no cycle can be odd since it has to cross an even number of times the edges between $K_n^*$ and $(K_n^*)'$ and since edges labeled $i$ inside $K_n^*$ (and inside $(K_n^*)'$) are not adjacent). So the edges labeled $i$ in $K_{2n}^*$ are the union of two disjoint perfect matchings.

Let us now finally show that the colors and labels given to the edges of $K_{2n}^*$ form a feasible schedule. This is certainly true for the $n$ perfect matchings defined on the edges of $K_{n,n}$ by the construction of Proposition 3; in each such matching all $n$ edges have different labels. Then we construct the perfect matchings in $K_n^* \bigcup (K_n^*)'$ by taking the matchings $F_j$ in the factorization of $K_n^*$ corresponding to the initial schedule that was assumed to exist and associating the corresponding perfect matching $F_j'$ in $(K_n^*)'$.

It just remains to verify that condition (b) on parallel edges is still verified. This can be seen as follows: for $K_8^*$ the initial labeling is such that in every family of parallel edges the difference in the labels is one. Then one may modify the label of one or of both edges in a pair of parallel edges; but the difference in the labels of parallel edges will always keep the same parity (odd) since the only changes in labels are additions of an even quantity $n/2$ and computations modulo an even integer $n$. So we see that the labels in each pair of parallel edges will remain different.

We now have obtained a feasible schedule for $K_{2n}^*$ and Proposition 2 is proved. □

**Remark 4.** It should also be observed that the above construction produces a schedule such that for any team the two games played on the same stadium involve another team of the same subleague for one game and a team of the other subleague for the second game.

In terms of graphs we may also formulate the existence of feasible schedules in the following way.

**Proposition 5.** *Let $K^*_{2n}$ be a complete graph on $2n$ nodes where the edges of an arbitrary perfect matching have been doubled. Assume $n = 2^p$ (with $p \geqslant 2$ integral).*

*Then there exists a labeling $(\alpha(e),\ \beta(e))$ of each edge $e$ such that*

(i) $\alpha(e) \in \{1, \dots, 2n\}$, $\beta(e) \in \{1, \dots, n\}$.
(ii) *for any two edges $e, e'$ $(\alpha(e), \beta(e)) \neq (\alpha(e'), \beta(e'))$.*
(iii) *all edges $e$ with the same label $\alpha(e) = \alpha$ form a perfect matching (for any $\alpha(e) \in \{1, \dots, 2n\}$).*
(iv) *for any node $v$ and any $\beta$ $(1 \leqslant \beta \leqslant n)$, there are exactly two edges $e, e'$ adjacent to $v$ for which*

$$\beta(e) = \beta(e') = \beta.$$

(v) *if $e, e'$ are parallel edges, then $\beta(e) \neq \beta(e')$.*

Another formulation based on the above construction (without condition $(v)$) would be: There exists a labeling $(\alpha(e), \beta(e))$ of each edge $e$ such that

(I) $\alpha(e), \beta(e) \in \{1, \dots, 2n\}$.
(II) for any two edges $e, e'$ $(\alpha(e), \beta(e)) \neq (\alpha(e'), \beta(e'))$.
(III) all edges $e$ with the same label $\alpha(e) = \alpha$ form a perfect matching $(1 \leqslant \alpha \leqslant 2n)$.
(IV) all edges $e$ with the same label $\beta(e) = \beta$ form a perfect matching $(1 \leqslant \beta \leqslant 2n)$.

In some sense the labels $\alpha(e)$ and $\beta(e)$ define two "orthogonal" edge colorings of $K^*_{2n}$.

## 5. Final remarks

The construction procedure given in Section 4 applies to the special case where $2n = 2^p$; this is due to its inductive nature. Feasible schedules for $2n = 16$ teams by integer programming techniques have recently been obtained, see [12]; notice that solutions exist for the classical case without additional round.

Feasible schedules have been obtained for $2n = 10$ or $2n = 12$ and our construction is a priori not able to handle these cases. Further research is needed to develop a general construction procedure based on semi-leagues which could provide feasible schedules whenever they exist.

**Acknowledgements**

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## Appendix A. The case of $2n = 6$ teams

For a league of $2n = 6$ teams playing on $n = 3$ stadiums, there exists a schedule in $2n - 1 = 5$ days where each team plays at most twice in each stadium. It is given in [3]:

$$\begin{array}{llll} F_1 = [2,4], & [3,1], & [6,5] \\ F_2 = [5,3], & [4,1], & [6,2] \\ F_3 = [5,4], & [3,6], & [1,2] \\ F_4 = [2,3], & [4,6], & [1,5] \\ F_5 = [1,6], & [2,5], & [3,4] \end{array}$$

In each $F_i$, the game $j$ is played in stadium $j$ $(j = 1, 2, 3)$.

However, for the problem discussed in this note (factorization of $K^*_{2n}$) there is no schedule satisfying requirements (a) and (b).

This can be seen as follows: $K^*_6$ has three nonisomorphic factorizations that are given in Fig. A.1. The games to be played twice are [1,2], [3,4], [5,6].

We let $D$ be the set of double edges [1,2], [3,4], [5,6] in $K^*_6$. In any edge 6-coloring $F_1, \ldots, F_6$ of $K^*_6$ we have $|D \cap F_i| \neq 2$, since an $F_i$ which would contain two edges of $D$ would also contain a third edge of $D$. So we have $|D \cap F_i| \in \{0, 1, 3\}$. If we associate to an edge 6-coloring $F_1, \ldots, F_6$ the values $|D \cap F_1| \leqslant |D \cap F_2| \leqslant \cdots \leqslant |D \cap F_6|$ the only cases are (A) (1, 1, 1, 1, 1, 1), (B) (0, 0, 1, 1, 1, 3) and (C) (0, 0, 0, 0, 3, 3).

Let now $H_j$ represent the games played in stadium $j$ for $j = 1, 2, 3$.

**Proposition 6.** *If there exists a feasible schedule for $K^*_6$, then each $H_j$ is a $C_6$ (cycle on 6 nodes).*

**Proof.** $H_j$ is a 2-factor in $K^*_6$ (all degrees are 2 since every team plays exactly two games in each stadium); $H_j$ cannot consist of a $C_4$ (cycle on 4 nodes) and a double edge ([1,2], [3,4] or [5,6]), since this would mean that (b) is violated.

$H_j$ cannot consist of two triangles; this can be seen as follows: assume $H_j$ consists of two triangles $T_1$ and $T_2$; each one contains at most one of the edges of $D$; so we have $|H_j \cap D| \leqslant 2$.

Each triangle can contain at most one pair of nodes linked by double edges. Assume w.l.o.g. that $T_1 \ni 1, 2$ linked by an edge in $H_j$ and an edge in $H_\ell$. The edge [1,2] in $H_l$ will be contained in a matching $F_k$ which will use at most one edge $e$ of $T_2 \cap H_j$. After removal of $F_k$ we have a triangle $T_1$ with three edges in $H_j$ (and no double edges between nodes of $T_1$) and at least two edges in $T_2 \cap H_j$. Let $e \in T_2 \cap H_j$; it will belong to a matching $F_{\overline{k}}$ which will also contain an edge between a node of $T_2$ and a node of $T_1$ (belonging to some

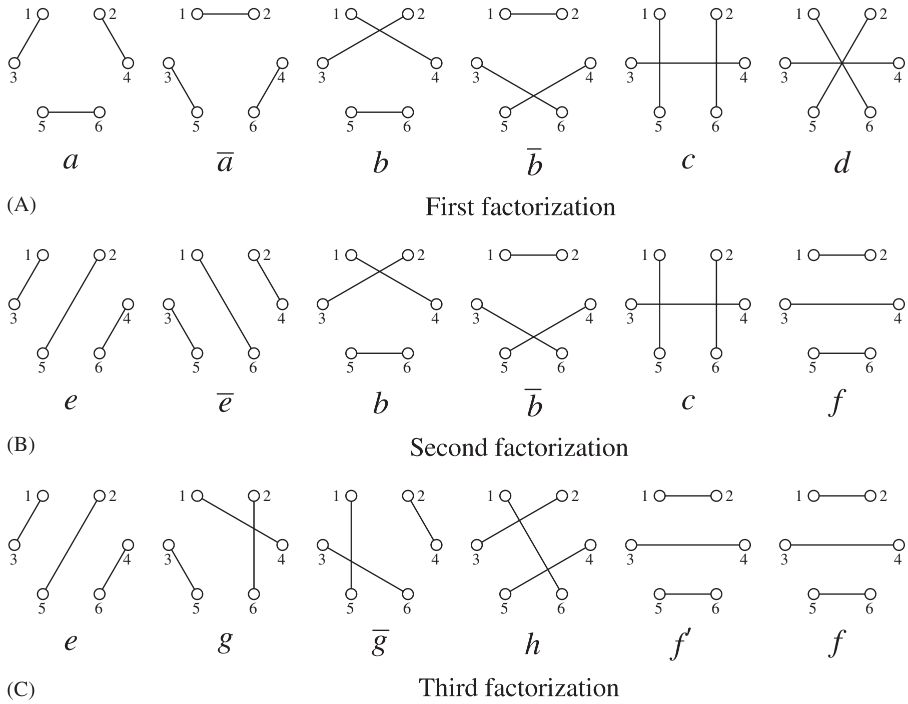


Fig. A.1. The factorization of $K_6^*$. (A) First factorization, (B) Second factorization, (C) Third factorization.

$H_{j_1}$ with $j_1 \neq j$); the two remaining nodes of $T_1$ are linked by only one edge and this edge is in $H_j$.

This is impossible (two games in stadium $j$ on day $\overline{k}$). □

**Proposition 7.** *There is no feasible schedule associated to the factorization C of* $K_6^*$.

**Proof.** Consider the factorization $C$ (associated to the values (0, 0, 0, 0, 3, 3) of $|D \cap F_i|$) (see Fig. A.1. C). It consists of matchings $F_e$, $F_g$, $F_{\overline{g}}$, $F_h$, $F_{f'}$, $F_f$; w.l.o.g. we can assume that for $F_{f'}$ : $12 \in H_1$, $34 \in H_2$, $56 \in H_3$ and for $F_f$ : $12 \in H_2$, $34 \in H_3$ and $56 \in H_1$. Now consider $F_h = 16, 23, 45$.

*Case* 1: $16 \in H_1$: Then $23, 45, 15, 26, 34 \notin H_1$. But then 4 cannot play in stadium 1 against 1, 6 (already 2 games in stadium 1), nor 3 (games already in $H_2$ and $H_3$), nor 5 (since this would give $45, 16 \in H_1 \cap F_h$). It can play only against 2, but 4 has two games to play in stadium 1.

*Case* 2: $16 \in H_2$: Then $23, 45 \notin H_2$; $26 \notin H_2$ from Proposition 6. Now $13, 14, 15 \notin H_2$. Looking at $F_g$ we see that $35 \in H_2$. But then the only way to complete $H_2$ is to have games 25 and 46 in stadium 2 (this is not possible since both occur on the same day in $F_e$) or games 24 and 56 in stadium 2 (this is again impossible since the two games between 5 and 6 are in $H_1$ and in $H_3$).

*Case* 3: $16 \in H_3$: From Proposition 6, $15 \notin H_3$ and also 26, 36, $46 \notin H_3$. Since $16 \in F_h$ we also have 23, $45 \notin H_3$. But now 15, $36 \in F_{\overline{g}}$ imply $24 \in H_3$. But then 2 can only play the second game in $H_3$ against 5, so $25 \in H_3$. The only remaining teams (having one more game to play in stadium 3 are 1 and 3); they cannot play since $13 \in F_e$ which contains also $25 \in H_3$. □

**Proposition 8.** *There is no feasible schedule associated to the factorization B of* $K_6^*$.

**Proof.** Consider the factorization $B$ (associated to the values (0, 0, 1, 1, 1, 3) of $|D \cap F_i|$) (see Fig. A.1. B). It consists of matchings $F_e, F_{\overline{e}}, F_b, F_{\overline{b}}, F_c, F_f$; w.l.o.g. we may assume for $F_f$ : $12 \in H_1$, $34 \in H_2$, $56 \in H_3$ and for $F_c$ : $15 \in H_1$, $26 \in H_2$, $34 \in H_3$. Now from Proposition 6, we have $25 \notin H_1$ also 13, 14, $16 \notin H_1$. From $F_e$ we have $46 \in H_1$.

*Case* 1: $13 \in H_2$: Then $25 \in H_3$ (from $F_e$). We have 23, 35, $36 \notin H_2$. Also $14 \notin H_2$ from Proposition 6. From $F_b$ we have $56 \in H_2$. But then $16 \notin H_2$, which implies $16 \in H_3$. It follows $12 \notin H_3$ so $12 \in H_2$. In order to have a $C_6$ for $H_2$ we should have $45 \in H_2$, but 45, $12 \in F_{\overline{b}}$ and we cannot have these two games on the same day.

*Case* 2: $13 \in H_3$: Then $25 \in H_2$. From Proposition 6, $56 \notin H_2$ so from (b) we must have $56 \in H_1$. But then $36 \notin H_1$ and team 3 could play only with team 2 in stadium 1 and it should play two games. □

**Proposition 9.** *There is no feasible schedule associated to the factorization A of* $K_6^*$.

**Proof.** Consider the factorization $A$ (associated to the values (1, 1, 1, 1, 1, 1) of $|D \cap F_i|$) (see Fig. A.1.A). It consists of matchings $F_a, F_{\overline{a}}, F_b, F_{\overline{b}}, F_c, F_d$. w.l.o.g. we can assume for $F_a$ : $13 \in H_1$, $24 \in H_2$, $56 \in H_3$ and for $F_b$ $56 \in H_1$.

*Case* 1: $14 \in H_2$: Then from $F_b$ $23 \in H_3$. But then $12 \notin H_2$ from Proposition 6. Also 34, 45, $46 \notin H_2$. Since $13 \notin H_2$, we must have 35, $36 \in H_2$ (because 15, $26 \in F_a$ and so they are scheduled on the same day). So we must have 25, $16 \in H_2$, but this is again impossible since 25, $16 \in F_d$.

*Case* 2: $14 \in H_3$: Then from $F_b$ $23 \in H_2$. From Proposition 6, we have $34 \notin H_2$ hence $34 \in H_1$ and $34 \in H_3$. Now 12, 25, $26 \notin H_2$ and it follows $12 \in H_1$, $12 \in H_3$. From $F_d$ we must have $16 \in H_2$.

If $26 \in H_3$ or $H_1$, then from $F_c$ we have $15 \in H_2$; we cannot have 36, $45 \in H_2$ because these games are played the same day (36, $45 \in F_{\overline{b}}$; so we must have 35, 46 but these are again played the same day (35, $46 \in F_{\overline{a}}$). □

Since all cases have been examined, we have established that there is no feasible solution for $K_6^*$.

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