The Categorical Product of two 5-chromatic digraphs can be 3-chromatic.

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Abstract

We provide an example of a 5-chromatic oriented graph D such that the categorical product of D and TT_5 is 3-chromatic, where TT_5 is the transitive tournament on 5 vertices.

The transitive tournament TT_5 is the digraph on the vertex set $\{1,2,3,4,5\}$ and edge set $\{(i,j) : i < j\}$. Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be two digraphs. The *categorical product* of D_1 and D_2 is the digraph $D_1 \times D_2 = (V_1 \times V_2, E)$ where $E = \{((x,y),(z,t)) : (x,z) \in E_1 \text{ and } (y,t) \in E_2\}$. Motivated by Hedetniemi's conjecture (see Sauer [5] for a survey), the following function was defined:

$$\delta(n) := \min\{\chi(D_1 \times D_2) : \chi(D_1) = \chi(D_2) = n \text{ and } D_1, D_2 \in \mathcal{D}\}$$

where $\mathcal{D}$ is the class of finite digraphs and $\chi(D)$ is the chromatic number of D. The first striking result concerning this function was proved by Poljak and Rödl [4]: either δ is bounded by 4 or it tends to infinity. Later on, Poljak [3], and independently Zhu [6] proved that 4 can be replaced by 3. When δ is restricted to undirected graphs, Hedetniemi [2] conjectured that $\delta(n) = n$. El-Zahar and Sauer proved in [1] that $\delta(4) = 4$ for undirected graphs. For directed graphs, it is known that $\delta(4) = \delta(3) = 3$.

Theorem 1 $\delta(5) = 3$

Proof. The oriented graph D depicted in Fig. 1 is 5-chromatic. Observe for this that the twelve vertices with degree 8 induce a graph G which is uniquely 4 colorable up to a rotation. Indeed the stability of G is 3 and the color classes must consist of three consecutive vertices in the cyclic order. Now, one of the three vertices with degree 4 has to have one neighbour in each of the four color classes. Thus D is not 4-colorable. The 3-coloration f of $D \times TT_5$ is given by the label of the vertices of D: when a vertex x of D is labelled by c^3a^2, for instance, we mean that $f(x,1) = c, f(x,2) = c, f(x,3) = c, f(x,4) = a, f(x,5) = a$. To see that f is a good coloration, note that when there is an edge from x to y (both seen as words of length 5 on the alphabet $\{a,b,c\}$), the i$^{\text{th}}$ letter of x is not equal to the j$^{\text{th}}$ letter of y for every $j > i$. $\square$

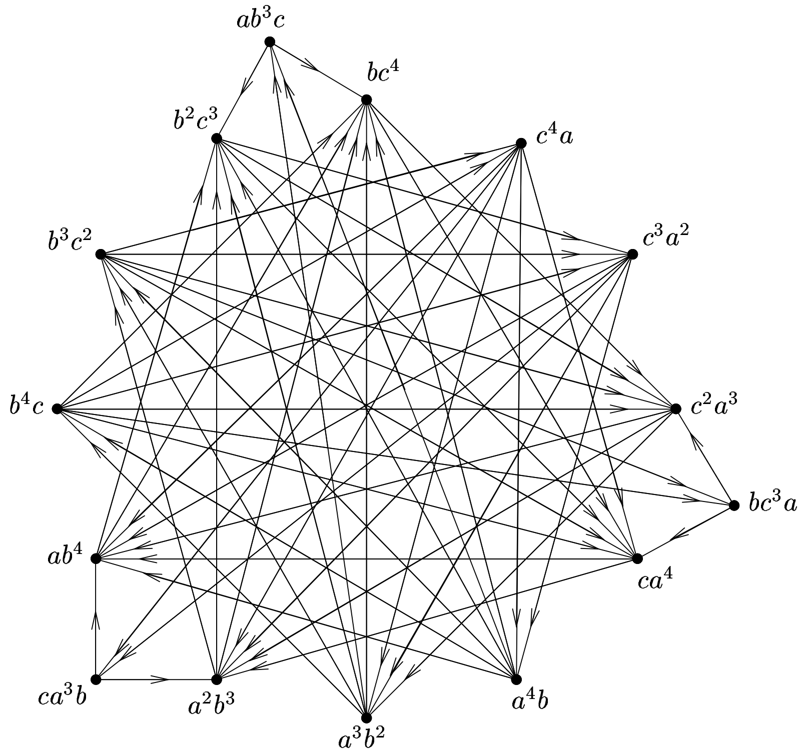

Fig. 1

This example is a subgraph of the graph K_3^{TT5} which has chromatic number 5 (this is not hard to check). A short analysis also gives that $\chi(K_3^{TT_4}) = 6$ and $\chi(K_3^{TT_k}) = 4$ for all $k > 5$. We were not able to find an infinite family of 5-chromatic digraphs (D_1, D_2) such that $\chi(D_1 \times D_2) = 3$.

References

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