# Spanning a strong digraph by $\alpha$ circuits: A proof of Gallai's conjecture.

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**Abstract**

In 1963, Tibor Gallai [9] asked whether every strongly connected directed graph $D$ is spanned by $\alpha$ directed circuits, where $\alpha$ is the stability of $D$. We give a proof of this conjecture.

## 1 Coherent cyclic orders.

In this paper, circuits of length two are allowed. Since loops and multiple arcs play no role in this topic, we will simply assume that our digraphs are loopless and simple. A directed graph (digraph) is *strongly connected*, or simply *strong*, if for all vertices $x, y$, there exists a directed path from $x$ to $y$. A *stable set* of a directed graph $D$ is a subset of vertices which are not pairwise joined by arcs. The *stability* of $D$, denoted by $\alpha(D)$, is the number of vertices of a maximum stable set of $D$. It is well-known, by the Gallai-Milgram theorem [10] (see also [1] p. 234 and [3] p. 44), that $D$ admits a vertex-partition into $\alpha(D)$ disjoint paths. We shall use in our proof a particular case of this result, known as Dilworth's theorem [8]: a partial order $P$ admits a vertex-partition into $\alpha(P)$ chains (linear orders). Here $\alpha(P)$ is the size of a maximal antichain. In [9], Gallai raised the problem, when $D$ is strongly connected, of spanning $D$ by a union of circuits. Precisely, he made the following conjecture (also formulated in [1] p. 330, [2] and [3] p. 45):

**Conjecture 1** *Every strong digraph with stability $\alpha$ is spanned by the union of $\alpha$ circuits.*

The case $\alpha = 1$ is Camion's theorem [6]: Every strong tournament has a hamilton circuit. The case $\alpha = 2$ is a corollary of a result of Chen and Manalastas [7] (see also Bondy [4]): Every strong digraph with stability two is spanned by two circuits intersecting each other on a (possibly empty) path. In [11] was proved the case $\alpha = 3$. In the next section of this paper, we will give a proof of Gallai's conjecture for every $\alpha$.

Let $D$ be a strong digraph on vertex set $V$. An enumeration $E = v_1, \dots, v_n$ of $V$ is *elementary equivalent* to $E'$ if one the following holds: $E' = v_n, v_1, \dots, v_{n-1}$, or $E' = v_2, v_1, v_3, \dots, v_n$ if neither

$v_1v_2$ nor $v_2v_1$ is an arc of $D$. Two enumerations $E, E'$ of $V$ are *equivalent* if there is a sequence $E = E_1, \dots, E_k = E'$ such that $E_i$ and $E_{i+1}$ are elementary equivalent, for $i = 1, \dots, k-1$. The classes of this equivalence relation are called the *cyclic orders* of $D$. Roughly speaking, a cyclic order is a class of enumerations of the vertices on the integers modulo $n$, where one stay in the class while switching consecutive vertices which are not joined by an arc. We fix an enumeration $E = v_1, \dots, v_n$ of $V$, the following definitions are understood with respect to $E$. An arc $v_iv_j$ of $D$ is a *forward arc* if $i < j$, otherwise it is a *backward arc.* A directed path of $D$ is a *forward path* if it only contains forward arcs. The *index* of a directed circuit $C$ of $D$ is the number of backward arcs of $C$, we denote it by $i_E(C)$. This correspond to the winding number of the circuit. Observe that $i_E(C) = i_{E'}(C)$ if $E'$ is elementary equivalent to $E$. Consequently, the index of a circuit is invariant in a given cyclic order $\mathcal{C}$, we denote it by $i_{\mathcal{C}}(C)$. By extension, the index $i_{\mathcal{C}}(\mathcal{S})$ of a set of circuits $\mathcal{S}$ is the sum of the indices of the circuits of $\mathcal{S}$. A circuit is *simple* if it has index one. A cyclic order $\mathcal{C}$ is *coherent* if every arc of $D$ is contained in a simple circuit, or, equivalently, if for every enumeration $E$ of $\mathcal{C}$ and every backward arc $v_jv_i$ of $E$, there exists a forward path from $v_i$ to $v_j$. We denote by $cir(D)$ the set of all directed circuits of $D$.

**Lemma 1** *Every strong digraph has a coherent cyclic order.*

**Proof.** Let us consider a cyclic order $\mathcal{C}$ which is minimum with respect to $i_{\mathcal{C}}(cir(D))$. We suppose for contradiction that $\mathcal{C}$ is not coherent. There exists an enumeration $E = v_1, \dots, v_n$ and a backward arc $a = v_jv_i$ which is not in a simple circuit. Assume moreover that $E$ and $a$ are chosen in order to minimize $j - i$. Let $k$ be the largest integer $i \leq k < j$ such that there exists a forward path from $v_i$ to $v_k$. Observe that $v_k$ has no out-neighbour in $]v_k, v_j]$. If $k \neq i$, by the minimality of $j - i$, $v_k$ has no in-neighbour in $]v_k, v_j]$. In particular the enumeration $E' = v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_j, v_k, v_{j+1}, \dots, v_n$ is equivalent to $E$, and contradicts the minimality of $j - i$. Thus $k = i$, and by the minimality of $j - i$, there is no in-neighbour of $v_i$ in $]v_i, v_j[$. In particular the enumeration $E' = v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_i, v_j, \dots, v_n$ is equivalent to $E$. Observe now that in $E'' = v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_j, v_i, \dots, v_n$, every circuit $C$ satisfies $i_{E''}(C) \leq i_{E'}(C)$, and the inequality is strict if the arc $a$ belongs to $C$, a contradiction. □

A direct corollary of Lemma 1 is that every strong tournament has a hamilton circuit, just consider for this any coherent cyclic order.

## 2 Cyclic stability versus spanning circuits.

The *cyclic stability* of a coherent cyclic order $\mathcal{C}$ is the maximum $k$ for which there exists an enumeration $v_1, \dots, v_n$ of $\mathcal{C}$ such that $\{v_1, \dots, v_k\}$ is a stable set of $D$. We denote it by $\alpha(\mathcal{C})$, observe that we clearly have $\alpha(\mathcal{C}) \leq \alpha(D)$.

**Lemma 2** *Let $D$ be a strong digraph and $v_1, \dots, v_n$ be an enumeration of a coherent cyclic order $\mathcal{C}$ of $D$. Let $X$ be a subset of vertices of $D$ such that there is no forward path between two distinct vertices of $X$. Then $|X| \leq \alpha(\mathcal{C})$.*

**Proof.** We consider an enumeration $E = v_1, \dots, v_n$ of $\mathcal{C}$ such that there is no forward path between two distinct vertices of $X$, and chosen in such a way that $j - i$ is minimum, where $v_i$ is the first element of $X$ in the enumeration, and $v_j$ is the last element of $X$ in the enumeration. Suppose for contradiction that $X \neq \{v_i, \dots, v_j\}$. There exists $v_k \notin X$ for some $i < k < j$. There cannot exist both a forward path from $X \cap \{v_i, \dots, v_{k-1}\}$ to $v_k$ and a forward path from $v_k$ to $X \cap \{v_{k+1}, \dots, v_j\}$. Without loss of generality, we assume that there is no forward path from $X \cap \{v_i, \dots, v_{k-1}\}$ to $v_k$. Suppose moreover that $v_k$ is chosen with minimum index $k$. Clearly, $v_k$ has no in-neighbour in $\{v_i, \dots, v_{k-1}\}$, and since $\mathcal{C}$ is coherent, $v_k$ has no out-neighbour in $\{v_i, \dots, v_{k-1}\}$. Thus the enumeration $v_1, \dots, v_{i-1}, v_k, v_i, \dots, v_{k-1}, v_{k+1}, \dots, v_n$ belongs to $\mathcal{C}$, contradicting the minimality of $j - i$. Consequently, $X = \{v_i, \dots, v_j\}$, and there is no forward arcs, and then no backward arcs, between the vertices of $X$. Considering now the enumeration $v_i, \ldots, v_n, v_1, \ldots, v_{i-1}$, we conclude that $|X| \leq \alpha(\mathcal{C})$. $\square$

Let $P = x_1, \ldots, x_k$ be a directed path, we call $x_1$ the *head* of $P$ and $x_k$ the *tail* of $P$. We denote the restriction of $P$ to $\{x_i, \ldots, x_j\}$ by $P[x_i, x_j]$.

**Theorem 1** *Let $D$ be a strong digraph with a coherent cyclic order $\mathcal{C}$. The minimal $i_{\mathcal{C}}(\mathcal{S})$, where $\mathcal{S}$ is a spanning set of circuits of $D$ is equal to $\alpha(\mathcal{C})$.*

**Proof.** We consider a coherent cyclic order $\mathcal{C}$ of $D$ with cyclic stability $k := \alpha(\mathcal{C})$. Let $E = v_1, \ldots, v_n$ be an enumeration of $\mathcal{C}$ such that $S = \{v_1, \ldots, v_k\}$ is a stable set of $D$. Clearly, if a circuit $C$ contains $q$ vertices of $S$, the index of $C$ is at least $q$. In particular the inequality $i_{\mathcal{C}}(\mathcal{S}) \geq k$ is satisfied for every spanning set of circuits of $D$. To prove that equality holds, we consider an auxiliary acyclic digraph $D'$ on vertex set $V \cup \{v'_1, \ldots, v'_k\}$ which arc set consists of every forward arc of $E$ and every arc $v_i v'_j$ for which $v_i v_j$ is an arc of $D$. We call $T'$ the transitive closure of $D'$. Let us prove that the size of a maximal antichain in the partial order $T'$ is exactly $k$. Consider such an antichain $A$, and set $A_1 := A \cap \{v_1, \ldots, v_k\}$, $A_2 := A \cap \{v_{k+1}, \ldots, v_n\}$ and $A_3 := A \cap \{v'_1, \ldots, v'_k\}$. Since one can arbitrarily permute the vertices of $S$ in the enumeration $E$ and still remain in $\mathcal{C}$, we may assume that $A_3 = \{v'_1, \ldots, v'_j\}$ for some $0 \leq j \leq k$. Since every vertex is in a simple circuit, there is a directed path in $D'$ from $v_i$ to $v'_i$, and consequently we cannot both have $v_i \in A$ and $v'_i \in A$. Clearly, the enumeration $E' = v_{j+1}, \ldots, v_n, v_1, \ldots, v_j$ belongs to $\mathcal{C}$. By the fact that $A$ is an antichain of $T'$, there is no forward path joining two elements of $(A \cap V) \cup \{v_1, \ldots, v_j\}$ in $E'$, and thus, by Lemma 2, $|A| = |(A \cap V) \cup \{v_1, \ldots, v_j\}| \leq k$. Observe also that $\{v_1, \ldots, v_k\}$ are the sources of $T'$ and $\{v'_1, \ldots, v'_k\}$ are the sinks of $T'$, and both are maximal antichains of $T'$. We apply Dilworth's theorem in order to partition $T'$ into $k$ chains (thus starting in the set $\{v_1, \ldots, v_k\}$ and ending in the set $\{v'_1, \ldots, v'_k\}$), and by this, there exists a spanning set $P_1, \ldots, P_k$ of directed paths of $D'$ with heads in $\{v_1, \ldots, v_k\}$ and tails in $\{v'_1, \ldots, v'_k\}$. We can assume without loss of generality that the head of $P_i$ is exactly $v_i$, for all $i = 1, \ldots, k$. Let us now denote by $\sigma$ the permutation of $\{1, \ldots, k\}$ such that $v'_{\sigma(i)}$ is the tail of $P_i$, for all $i$. Assume that among all spanning sets of paths, we have chosen $P_1, \ldots, P_k$ (with respective heads $v_1, \ldots, v_k$) in such a way that the permutation $\sigma$ has a maximum number of cycles. We claim that if $(i_1, \ldots, i_p)$ is a cycle of $\sigma$ (meaning that $\sigma(i_j) = i_{j+1}$ and $\sigma(i_p) = i_1$), then the paths $P_{i_1}, \ldots, P_{i_p}$ are pairwise vertex-disjoint. If not, suppose that $v$ is a common vertex of $P_{i_l}$ and $P_{i_m}$, and replace $P_{i_l}$ by $P_{i_l}[v_{i_l}, v] \cup P_{i_m}[v, v_{\sigma(i_m)}]$ and $P_{i_m}$ by $P_{i_m}[v_{i_m}, v] \cup P_{i_l}[v, v_{\sigma(i_l)}]$. This is a contradiction to the maximality of the number of cycles of $\sigma$. Now, in the set of paths $P_1, \ldots, P_k$, contract all the pairs $\{v_i, v'_i\}$, for $i = 1, \ldots, k$. This gives a spanning set $\mathcal{S}$ of circuits of $D$ which satisfies $i_{\mathcal{C}}(\mathcal{S}) = k$. $\square$

**Corollary 1.1** *Every strong digraph $D$ is spanned by $\alpha(D)$ circuits.*

**Proof.** By Lemma 1, $D$ has a coherent cyclic order $\mathcal{C}$. By Theorem 1, $D$ is spanned by a set $\mathcal{S}$ of circuits such that $|\mathcal{S}| \leq i_{\mathcal{C}}(\mathcal{S}) = \alpha(\mathcal{C}) \leq \alpha(D)$. $\square$

We now establish the arc-cover analogue of Theorem 1. Again, a minimax result holds.

# 3 Cyclic feedback arc set versus arc cover.

Let $\mathcal{C}$ be a cyclic order of a strong digraph $D$. We denote by $\beta(\mathcal{C})$ the maximum $k$ for which there exists an enumeration of $\mathcal{C}$ with $k$ backward arcs. We call $k$ the *maximal feedback arc set* of $\mathcal{C}$. Since every vertex of $D$ has indegree at least one, we clearly have $\alpha(\mathcal{C}) \leq \beta(\mathcal{C})$.

**Theorem 2** *Let $D = (V, A)$ be a strong digraph with a coherent cyclic order $\mathcal{C}$. The minimal $i_{\mathcal{C}}(\mathcal{S})$, where $\mathcal{S}$ is a set of circuits which covers the arc set of $D$, is equal to $\beta(\mathcal{C})$.*

**Proof.** If $\mathcal{S}$ is a set of circuits which spans the arcs of $D$, every backward arc in any enumeration of $\mathcal{C}$ must be in a circuit of $\mathcal{S}$. In particular the inequality $i_{\mathcal{C}}(\mathcal{S}) \geq \beta(\mathcal{C})$ clearly holds. Let $D'$ be the subdivision of $D$, i.e. the digraph with vertex set $V \cup A$ and arc set $\{(v,e) : v$ is the head of $e$ and $e \in A\} \bigcup \{(e,v) : v$ is the tail of $e$ and $e \in A\}$. There is a one-to-one correspondence $\phi$ between the circuits of $D'$ and the circuits of $D$. Let $E = v_1, \dots, v_n$ be an enumeration of $\mathcal{C}$ with backward arc set $\{e_1, \dots, e_k\}$, where $k = \beta(\mathcal{C})$. Consider the enumeration $E'$ of $D'$ given by

$$E' := e_1, \dots, e_k, v_1, f_1^1, \dots, f_{n_1}^1, v_2, f_1^2, \dots, f_{n_2}^2, \dots, v_n$$

where $\{f_1^i, \dots, f_{n_i}^i\}$ is the set of forward arcs in $E$ with head $v_i$. Let $\mathcal{C}'$ be the cyclic order of $E'$. The index in $\mathcal{C}'$ of a circuit $C$ of $D'$ is equal to the index in $\mathcal{C}$ of $\phi(C)$, thus $\mathcal{C}'$ is coherent. Let $F'$ be any enumeration of $\mathcal{C}'$. We denote by $F$ the enumeration induced by $F'$ on $V$. Since $\mathcal{C}'$ is coherent, if $e := xy$ is a forward arc of $F$, $xey$ is a forward path of $F'$. In particular, $F$ belongs to $\mathcal{C}$ (since one cannot switch $x$ and $y$ in $F'$). Moreover, if $e := xy$ is a backward arc of $F$, exactly one of $xe$ or $ey$ is a backward arc of $F'$. Thus, we have $\beta(\mathcal{C}') \leq \beta(\mathcal{C})$. By Theorem 1, the vertex set of $D'$ is spanned by a set of circuits $\mathcal{S}'$ with $i_{\mathcal{C}'}(\mathcal{S}') \leq \alpha(\mathcal{C}') \leq \beta(\mathcal{C}') \leq \beta(\mathcal{C})$. To conclude, observe that $\mathcal{S} := \{\phi(C), C \in \mathcal{S}'\}$ is a set of circuits which covers the arc set of $D$ and verify $i_{\mathcal{C}}(\mathcal{S}) = i_{\mathcal{C}'}(\mathcal{S}') \leq \beta(\mathcal{C})$. □

## 4 Longest circuit versus minimum cyclic coloration.

In this section, we present a third min-max theorem which consists of a fractional version of a theorem of J.A. Bondy ([5]). Our proof is similar to the classical proof on the circular chromatic number in the non-oriented case, see X. Zhu ([12]) for a survey.
The *cyclic chromatic number* of a coherent cyclic order $\mathcal{C}$ of a strong digraph $D$, denoted by $\chi(\mathcal{C})$, is the minimum $k$ for which there exists an enumeration $E = v_1, \dots, v_{i_1}, v_{i_1+1}, \dots, v_{i_2}, v_{i_2+1}, \dots, v_{i_k}$ of $\mathcal{C}$ for which $v_{i_j+1}, \dots, v_{i_{j+1}}$ is a stable set for all $j = 0, \dots, k-1$ (with $i_0 := 0$).
Under the same hypothesis, the *circular chromatic number* of $\mathcal{C}$, denoted by $\chi_c(\mathcal{C})$ is the infimum of the numbers $r \geq 1$ for which $\mathcal{C}$ admits an $r$-circular coloration. A mapping $f : V \to [0, r[$ is called an *$r$-circular coloration* if $f$ verifies:

**1)** If $x$ and $y$ are linked in $D$, then $1 \leq |f(x) - f(y)| \leq r - 1$.
**2)** If $0 \leq f(v_1) \leq f(v_2) \leq \dots \leq f(v_n) < r$, then $v_1, \dots, v_n$ must be an enumeration of $\mathcal{C}$. Such an enumeration is called *related* to $f$.

As usual, it is convenient to represent such an application as a mapping from $V$ into a circle of the euclidean plane with circumference $r$. Condition **1)** asserts then that two linked vertices have distance at least 1 on this circle. And by condition **2)**, the vertices of $D$ are placed on the circle according to an enumeration of the cyclic order $\mathcal{C}$. By compactness of this representation, the infimum used in the definition of $\chi_c$ is a minimum, that is to say that there exists a $\chi_c(\mathcal{C})$-circular coloration of $\mathcal{C}$.
Note that the enumeration given by **2)** is possibly not unique. Indeed, two vertices of $V$ may have the same image by $f$. In this case, these two vertices are not linked in $D$ (because of **1)**) and so, the two enumerations are equivalent. Moreover, two enumerations related to $f$ have same sets of forward arcs and backward arcs.

**Lemma 3** *For $D$ a strong digraph and $\mathcal{C}$ a coherent cyclic order of $D$, we have $\lceil \chi_c(\mathcal{C}) \rceil = \chi(\mathcal{C})$.*

**Proof.** First, if $E = v_1, \dots, v_{i_1}, v_{i_1+1}, \dots, v_{i_2}, v_{i_2+1}, \dots, v_{i_k}$ is an enumeration of $\mathcal{C}$ which realizes $\chi(\mathcal{C})$, we can easily check that $f : V \to [0, k[$ defined by $f(v_p) = j$ if $i_j + 1 \leq p \leq i_{j+1}$ for $0 \leq j \leq k-1$ with $i_0 = 0$ is a $k$-circular coloration of $\mathcal{C}$. So, we get the inequality $\chi_c(\mathcal{C}) \leq \chi(\mathcal{C})$.
Conversely, the existence of an enumeration of $\mathcal{C}$ which realizes $\chi_c(\mathcal{C})$ will achieve the bound. Indeed, fix

$f$ an $r$-circular coloration of $\mathcal{C}$ with $r = \chi_c(\mathcal{C})$ and $E$ an enumeration of $\mathcal{C}$ related to $f$. By definition of $f$, for every integer $0 \leq j \leq \lceil r \rceil - 1$, the set $\{v \in V : f(v) \in [j, j+1[\}$ is a stable set of $D$ and forms an interval on $E$, so we get $\chi(\mathcal{C}) \leq \lceil \chi_c(\mathcal{C}) \rceil$. □

The following Lemma gives a criterion to decide whether an $r$-circular coloration $f$ is best possible or not. We define an auxiliary digraph $D_f$ with vertex set V and arc set $\{xy \in E(D) : f(y) - f(x) = 1 \text{ or } f(x) - f(y) = r - 1\}$. Observe that the arcs $xy$ of $D_f$ with $f(y) - f(x) = 1$ (resp. $f(x) - f(y) = r - 1$) are forward (resp. backward) in any enumeration related to $f$.

**Lemma 4** *If $f$ is an $r$-circular coloration of $\mathcal{C}$ with $r > 2$ for which $D_f$ is an acyclic digraph, then we can provide a real number $r' < r$ such that $\mathcal{C}$ admits an $r'$-circular coloration.*

**Proof.** First of all, if a vertex $x$ of $D$ has an out-neighbour $y$ with $f(x) - f(y) = 1$, by property **1)** of $f$ and coherence of $\mathcal{C}$, the arc $yx$ must be also in $D$, and similarly if $x$ has an out-neighbour $y$ with $f(y) - f(x) = r - 1$, the arc $yx$ must be in $D$. So, a vertex $x$ with in-degree 0 (resp. out-degree 0) in $D_f$ has no neighbour $z$ with $f(x) - f(z) = 1$ modulo $r$ (resp. $f(z) - f(x) = 1$ modulo $r$). Then, if $E(D_f) = \emptyset$, it is easy to provide an $r'$-circular coloration $f'$ of $\mathcal{C}$ with $r' < r$. Just multiply $f$ by a factor $1 - \epsilon$ with $\epsilon > 0$ and $\epsilon$ small enough.
Now, amongst the $r$-circular colorations $f$ of $\mathcal{C}$ for which $D_f$ is acyclic, choose one with minimal number of arcs for $D_f$. Assume that $E(D_f) \neq \emptyset$. We can choose a vertex $x$ of $D_f$ with in-degree 0 and out-degree at least 1. Denote by $y_1, \ldots, y_p$ the out-neighbours of $x$ in $D_f$, we have for all $i$, $f(y_i) = f(x) + 1$ modulo $r$. By definition of $f$, $x$ has no neighbour $z$ such that $f(x) - f(z) < 1$ or $f(z) - f(x) > r - 1$ and, moreover, since $x$ has in-degree 0 in $D_f$, by the previous remark, $x$ has no neighbour $z$ with $f(x) - f(z) = 1$ or $f(z) - f(x) = r - 1$. Observe that none of the $y_i$ verifies this, because $r > 2$. So, we can provide an $r$-circular coloration $f'$ derived from $f$ just by changing the value of $f(x)$: choose $f'(x) = f(x) - \epsilon$ modulo $r$ with $\epsilon > 0$ and such that no neighbour of $x$ has an image by $f$ in $[f'(x) - 1, f'(x)] \cup [r - 1 + f'(x), r[$. We check that $E(D_{f'}) = E(D_f) \setminus \{xy_i : i = 1, \ldots, p\}$, which contradicts the choice of $f$.
So, $E(D_f) = \emptyset$ and we provide an $r'$-circular coloration of $\mathcal{C}$ with $r' < r$ as previously. □

Finally, we can state a third min-max theorem about cyclic orders. For this, we define, for a fixed cyclic order $\mathcal{C}$, the *cyclic length* of a circuit $C$ of $D$, denoted by $l_{\mathcal{C}}(C)$, as the number of vertices of $C$ in $D$, divided by the index of $C$ in $\mathcal{C}$.

**Theorem 3** *Let $\mathcal{C}$ be a coherent cyclic order of a strong digraph $D$. The maximal $l_{\mathcal{C}}(C)$, where $C$ is a circuit of $D$ is equal to $\chi_c(\mathcal{C})$.*

**Proof.** Consider an $r$-circular coloration $f$ of $\mathcal{C}$ with $r = \chi_c(\mathcal{C})$ and $E = v_1, \ldots, v_n$ an enumeration of $\mathcal{C}$ related to $f$. For a circuit $C$ in $D$, we compute the length $l$ of the image of $C$ by $f$:

$$l := \sum_{\substack{xy \in E(C) \\ xy \text{ forward in } E}} (f(y) - f(x)) + \sum_{\substack{xy \in E(C) \\ xy \text{ backward in } E}} (r + f(y) - f(x))$$

A straighforward simplification of the sum gives $l = r.i_{\mathcal{C}}(C)$. Furthermore, condition **1)** of the definition of $f$ implies that $f(y) - f(x) \geq 1$ if $xy \in E(C)$ and $xy$ is forward in $E$ (i.e. $f(x) < f(y)$) and $r + f(y) - f(x) \geq 1$ if $xy \in E(C)$ and $xy$ is backward in $E$ (i.e. $f(y) < f(x)$). So, we have $c = r.i_{\mathcal{C}}(C) \geq l(C)$, hence $r \geq l_{\mathcal{C}}(C)$, and the inequality $\chi_c(\mathcal{C}) \geq \max\{l_{\mathcal{C}}(C) : C \text{ circuit of } D\}$ holds.

To get the equality, we have to find a circuit $C$ of $D$ such that $l_{\mathcal{C}}(C) = r$.
First of all, since $\mathcal{C}$ is coherent, it has a circuit of index 1 and, then the cyclic length of this circuit is greater or equal to 2. Thus, the previous inequality gives $r \geq 2$ and so, states the case $r = 2$. From now on, assume that $r > 2$, Lemma 4 asserts that there exists a circuit $C$ in the digraph $D_f$. So,

now, the inequalities provided in the direct sens of the proof are equalities: the image of every arc of $C$ by $f$ has length 1 if the arc is forward or $r-1$ if the arc is backward. So, we have $l = l(C)$, and $l_{\mathcal{C}}(C) = l(C)/i_{\mathcal{C}}(C) = r$, which achieves the bound. □

A corollary of Theorem 3 is an earlier result of J.A. Bondy, known since 1976. The *chromatic number* of a digraph $D$, denoted by $\chi(D)$, is the minimal number $k$ such that the vertices of $D$ admit a partition into $k$ stable sets of $D$. Clearly, $\chi(D) \leq \chi_c(\mathcal{C})$.

**Corollary 3.1** (Bondy [5]) *Every strong digraph $D$ has a circuit with length at least $\chi(D)$.*

**Proof.** Consider a coherent cyclic order $\mathcal{C}$ for $D$ and apply Theorem 3 to provide a circuit $C$ with $l_{\mathcal{C}}(C) = \chi_c(\mathcal{C})$. Since by Lemma 3 $\lceil l_{\mathcal{C}}(C) \rceil = \lceil \chi_c(\mathcal{C}) \rceil = \chi(\mathcal{C})$, we get $\chi(D) \leq \chi(\mathcal{C}) = \lceil l_{\mathcal{C}}(C) \rceil \leq l(C)$. □

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