- Some problems in graph theory and graphs algorithmic theory -

- Habilitation à diriger des recherches -

Stéphane Bessy
Université Montpellier 2 - CNRS, LIRMM.
e-mail: Stephane.Bessy@lirmm.fr.

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Contents

Foreword

This document is a long abstract of my research work, concerning graph theory and algorithms on graph. It summarizes some results, gives ideas of the proof for some of them and presents the context of the different topics together with some interesting open questions connected to them. This is an overview of ten selected papers which have been published in international journals or are submitted and which are included in the annex. This document is organized as follow: the first part precises the notations used in the rest of the paper; the second part deals with some problems on cycles in digraphs, a topic I am working on for almost 10

years; the third part is an overview of two graph coloring problems and one problem on structures in colored graphs; finally the fourth part focus on some results in algorithmic graph theory, mainly in parametrized complexity. I mainly worked in this last field with A. Perez who I co-supervised with C. Paul during his PhD Thesis.
To conclude, I would like to mention that this work is the result of different collaborations and each result is then a collective work, with me as common link. Doing research in graph theory is for me a great pleasure, and a job, and meeting people from various place to work with them is an also great pleasure. I would like to thank them for the nice moments we spent working together: J. Bang-Jensen, E. Birmelé, F.V. Fomin, S. Gaspers, F. Havet, C. Lepelletier, N. Lichiardopol, C. Paul, A. Perez, S. Saurabh, J.-S. Sereni and S. Thomassé.

1 Introduction

Almost all the definitions given below are standard and can be found in classical books on Graph Theory (see [47], [30] or [8]) or Parametrized Complexity Theory (see [55], [63] or [105]). We give them to precise the notations used in this document.

1.1 Basic definitions on graphs

1.1.1 Graphs

For a set X, we denote by $[X]^2$ the set of 2-element subsets of X. A *graph* G is a pair $(V(G), E(G))$ consisting of a finite set $V(G)$, called the *vertex set* of G, and a set $E(G)$, subset of $[V(G)]^2$, called the *edge set* of G. Classically, the cardinality of $V(G)$ and $E(G)$ are respectively denoted by $n(G)$ and $m(G)$. For notational simplicity, we write uv an unordered pair $\{u, v\}$ of $E(G)$. Two vertices x and y which belong to an edge e are *adjacent*, and x and y are the *ends* of e. Furthermore, we say that e is *incident* to x and y.
The set of vertices which are adjacent to a specified vertex x is the *neighborhood* of x and is denoted by $N_G(x)$. So, we call a *neighbor* of x a vertex which is adjacent to it. The *degree* of a vertex x, denoted by $d_G(x)$, is the cardinality of its neighborhood. Finally, when two adjacent vertices x and y have the same neighborhood in $V \setminus \{x, y\}$, we say that x and y are *true twins*. When no confusion can occur, we will forget the reference to the background graph in all the previous notations.
A graph $H = (V(H), E(H))$ is a *subgraph* of a graph $G = (V(G), E(G))$ if we have $V(H) \subseteq V(G)$ and $E(H) \subseteq V(G)$. If H is a subgraph of G with $V(H) = V(G)$, we say that H is a *spanning* (or *covering*) subgraph of G. And, if H is a subgraph of G with $E(H) = E(G) \cap [V(H)]^2$, we say that H is an *induced* subgraph of G. For X a subset of the vertex set of G, the *induced subgraph of G on X*, denoted by $G[X]$, is the induced subgraph of G which has X as vertex set. We denote also by $G \setminus X$ the induced subgraph of G on $V(G) \setminus X$. And, if F is a subset of the edge-set of G, we denote by $G - F$ the subgraph of G with vertex set $V(G)$ and edge set $E(G) \setminus F$. Finally, we say that two graphs G and H are *isomorphic* if there exists a bijection f from $V(G)$ to $V(H)$ such that for every vertices x and y of G, $xy \in E(G)$ if, and only if, $f(x)f(y) \in E(H)$. And, an *homomorphism* from G to H is a mapping f from $V(G)$ to $V(H)$ such that if xy is an edge of G, then $f(x)f(y)$ is an edge of H.

In Section 4.2, we will deal with multi-graphs (i.e. graphs with a multiset for edge set), but anywhere else in this document, the considered graphs are simple.

1.1.2 Some special graphs

The following definitions deal with some remarkable subgraphs. The *complete graph* on n vertices, denoted by K_n is the graph on vertex set $V(K_n) = \{v_1, \ldots, v_n\}$, and with edge set $[V(K_n)]^2$, meaning that K_n contains all the possible edges on its vertex set. A *clique* of a graph G is a subset of its vertex set which induces on G a graph isomorphic to a complete graph. Similarly, the *empty graph* on n vertices (which is not totally empty) is the graph on n vertices and with an empty edge set. An *independent set* of a graph G is a subset of its vertex set which induces on G a graph isomorphic to an empty graph. The *path graph* on k vertices, denoted by P_k is the graph on vertex set $V(P_k) = \{v_1, \ldots, v_k\}$, and with edge set $\{v_iv_{i+1} \ : \ i = 1, \ldots, k-1\}$. The vertices v_1 and v_k are called the *ends* of P_k. A *path* of a graph G is a subgraph of G which is isomorphic to a path graph. Finally, the *cycle graph* on k vertices, denoted by C_k is the graph on vertex set $\{v_1, \ldots, v_k\}$, and with edge set $\{v_i v_{i+1} \ : \ i = 1, \ldots, k-1\} \cup \{v_1 v_k\}$. A *cycle* of a graph G is a subgraph of G which is isomorphic to a cycle graph. Sometimes, we will use the term *k-cycle* to precise that the considered cycle has k vertices. A *hamiltonian* graph is a graph which admits a spanning cycle, an *acyclic graph* is a graph which contains no cycle, and a *chordal graph* is a graph with no induced cycle of size more than three. Finally, a *matching* in a graph is a set of pairwise disjoint edges of this graph.

Now, through these structures, we define some properties of graphs. First, a graph is *connected* if for every pair of vertices x and y of G, there exists a path in G with ends x and y. A *tree* is a connected graph without cycle. It is well known that a graph is connected if, and only if, it contains a spanning tree. A graph is *bipartite* if its vertex set admits a partition into two independent sets. The *complete bipartite* $K_{p,q}$ is the graph on vertex set $\{v_1, \ldots, v_p, w_1, \ldots, w_q\}$, and with edge set $\{v_i w_j \ : \ i = 1, \ldots, p \text{ and } j = 1, \ldots, q\}$. Finally, we say that a graph G is *planar* if there exists a *plane representation* of G, i.e. a drawing of G in the plane such that its edges intersect only at their ends.

1.2 Directed graphs

A *directed graph* (or *digraph*) D is a pair $(V(D), E(D))$ consisting of a finite set $V(D)$, also named the *vertex set* of D, and a subset $A(D)$ of $V(D) \times V(D)$, named the *arc set* of D. For simplicity, we also denote by xy an arc (x, y) of D, but this time the order in the notation matters. We say that x is the *tail* of the arc xy and y its *head*. We obtain then a different notion of neighborhood than in the non-oriented case. For a vertex x of D, the *out-neighborhood* (resp. *in-neighborhood*) of x, denoted by $N_D^+(x)$ (resp. $N_D^-(X)$) is the set of all vertices y in $V - x$ such that xy (resp. yx) is an arc of $A(D)$. The *out-degree* (resp. *in-degree*) of a vertex x, denoted by $d_D^+(x)$ (resp. $d_D^-(x)$) is the cardinality of its out-neighborhood (resp. in-neighborhood). The notions of sub-digraph, induced and spanning sub-digraph, homomorphism and isomorphism for digraphs are similar to those from graphs. For digraphs, paths and cycles are always directed. Namely, the *(directed) path* on k vertices has vertex set $\{v_1, \ldots, v_k\}$, and arc set $\{v_i v_{i+1} \ : \ i = 1, \ldots, k-1\}$. The vertex v_1 is the *beginning* of the path and v_k is its *end*. Similarly, the *(directed) cycle* (or *circuit*) on k vertices has vertex set $\{v_1, \ldots, v_k\}$, and arc set $\{v_i v_{i+1} \ : \ i = 1, \ldots, k-1\} \cup \{v_k v_1\}$. An *acyclic digraph* is a digraph which contains no (directed) cycle. A digraph $D = (V, A)$ is *strongly connected* (or just *strong*) if there exists a path from x to y in D for every choice of distinct vertices x and y of D. A *feedback-vertex-set* (resp. *feedback-arc-set*) is a set X of vertices (resp. arcs) of D such that $D \setminus X$ (resp. $D - X$) is acyclic. The *underlying graph* of a digraph D, denoted by $UG(D)$, is the (non-oriented) graph obtained from D by suppressing the orientation of each arc and deleting multiple edges.
To conclude on directed graph, a *tournament* is an orientation of a complete graph, that is a digraph D such that for every pair $\{x, y\}$ of distinct vertices of D either $xy \in A(D)$ or $yx \in A(D)$, but not both. Finally, the *complete digraph*, denoted by $K_n^\star$, is the digraph on n vertices containing all the possible arcs, i.e. obtained from the complete graph by replacing each edge by a directed cycle of size two.

1.3 Some graphs invariants

The *independence number* of a graph (or a digraph) G is the size of a largest independent set of G. We denote it by $\alpha(G)$.

A *vertex-cut* in a graph G is a subset X of vertices of G such that $G \setminus X$ is not connected. A graph G is *k-vertex-connected* if all its vertex-cut have size at least k, and the minimal size of one of its vertex-cut is the *vertex-connectivity* of G and it is denoted by $\kappa(G)$. This definition extends to *edge-cut*, which is a subset F of the edge-set of G such that $G - F$ is not connected. The *edge-connectivity*, defined similarly than the vertex-connectivity, of a graph G is denoted by $\lambda(G)$. These definitions extend also to digraphs, where the notion of strong connectivity stands for connectivity. Remark that, given any two vertices x and y in a graph, if there exists p paths from x to y in G, vertex-disjoint except in their ends, x and y, it is not possible to find a vertex-cut of G with size less than p that separate x from y. In fact, Menger's Theorem states that this fact characterizes the vertex-connectivity of a graph.

Theorem 1 (*'Menger's Theorem'*, K. Menger, 1927, [103])**.** *Let G be graph. The vertex-connectivity of G is p if, and only if, for every pair of vertices x and y of G, there exists p paths from x to y in G, vertex-disjoint* *except in their ends.*

This theorem also holds for edge-connectivity and oriented case (where the paths are oriented paths).

Finally, we define the general notion of proper coloring of a graph. A *k-coloring* of a graph G is a mapping c from $V(G)$ to the set $\{1,\dots,k\}$ such that if xy is an edge of G, then the values $c(x)$ and $c(y)$ are distinct. Equivalently, a k-coloring of G is a vertex-partition of G into k independent sets. The *chromatic number* of G, denoted by $\chi(G)$ is the minimum number k such that G admits a k-coloring. Coloring edges of graph leads to similar definitions. A *k-edge-coloring* of G is a mapping c' from $E(G)$ to the set $\{1,\dots,k\}$ such that if two edges e and f have a common extremity, then the values $c'(e)$ and $c'(f)$ are distinct. Equivalently, a k-edge-coloring of G is an edge-partition of G into k matchings. Similarly to the vertex case, the *edge-chromatic number* of G, denoted by $\chi'(G)$ is the minimum number k such that G admits a k-edge-coloring.

1.4 Algorithmic basics

We refer to [45] for general definition on polynomially solvable problems and *NP*-complete problems. For what we need, we focus on algorithms on graphs only, and will give the definition of *FPT* algorithms in this context.

When a problem turns out to be *NP*-complete, a lot of algorithmic tools have been developed to find a acceptable solution to it, for instance, approximation algorithms, randomized algorithms, exact algorithms with exponential time... An *FPT* algorithm can be viewed as a member of this late class. The principle of such an algorithm is to contained the exponential explosion to a special parameter. Namely, a problem *parameterized* by some integer k (i.e. its input is a graph G and an integer k) is said to be *fixed-parameter tractable* (*FPT* for short) whenever it can be solved in time $f(k)\cdot n^c$ for some constant $c>0$. As one of the most powerful technique to design fixed-parameter algorithms, *kernelization algorithms* have been extensively studied in the last decade (see [26] for a survey). A *kernelization algorithm* is a polynomial-time algorithm (called a series of *reduction rules*) that given an instance (G,k) of a parameterized problem P computes an instance (G',k') of P such that (i) (G,k) is a Yes-instance if and only if (G',k') is a Yes-instance and (ii) $|G'|\leq h(k)$ for some computable function $h()$ and $k'\leq k$. The instance (G',k') is called a *kernel* of P. Kernelization can be viewed as a sort of pre-processing algorithm that reduces the size of any instance. We say that (G',k') is a polynomial kernel if the function $h()$ is a polynomial. It is well-known that a parameterized problem is *FPT* if and only if it has a kernelization algorithm [105]. But the proof of this equivalence provides standard kernels of super-polynomial size (in the size of $f(k)$, precisely). So, to design efficient fixed-parameter algorithms, a kernel of small size - polynomial (or even linear) in k - is highly desirable. However, recent results give evidence that not every parameterized problem admits a polynomial kernel, unless something very unlikely happens in the polynomial hierarchy, see [27]. On the positive side, notable kernelization results include a $2k$ kernel for Vertex Cover [46], a $4k^2$ kernel for Feedback-Vertex-Set [126] and a $2k$ kernel for Cluster Editing [41].

2 Problems on circuits in digraphs

In this section, we are concerned with digraphs and every concept discussed deals with directed graphs. We are interested in finding cycles in strongly connected digraphs. As trees is a cornerstone for connectivity in graphs, cycles have this central place for strong digraph. Cycles are the simplest oriented structure in which starting from any vertex it is possible to reach any other vertex.

In a strong digraph, every arc is contained in a cycle and so, there exists a family of cycles which union is the whole digraph. Problems considered here deal with finding a family of cycles in a strong digraph D which satisfies certain properties. For instance, if we want a minimum (in cardinality) family of cycles which union spans D, we obtain a *covering problem* (Section 2.1). One of the tools for this problem is *cyclic orders* for strong digraphs, which we develop with S. Thomassé (Section 2.2). Another important class of problems, the *packing problems*, deals with finding a maximum family of disjoint cycles in a digraph (Section 2.3). Some results which appear in the two first sections were already mentioned in my PhD Thesis. I decide to present them because they have led to some tools and problems on which I am still working.

2.1 Covering by directed cycles[1]

Let $D = (V, E)$ be a strong digraph. We are mainly concerned here in finding a family $\mathcal{F}$ of cycles of D which union covers all the vertices of D. The most natural problem in this context is to ask for a family $\mathcal{F}$ with minimal cardinality. For long, this problem is known to be easy on tournaments, as stated by a result of P. Camion.

Theorem 2 (*'Camion's Theorem'*, P. Camion, 1959, [35]). *Every strongly connected tournament is hamiltonian.*

For general digraphs, there is no hope to have such an exact value or even a polynomial time process to compute the minimum number of cycles needed in $\mathcal{F}$. Indeed, as this problem contains the HAMILTONIAN CYCLE PROBLEM for digraphs, it turns out to be *NP*-complete, and so we ask for upper bounds on the cardinality of $\mathcal{F}$.
Let us mention the closely related problem consisting in finding a family of paths, instead of cycles, which union covers all the vertices of a digraph. A well-known bound on the cardinality of such a family is given by a Theorem of T. Gallai and A. Milgram, which provides not only a covering by paths, but a partition of the digraph into paths.

Theorem 3 (*'paths partition'*, T. Gallai and A. Milgram, 1960, [67]). *Every digraph D admits a vertex-partition into at most $\alpha(D)$ paths.*

Thank to the above result on paths and Camion's theorem on tournament, T. Gallai conjectured in 1964 that the independence number could also be an upper bound for the minimum number of cycles which cover the vertices of a digraph [65]. During my PhD thesis, with S. Thomassé, we proved this conjecture and obtained the following.[2]

Theorem 4 (S. Bessy, S. Thomassé, 2003, [22]). *Every strong digraph D contains a family of at most $\alpha(D)$ cycles which union covers the vertices of D.*

In 1995, Gallai's conjecture was refined by A. Bondy [29] (stating explicitly a remark of C.C. Chen and P. Manalastas [40]) who asked for some control on the cycles. More precisely: a strong digraph $D = (V, E)$ is a *k-handle* if $k = |E| - |V| + 1$ (a 0-handle is simply a single vertex). A *handle* is a directed path $H := x_1, \dots, x_l$ in which we allow $x_1 = x_l$. The vertices x_1 and x_l are the *extremities* of the handle H, and its other vertices are its *internal* vertices. For a subdigraph H of D, an *H-handle* is a handle of D with its extremities in $V(H)$ and its internal vertices disjoint from H. Finally, a *handle basis* of D (or *ear decomposition*, see [8]) is a sequence $H_0, H_1, \dots, H_k$ of handles of D such that H_0 is a single vertex, H_i is a $(\cup\{H_j : j < i\})$-handle for all $i = 1, \dots, k$ and $D = \cup\{H_i : i = 0, \dots, k\}$. Clearly, a digraph has a handle basis $H_0, \dots, H_k$ if and only if D is a k-handle. In this context the conjecture of A. Bondy is the following.

Conjecture 5 (*'Bondy's Conjecture'*, A. Bondy, 1995, [29]). The vertices of every strong digraph D can be covered by the disjoint union of some k_i-handles, where $k_i > 0$ for all i, and the sum of the k_i being at most $\alpha(D)$.

As it is possible to cover every k-handle by k cycles, Bondy's conjecture is stronger than the result of Theorem 4. For $k = 1$, Bondy's conjecture is true by Camion's Theorem. Furthermore, for $k = 2$, it has been solved by C.C. Chen and P. Manalastas [40], and by S. Thomassé for $k = 3$ [125]. The techniques used in the proof of Theorem 4 seem to be useless to tackle Bondy's Conjecture. However, using a different approach, with S. Thomassé, we obtained a result closely related to this conjecture.

Theorem 6 (S. Bessy, S. Thomassé, 2003, [21]). *Every strong digraph D is spanned by a k-handle, with $k \leq 2\alpha(D) - 1$.*

[1] This subsection and the next one are linked with the paper: S. Bessy and S. Thomassé, Spanning a strong digraph with alpha cyles: a conjecture of gallai. *Combinatorica*, 27(6):659–667, 2007.

[2] I highlight the results appearing in the papers which form my habilitation.

In other words, every strong digraph D admits a spanning strong subdigraph with at most $n+2\alpha(D)-2$ arcs. The problem of finding such a subdigraph with a minimum number of arcs is a classical problem in graph theory, named the MSSS PROBLEM (for minimal strong spanning subdigraph). This problem is also *NP*-complete (it also contains the HAMILTONIAN CYCLE PROBLEM) and the best known approximation algorithm, found by A. Vetta in 2001 [131], achieves a $\frac{3}{2}$ factor of approximation. A nice question is to look at what happens on some particular classes of digraphs, and particularly the following one.

Problem 7. Is there an approximation algorithm for the MSSS PROBLEM with a better factor than $\frac{3}{2}$? What about if we restrict the instances to the class of planar strong digraph?

2.2 Cyclic order of strong digraphs

In this section is presented a tool that we developed in order to prove Theorem 4. The notion of cyclic order allows, in some sense, cyclic statements of classical results on paths a digraphs.

Let D be a strong digraph on vertex set V. If $E = v_1, \ldots, v_n$ is an enumeration of V, for any $k \in \{2, \ldots, n\}$, the enumeration $v_k, \ldots, v_n, v_1, \ldots v_{k-1}$ is obtained by *rolling* E. Two enumerations of V are *equivalent* if we can pass from one to the other by a sequence of the following operations: rolling and exchanging two consecutive but not adjacent vertices. The classes of this equivalence relation are called the *cyclic orders* of D. Roughly speaking, a cyclic order is a class of enumerations of the vertices on a circle, where one stay in the class while switching consecutive vertices which are not joined by an arc. We fix an enumeration $E = v_1, \ldots, v_n$ of V, the following definitions are understood with respect to E. An arc v_iv_j of D is a *forward arc* if $i < j$, otherwise it is a *backward arc*. With respect to E, the *index* of a cycle C of D is the number of backward arcs containing in C, we denote it by $i_E(C)$. This corresponds to the winding number of the cycle. Observe that $i_E(C) = i_{E'}(C)$ if E and E' are equivalent. Consequently, the index of a cycle is invariant in a given cyclic order $\mathcal{O}$ and we denote it by $i_{\mathcal{O}}(C)$. A circuit is *simple* if it has index one. A cyclic order $\mathcal{O}$ is *coherent* if every arc of D is contained in a simple circuit.
The following lemma states the existence of coherent cyclic orders. It is proved by considering a cyclic order which minimizes the sum of the cyclic indices of all the cycles of D.

Lemma 8 (D.E. Knuth, 1974, [91] / S. Bessy, S. Thomassé, 2007, [22])**.** *Every strong digraph has a coherent cyclic order.*

As mentioned by A. Bondy in [30], this result was found originally by D.E. Knuth in 1974 [91] (we ignored it when we settled Lemma 8), in a different context, and no link with Gallai's conjecture have been done.

Now, we describe two min-max relations in context cyclic orders. As previously said, cyclic orders can be understood as a cyclic version of classical theorems on paths in digraphs. First, we give a cyclic version of Gallai-Milgram's paths partition Theorem. Given $\mathcal{O}$ a cyclic order of a strong digraph D, we denote by $\alpha(\mathcal{O})$ the size of a maximum *cyclic independent set* of $\mathcal{O}$, that is an independent set of D which is consecutive in an enumeration of $\mathcal{O}$. The following theorem provides a family of cycles which cover all the vertices of the considered digraph D.

Theorem 9 (S. Bessy, S. Thomassé, 2007, [22])**.** *Let D be a strong digraph with a coherent cyclic order $\mathcal{O}$. The minimal $\sum_{C\in\mathcal{R}} i_{\mathcal{O}}(C)$, where $\mathcal{R}$ is a spanning set of cycles of D is equal to $\alpha(\mathcal{O})$.*

The proof uses a very basic algorithmic process. Let us briefly explain it. We start by computing greedily a cyclic independent set X of $\mathcal{O}$ and consider an enumeration E of $\mathcal{O}$ where X stands at the beginning of E. Then, in a digraph build from the transitive closure of the acyclic digraph formed by the forward arcs of E, we apply Dilworth's Theorem [48] (a classical version of Gallai-Milgram's Theorem for orders). So, either we find a larger cyclic independent set than X, and we go on the process, or we find a set of $|X|$ paths covering this digraph, and we stop and show that these paths can be turned into cycles covering D.
Remark that, as $\alpha(\mathcal{O})$ is the size of an independent set of D, we find a set of at most $\alpha(D)$ cycles which covers the vertices of D. This gives a proof of Gallai's Conjecture. However, we have no control on the

number of arcs involved in this covering and then no bound on the sum of the k-handles needed to cover D, as asked by Bondy's Conjecture. On the other hand, in the proof of Theorem 9, as previously explained, we obtained a family $\mathcal{F}$ of k cycles, with $k \leq \alpha(D)$, and a set X of k vertices such that the union of the cycles of $\mathcal{F}$ minus X forms an acyclic digraph D'. In D', the cycles of $\mathcal{F}$ become paths. So, the main challenge to attempt resolving Bondy's Conjecture could be the following.

Problem 10. Is it possible to 'uncross' the k paths in D' in order to reduce their length and, then, the total number of arcs involved in the covering?

The second min-max theorem which we found in the field of cyclic order can be viewed as a cyclic version of the Gallai-Roy's Theorem.

Theorem 11 (*'The Gallai-Roy Theorem'*, T. Gallai, 1966, [66] and B. Roy, 1967, [114])**.** *Every digraph D contains a directed path on $\chi(D)$ vertices.*

The cyclic version is the following. Given an enumeration $E = v_1, \dots, v_n$ of the vertices of a digraph D, a *coloring of E* into r colors is a partition of V into r sets $V_1, \cdots, V_r$ such that for every j, V_j are an independent set of D and V_j are consecutive on E (i.e. there exist integers $i_0 = 0 < i_1 < \cdots < i_r = n$ such that $V_j = \{v_{i_{j-1}+1}, \cdots, v_{i_j}\}$ for all $j \in \{1, \dots, r\}$). The *chromatic number of E* is the minimum value r for which there exists a r-coloring of E. For a cyclic order $\mathcal{O}$ of D, the *cyclic chromatic number of $\mathcal{O}$*, denoted by $\chi(\mathcal{O})$ is the minimum value of the chromatic number of an enumeration belonging to $\mathcal{O}$. Finally, for a cycle C of D and a cyclic order $\mathcal{O}$ of D, the *cyclic length* of C is the value $|C|/i_{\mathcal{O}}(C)$. We have the following min-max relation.

Theorem 12 (S. Bessy, S. Thomassé, 2007, [22])**.** *Let $\mathcal{O}$ be a coherent cyclic order of a strong digraph D. The maximal $\lfloor l_{\mathcal{O}}(C) \rfloor$, where C is a circuit of D is equal to $\chi(\mathcal{O})$.*

There even exists a fractional version of this theorem (see [22]). It is similar (but oriented) to the classical result on the circular chromatic number for non-oriented graphs (see the survey of X. Zhu [137], for instance). As a corollary of Theorem 12, we obtain a classical theorem of A. Bondy.

Theorem 13 (A. Bondy, 1976, [28])**.** *Every strong digraph D contains a cycles on at least $\chi(D)$ vertices.*

To conclude this section, remark that we established some results on cyclic orders using graph theoretical tools. However, there exists proofs of these results (and even others) obtained by techniques from linear programming or polyhedral combinatorial optimization (see the work of P. Charbit and A. Sebö [37] and A. Sebö [117]). In particular, using these techniques, A. Sebö gave in [117] a cyclic version of Menger's Theorem (we missed it...). Let $\mathcal{O}$ be a cyclic order of a strong digraph D. A *cyclic feedback-vertex-set* of $\mathcal{O}$ is a set U of vertices of D such that for every cycle C of D, we have $|V(C) \cap U| \geq i_{\mathcal{O}}(C)$. A. Sebö establishes the following.

Theorem 14 (A. Sebö, 2007, [117])**.** *Let $\mathcal{O}$ be a cyclic order of a strong digraph D. The minimum cardinality of a cyclic feedback-vertex-set is equal to the maximum of $\sum_{C \in \mathcal{R}} i_{\mathcal{O}}(C)$, where $\mathcal{R}$ is a set of vertex-disjoint cycles of D.*

A. Sebö gives also in [117] an 'arc version' of this result and weighted analogous of these two statements. Obviously, we can wonder if there exists other theorems on paths which can be turn into a cyclic form.

Problem 15. Is there other results on paths in digraphs which admits a cyclic equivalent?

2.3 Packing of directed cycles[3]

In this section, we are concerned about a converse problem (in a certain way) of the previous one. Given a digraph D, we denote by $\nu_0(D)$ (resp. $\nu_1(D)$) the maximum number of vertex-disjoint (resp. arc-disjoint)

[3]This subsection is linked with the papers: S. Bessy, N. Lichiardopol, and J.S. Sereni, Two proofs of the bermond-thomassen conjecture for tournaments with bounded minimum in-degree. *Discrete Mathematics*, 310:557–560, 2010 and J. Bang-Jensen, S. Bessy, and S. Thomassé, Disjoint 3-cycles in tournaments: a proof of the bermond-thomassen conjecture for tournaments, *submitted*, 2011

circuits in D. The problems dealing with *circuits packing* in digraphs consist in computing, or finding bounds on ν_0 and ν_1.
In addition, we define $\tau_0(D)$ (resp. $\tau_1(D)$) to be the minimum size of a feedback-vertex-set of D (resp. feedback-arc-set of D). It is clear that τ_0 (resp. τ_1) is a natural lower bound of for the packing number ν_0 (resp. ν_1). The converse is not true, but it is possible to bound above ν_0 (resp. ν_1) by a function of τ_0 (resp. τ_1). This statement was conjectured in 1973 by D.H Younger [136], and the first case of this conjecture 'is τ_0 bounded when $\nu_0 = 1$?', was solved by W. McCuaig.

Theorem 16 (W. McCuaig, 1991, [102])**.** *If D is a digraph with no two vertex-disjoint cycles, then there exists a set X of at most 3 vertices such that $D \setminus X$ is acyclic.*

In fact, some years later, in 1996, Younger's Conjecture was settled by B. Reed, N. Robertson, P.D. Seymour and R. Thomas who proved the following.

Theorem 17 (*'Younger's Conjecture'*, B. Reed, N. Robertson, P.D. Seymour and R. Thomas, 1996, [112])**.** *There exist functions $f_0, f_1 : \mathbb{N} \to \mathbb{N}$ such that for every digraph D we have $\tau_0(D) \leq f_0(\nu_0(D))$ and $\tau_1(D) \leq f_1(\nu_1(D))$.*

More precisely, to prove Theorem 16, W. McCuaig gave a complete characterization of digraphs with no two disjoint cycles. To prove Theorem 17, the authors used Ramsey Theory, leading to exponential functions for f_0 and f_1. We can ask if there is possible other ways to prove these two theorems. In particular, as cyclic orders have strong link with cycles in digraphs, they could be useful for that.

Problem 18. Is it possible to prove Theorem 16 or Theorem 17 using cyclic orders of digraphs, and then, obtaining simpler proofs or better bounds on f_0 and f_1?

For the end of this subsection, we focus on different lower bounds for the maximum number of vertex-disjoint cycles in a digraph (i.e. ν_0). By Theorem 17, we know that if τ_0 is large enough, then ν_0 will be large also. There is no hope to easily obtain an exact value for τ_0, as it is known for long that computing τ_0 is an *NP*-hard problem (it has been proved by R.M. Karp in 1972 [85]). However, there is a natural and tractable lower bound for τ_0. For a digraph D, the *minimum out-degree* (resp. *minimum in-degree*) of D, denoted by $\delta^+(D)$ (resp. $\delta^-(D)$), is the minimum value over the out-degrees (resp. in-degrees) of the vertices of D. As there exists a set X of $\tau_0(D)$ vertices of D such that $D \setminus X$ is acyclic, there is a vertex x with out-degree 0 in $D \setminus X$ and thus, we have $\delta^+(D) \leq d^+(x) \leq \tau_0(D)$. So, using Theorem 17, we know that if δ^+ is large enough, ν_0 will be also large. This corollary of Theorem 17, was, in fact, directly proved by C. Thomassen in 1983 [127]. With J.C. Bermond, they raised a conjecture on the minimum value of δ^+ which insures $\nu_0 \geq k$.

Conjecture 19 (*'The Bermond Thomassen Conjecture'*, J.C. Bermond and C. Thomassen, 1981, [13])**.** If $\delta^+(D) \geq 2k-1$ then $\nu_0(D) \geq k$, what means that D contains at least k vertex-disjoint cycles.

Remark that the complete digraph (with all the possible arcs) is a sharp example for this statement. The conjecture is trivial for $k=1$ and it has been verified for general digraphs when $k=2$ by C. Thomassen [127] and $k=3$ by N. Lichiardopol, A. Pór and J.-S. Sereni [96]. Furthermore, N. Alon proved in 1996 that there exists a linear function of k that insure $\nu_0 \geq k$. Namely, using some probabilistic arguments, he proves the following.

Theorem 20 (N. Alon, 1996, [3])**.** *If $\delta^+(D) \geq 64k$ then $\nu_0(D) \geq k$.*

The status of the Bermond Thomassen Conjecture was even not known on tournaments. I have worked on this specific problem. In this case, as any cycle in tournament always contains a 3-cycle, we focus on disjoint 3-cycles. First, in 2005, with N. Lichiardopol and J.S. Sereni, we proved that the Bermond Thomassen Conjecture is true for regular tournaments. More precisely, we obtained the following result.

Theorem 21 (S. Bessy, N. Lichiardopol and J.S. Sereni, 2005, [18])**.** *If T is a tournament with $\delta^+(T) \geq 2k-1$ and $\delta^-(T) \geq 2k-1$ then, $\nu_0(T) \geq k$.*

More precisely, we proved that, given a collection $\mathcal{F}$ of $t < k$ disjoint 3-cycles of T, it is always possible to find a 3-cycle C of $\mathcal{F}$ such that $T[(V(T) \setminus V(\mathcal{F})) \cup V(C)]$ contains two disjoint 3-cycles. Then, removing

C from $\mathcal{F}$ and adding these two 3-cycles, we obtain a collection of $t+1$ disjoint 3-cycles of T. Unfortunately, this scheme of proof does not work if we remove the condition $\delta^-(T) \geq 2k-1$. However, some years later, in 2010, during a second attempt with J. Bang-Jensen and S. Thomassé, we finally proved the Bermond Thomassen Conjecture for tournaments.

Theorem 22 (J. Bang-Jensen, S. Bessy and S. Thomassé, 2010, [7]). *Every tournament T with $\delta^+(T) \geq 2k-1$ has k disjoint cycles each of which have length 3.*

Here, the proof is also based on the possibility to extend a family of disjoint 3-cycles. More precisely, given a collection $\mathcal{F}$ of $t < k$ disjoint 3-cycles of T, we proved that is always possible to find a larger family of disjoint 3-cycles intersecting $V(T) \setminus V(\mathcal{F})$ on a most four vertices. This method is similar to the one used in the proof of Theorem 21, but it allows more recombination possibilities on the 3-cycles when enlarging $\mathcal{F}$.

As previously mentioned, Bermond Thomassen Conjecture is sharp on complete digraphs. But for large k, we did not find any sharp example of this statement for tournaments, and then without 2-cycles. Indeed, such examples do not exist for tournament as we have shown by improving Theorem 22 for tournaments with large minimum out-degree. Roughly speaking, a tournament T with $\delta^+(T) > \frac{3}{2}k$ and k large enough contains k disjoint cycles of length 3. More precisely, we proved the following.

Theorem 23 (J. Bang-Jensen, S. Bessy and S. Thomassé, 2010, [7]). *For every value $\alpha > \frac{3}{2}$, there exists a constant k_α, such that for every $k \geq k_\alpha$, every tournament T with $\delta^+(T) \geq \alpha k$ has k disjoint 3-cycles.*

This statement is optimal for the value $\frac{3}{2}$, as shown by the family of regular tournament, i.e. tournament T that verify $d^+(x) = d^-(x)$ for every vertices of T. However, we do not know what happens for tournaments with $\delta^+ = \frac{3}{2}k$, but we conjecture that they also contain k disjoint 3-cycles. As when we forbid small cycles (of length 2) we can asymptotically improve the statement of the Bermond Thomassen Conjecture on tournaments and we conjecture that this could be also true for general digraphs.

Conjecture 24 (J. Bang-Jensen, S. Bessy and S. Thomassé, 2010, [7]). If a digraph D has no cycles of length less than g and minimum out-degree at least k, with k large enough, then D contains at least $\frac{g+1}{g} \cdot k$ disjoint cycles.

To conclude this subsection, I just would like to mention two nice conjectures dealing with feedback-arc-set in digraphs. As previously mentioned, we know that τ_1 is *NP*-hard to compute for digraph, and even for tournaments [9]. However there is a class of digraphs where τ_1 is not hard to compute. Indeed, for planar digraph, C.L. Lucchesi proved in 1976 [98] that is possible to compute a feedback-arc-set on a planar digraph in polynomial time. Furthermore, always for planar digraphs, the Lucchesi-Younger Theorem [99] asserts that $\tau_1 = \nu_1$. But even in the planar case, much remains unknown on feedback arc set, as shown by these two long-standing conjectures.

Conjecture 25 (*(weak form)* D.R. Woodall, 1978, [134]). Every planar strong digraph with no 2-cycles admits three disjoint feedback-arc-sets.

Conjecture 26 (V. Neumann-Lara, 1982, [104]). Every planar digraph D with no 2-cycles has a feedback-arc-set which forms a bipartite subdigraph of D.

3 Coloring and partitioning problems

During my research works, I also focused on some graph coloring problems or problems in colored graphs. These topics are structural and can be viewed as partitioning questions, as those from Section 2. However, they do not present such a unity and come from really different fields. The two first subsections deal with some questions arising in a modeling context. In Subsection 3.1, we use an arc-coloring model to obtain results on a function theory problem, and Subsection 3.2 deals with questions from a classical application of graph coloring theory: optimization in optical communication networks. Finally, in Subsection 3.3, we are interested in a problem from a slightly different context. Given a graph and a coloring of this graph, we look for a subgraph having some structural properties (e.g. a path, a cycle...) and some properties according to the coloring (e.g. monochromatic, bi-chromatic...).

3.1 Arc-coloring in digraphs[4]

Related to a function theory problem, with É. Birmelé and F. Havet, we have studied an extension of graph coloring to digraphs. The problem was initially proposed by A. El Sahili [58] and comes from a question arising in function theory. Namely, let f and g be two maps from a finite set A into a set B. Suppose that f and g are *nowhere coinciding*, that is for all $a \in A$, $f(a) \neq g(a)$. A subset A' of A is (f,g)-*independent* if $f(A') \cap g(A') = \emptyset$. We are interested in finding the minimum number of (f,g)-independent subsets needed to partition A in the case where every element of B has a bounded number of antecedents by the functions f and g. As shown by El-Sahili [59], this can be translated into an arc-coloring problem.

We focus on a special type of arc-coloring for digraphs, introduced by S. Poljak and V. Rödl in 1981 [108]. Other classical arc-colorings exist, see [72] for instance, but this one model the previous problem. So, here, an *arc-coloring* of a digraph D is an application c from the arc-set $A(D)$ into a set of colors S such that if the tail of an arc e is the head of an arc e' then $c(e) \neq c(e')$. In other words, the arcs from a same color class form a bipartite graph and are oriented from a part of the bipartition to the other one. The *arc-chromatic number* of D, denoted by $\chi_a(D)$, is the minimum number of colors used by an arc-coloring of D. Another way to define the notion of arc-coloring is the following: in an arc-coloring of D, for any arc xy, the set of colors appearing on the arcs with tail x must not be a subset of the set of colors appearing on the arcs with tail y (as it contains the color of xy). We denote by $\overline{H_k}$ the complementary of the hypercube of dimension k, i.e. $\overline{H_k}$ is the digraph with vertex set all the subsets of $\{1,\ldots,k\}$ and with arc set $\{XY\ :\ X \not\subseteq Y\}$. With the previous remark, a digraph D has an arc-coloring with k colors if, and only if, it admits an homomorphism to $\overline{H_k}$ (then, if an arc xy of D is mapped to an arc XY of $\overline{H_k}$, we color xy with an integer of $X \setminus Y$). Using this and Sperner's Lemma [123] to find a homomorphism into a complete subdigraph of $\overline{H_k}$, S. Poljack and V. Rödl obtained the following theorem.

Theorem 27 (S. Poljak and V. Rödl, 1981, [108])**.** *For every digraph D, we have* $\log(\chi(D)) \leq \chi_a(D) \leq \theta(\chi(D))$*, where* $\theta(k) = \min\{s\ :\ \binom{s}{\lfloor s/2 \rfloor} \geq k\}$.

Now, we come back to the function theory problem and the model proposed by A. El-Sahili in [59]. Let $D_{f,g}$ be the digraph defined by: $V(D_{f,g}) = B$ and $(b,b') \in E(D_{f,g})$ if there exists an element a in A such that $g(a) = b$ and $f(a) = b'$. Then, a (f,g)-independent subset of A corresponds to a set of arcs of $D_{f,g}$ which do not form paths of length more than one or cycles. And so, the minimum number of (f,g)-independent subsets needed to partition A, denoted by $\phi(f,g)$ is exactly the arc-chromatic number of $D_{f,g}$. Furthermore, we want to take into consideration in the model the number of antecedents by f and g for the elements of B. Precisely, let $\Phi(k)$ (resp. $\Phi^{\vee}(k,l)$) be the maximum value of $\phi(f,g)$ for two nowhere coinciding maps f and g from A into B such that for every z in B, $|g^{-1}(z)| \leq k$ (resp. either $|g^{-1}(z)| \leq k$ or $|f^{-1}(z)| \leq l$). The condition $f^{-1}(z)$ (resp. $g^{-1}(z)$) has at most k elements means that each vertex has in-degree (resp. out-degree) at most k in $D_{f,g}$. To turn these notions into digraphs context, A. El-Sahili [59] defines a k-*digraph* to be a digraph in which every vertex has out-degree at most k. Similarly, a $(k \vee l)$-*digraph* is a digraph in which every vertex has either out-degree at most k or in-degree at most l. Hence, $\Phi(k)$ (resp. $\Phi^{\vee}(k,l)$) is the maximum value of $\chi_a(D)$ for D a k-digraph (resp. a $(k \vee l)$-digraph).

So, motivated by the previous interpretation in function theory and by the corresponding coloring problem, we studied the behavior of the functions Φ and $\Phi^{\vee}$. The first results on these functions was given by A. El-Sahili, who proved the following.

Theorem 28 (A. El-Sahili, 2003, [59])**.** *We have* $\Phi^{\vee}(k,k) \leq 2k+1$.

[4]This subsection is linked with the paper: S. Bessy, E. Birmelé, and F. Havet, Arc-chromatic number of digraphs in which every vertex has bounded outdegree or bounded indegree, *Journal of Graph Theory*, 53(4):315–332, 2006.

Using Theorem 27, we improved this bound to the following.

Theorem 29 (S. Bessy, É. Birmelé and F. Havet, 2006, [14]). *We have* $\Phi(k) \leq \theta(2k)$ *if* $k \geq 2$*, and* $\Phi^{\vee}(k,l) \leq \theta(2k+2l)$ *if* $k+l \geq 3$.

As asymptotically the function θ is equivalent to the function log, we obtain quite better bounds than those from Theorem 28. Furthermore, we get examples proving that, up to constant multiplicative factor, the bounds given by Theorem 29 are optimal.
We have completed our work by finding some properties on the behavior of the functions Φ and $\Phi^{\vee}$.

Theorem 30 (S. Bessy, E. Birmelé and F. Havet, 2006, [14]). *For every* $k \geq 1$*, we have* $\Phi(k) \leq \Phi^{\vee}(k,0) \leq \cdots \leq \Phi^{\vee}(k,k) \leq \Phi(k)+2$ *and* $\Phi^{\vee}(k,1) \leq \Phi(k)+1$.

Moreover, we conjectured that the first inequality is not optimal, and that $\Phi^{\vee}$ is closer to Φ.

Conjecture 31 (S. Bessy, E. Birmelé and F. Havet, 2006, [14]). For every $k \geq 1$, we have $\Phi^{\vee}(k,1) = \Phi(k)$ and $\Phi^{\vee}(k,k) \leq \Phi(k)+1$.

Finally, we checked our conjecture for small values of k by computing exact values of $\Phi^{\vee}(k,l)$ and $\Phi(k)$ for $k \leq 3$ and $l \leq 3$. In particular, as a remarkable result we obtain the following.

Theorem 32 (S. Bessy, E. Birmelé and F. Havet, 2006, [14]). *We have* $\Phi^{\vee}(2,2) = 4$*, that is, every digraph with the property that* $d^{+}(x) \leq 2$ *or* $d^{-}(x) \leq 2$ *for each of its vertex* x *admits an arc-coloring with at most four colors.*

This statement was settled by using more sharpened homomorphisms from $(2 \vee 2)$-digraph into $\overline{H_4}$ than the usual mapping to a maximum complete subdigraph of $\overline{H_4}$ (given by Sperner's Lemma [123]).

3.2 WDM[5]

This subsection presents another application of graph coloring, in the domain of network optimization and design. I worked in that field while I was in postdoc in 2004 in the *Mascotte Team* at Sophia Antipolis, a research team led by J.C. Bermond and working on algorithms, discrete mathematics and combinatorial optimization with motivations coming from communication networks. During this year, with C. Lepelletier, a Master's degree student, we were interested in a problem arising in the design of optical networks. This topic has been of growing interest over the two last decades, using tools from graph theory and design theory (for instance, see [11], [75] or [10] for a background review of optical networks). The model considered here is valid for the so-called *wavelength division multiplexing* (or *WDM*) optical network. Such a network is modeled by a symmetric directed graph with arcs representing the fiber-optic links. A *request* in the network is an ordered pair of graph nodes, representing a possible communication in the network. A set of different requests is an *instance* in the network. For each request of the instance, we have to select a routing directed path to satisfy it, and the set of all selected paths forms a *routing set* according to the instance. To make the communications possible, a wavelength is allocated to each routing path, such that two paths sharing an arc do not carry the same wavelength; otherwise the corresponding communications could interfere. Given a routing set related to the wavelength assignment, we can define two classical invariants. The *arc-forwarding index* of the routing set is the maximum number of paths sharing the same arc. In the network, there is a general bound on the number of wavelengths which can transit at the same time in a fiber-optic link, corresponding to the admissible maximal arc-forwarding index. The other invariant, called the *optical index* of the routing set, is the minimum number of wavelengths to assign to the routing paths in order to ensure that there is no interference in the network. The main challenge is to provide, for a given instance, a routing set which minimizes the arc-forwarding index or the optical index, or both if possible.

[5]This subsection is linked with the paper: S. Bessy and C. Lepelletier, Optical index of fault tolerant routings in wdm networks, *Networks*, 56(2):95–102, 2010.

Our work is a contribution to a variant of this problem, introduced by J. Maňuch and L. Stacho [101], in which we focus on possible breakdowns of nodes in the network. Precisely, for a given fixed integer f, we have to provide, for every request, not just one directed path to satisfy it, but rather a set of $f+1$ directed paths with the same starting and ending nodes (corresponding to the request) and which are pairwise internally disjoint. In this routing, if f nodes break down, every request between the remaining nodes could still be satisfied by a previously selected routing path which contains no failed component. Such a routing set of directed paths is called an *f-fault tolerant routing* or an *f-tolerant routing*.
Considering the problematics developed in [101], we focused on the very special cases of complete symmetric directed graphs and complete balanced bipartite symmetric directed graphs. Moreover, we only studied the case of *all-to-all* communication, i.e., where the instance of the problem is the set of all ordered pairs of nodes of the network. So, in a all-to-all context, for a digraph D and a fixed positive integer f, an *f-tolerant routing* in D is a set of paths $\mathcal{R} = \{P_i(u,v) \ : \ u,v \in V, u \neq v, i = 0,\dots,f\}$ where, for each pair of distinct vertices $u,v \in V(D)$, the paths $P_0(u,v),\dots,P_f(u,v)$ are internally vertex disjoint. Note that such a set of paths exists if and only if the connectivity of the directed graph is large enough (at least $f+1$), which will be the case in complete and complete bipartite networks for suitable f.

The basic parameters for WDM optical networks, the arc-forwarding index and the optical index, are generalized in f-tolerant routings. The *load* of an arc in $\mathcal{R}$ is the number of directed paths of $\mathcal{R}$ containing it. By extension, the maximum load over all the arcs of D is the *load* of the routing, which is also called the *arc-forwarding index of* $\mathcal{R}$ and is denoted by $\pi(\mathcal{R})$. Finally, the *optical index of* $\mathcal{R}$, denoted $w(\mathcal{R})$, is the minimum number of wavelengths to assign to paths of $\mathcal{R}$ so that no two paths sharing an arc receive the same wavelength. In other words, $w(\mathcal{R})$ is exactly the chromatic number of the graph with vertex set $\mathcal{R}$ and where two paths of $\mathcal{R}$ are linked if they share the same arc of D (known as the *path graph* of $\mathcal{R}$). The goal, in that context, is to minimize $\pi(\mathcal{R})$ and $w(\mathcal{R})$. So the *f-tolerant arc-forwarding index* of D and the *f-tolerant optical index* of D are respectively defined by:

$$\pi_f(D) = \min_{\mathcal{R}} \pi(\mathcal{R})$$

$$w_f(D) = \min_{\mathcal{R}} w(\mathcal{R})$$

where the minima span all the possible routing sets $\mathcal{R}$. A routing set achieving one of the bounds is said to be *optimal for the arc-forwarding index* or *optimal for the optical index*, respectively.

For a routing set $\mathcal{R}$, all paths sharing the same arc must receive different wavelengths in the computation of $w(\mathcal{R})$. In particular, we have $\pi(\mathcal{R}) \leq w(\mathcal{R})$. By considering a routing set which is optimal for the optical index, we obtain $\pi_f(D) \leq w_f(D)$. The equality was conjectured by J. Maňuch and L. Stacho [101].

Conjecture 33 (J. Maňuch, L. Stacho, 2003, [101])**.** Let D be a symmetric directed k-vertex-connected graph. For any f, $0 \leq f < k$, we have $\pi_f(D) = w_f(D)$.

For $f = 0$ (without tolerating any faults), the conjecture was previously raised by B. Beauquier et al. [10] and plays a central role in the field of WDM networks.

Recall that we denote by $K_n^\star$ the *complete symmetric directed graph* on n vertices. In addition, the *complete balanced bipartite symmetric digraph* $K_{n,n}^\star$ is the directed graph on vertex set $X \cup Y$ with $X = \{x_1,\dots,x_n\}$ and $Y = \{y_1,\dots,y_n\}$ and arc set $\{xy, yx : x \in X, y \in Y\}$. Thus, we have considered the problem of computing exactly $w_f(K_n^\star)$ and $w_f(K_{n,n}^\star)$. It is easy to provide a lower bound for the arc-forwarding index of $K_n^\star$. Indeed, any two vertices x and y of $K_n^\star$ have to be linked in an f-tolerant routing by $f+1$ internally disjoint paths. If one of these paths has length one (the arc xy), all the others have length at least two, and at least $2f+1$ arcs are needed to ensure f-tolerant communication from x to y. So, by an average argument, one arc of $K_n^\star$ must have load at least $2f+1$, providing $\pi_f(K_n^\star) \geq 2f+1$. Similarly, we obtain an easy lower bound on $\pi_f(K_{n,n}^\star)$. In the case of $K_n^\star$, in 2005, A. Gupta, J. Maňuch and L. Stacho proved in [75] that this lower bound gives exactly the value of the arc-forwarding index. Indeed, they construct f-tolerant routings through families of independent idempotent Latin squares which are optimal for the arc-forwarding index.

Theorem 34 (A. Gupta, J. Maňuch, L.Stacho, 2005, [75])**.** *For every f with $0 \leq f \leq n-2$, we have $\pi_f(K_n^\star) = 2f+1$.*

They also partially bound the optical index of their f-tolerant routings, proving that $w_f(K_n^\star) \le 3f+1$ for some values of f. This result was improved in 2006 by J.H. Dinitz, A.C.H. Ling and D.R. Stinson [49], who gave a better multiplicative factor for some infinite sets of values of n and the optimal index up to an additive constant for another infinite set of values of n. We have improved these results and fixed this computation by showing that every f-tolerant routing set of $K_n^\star$ which is optimal for the arc-forwarding index is also optimal for the optical index. We thus prove Conjecture 33 for the complete digraphs.

Theorem 35 (S. Bessy, C. Lepelletier, 2007, [17])**.** *For every f, $0 \le f \le n-2$, and every f-tolerant routing set $\mathcal{R}$ of $K_n^\star$ with $\pi(\mathcal{R}) = \pi_f(K_n^\star) = 2f+1$, we have $w(\mathcal{R}) = 2f+1$. In particular, we have $w_f(K_n^\star) = \pi_f(K_n^\star) = 2f+1$.*

We obtained this result using an edge coloring model. We define a graph with vertex set the arcs of $K_n^\star$ and where to arcs of $K_n^\star$ are linked by an edge if they belong to a same path of the considered routing. Theorem 35 is then simply obtained by applying Vizing's Theorem [132] to this special graph.

A remaining important issue concerning f-tolerant routings for $K_n^\star$ is the design of the routings.

Problem 36. Is there a simple way (without using idempotent Latin square, for instance) to design in $K_n^\star$ optimal f-tolerant routings for the arc-forwarding index?

Moreover, we have computed the exact optical index of $K_{n,n}^\star$ and thus proved Conjecture 33 also for this family of graphs. This improves the result of A. Gupta, J. Maňuch and L. Stacho [75], where the upper bound given on the optical index of $K_{n,n}^\star$ is 20% higher than the conjectured optimal value. For that, we described a family of routings and shown that they are all optimal for the arc-forwarding index and the optical index.

Theorem 37 (S. Bessy, C. Lepelletier, 2007, [17])**.** *For any $n \ge 1$ and any f with $0 \le f \le n-1$, we have $w_f(K_{n,n}^\star) = \pi_f(K_{n,n}^\star)$.*

3.3 Substructures in colored graphs[6]

The problem studied in this section is not exactly a coloring problem, but concerns the existence of some structure in a colored graph. This problematic covers a broad range of problems, and with S. Thomassé, we focused on a conjecture of J. Lehel on the partition of a bi-colored complete graph into two monochromatic cycles. To be precise, we say that a colored graph has a *partition into p monochromatic cycles (or paths)* if it admits a vertex-partition into p subgraphs every one of which admits a spanning monochromatic cycle (or path).

Many questions deal with the existence of monochromatic paths and cycles in edge-colored complete graphs. For instance, in 1991, P. Erdős, A. Gyárfás and L. Pyber studied in [60] the minimal number of monochromatic cycles needed to partition the vertex set of the complete graph with edges colored with k colors. In 2006, A. Gyárfás, M. Ruszinkó, G.N. Sárközy and E. Szemerédi [77] proved that $O(k \log k)$ such cycles suffice to partition the vertices. One case which received a particular attention was the case $k = 2$, where we would like to cover a complete graph which edges are colored blue and red by two monochromatic cycles. A conjecture of Lehel, first cited in [6], asserts that a blue and a red cycle partition the vertices, where empty set, singletons and edges are allowed as cycles. This statement was proved for sufficiently large n by T. Łuczak, V. Rödl and E. Szemerédi [128], and more recently by P. Allen [2] with a better bound. Their proofs respectively use the Szemerédi Regularity Lemma and Ramsey's Theory to find useful partition of the vertex set of the colored complete graph. With S. Thomassé, we obtained a general proof of this statement.

Theorem 38 (S. Bessy, S. Thomassé, 2010, [23])**.** *Every complete graph with red and blue edges has a vertex partition into a red cycle and a blue cycle.*

[6]This subsection is linked with the paper: S. Bessy and S. Thomassé, Partitionning a graph into a cycle and an anticycle, a proof of lehel's conjecture, *Journal of Combinatorial Theory, Serie B*, 100(2):176–180, 2010.

Our proof is based on induction, using as starting point the proof of A. Gyárfás of the existence of one red cycle and one blue cycle covering the vertices and intersecting on at most one vertex (see [76]). For this, he considered a longest path consisting of a red path followed by a blue path. The nice fact is that such a path P is hamiltonian. Indeed, if a vertex v is not covered, it must be joined in blue to the origin a of P and in red to the end b of P. But then, one can cover the vertices of P and v using the edge ab. Consequently, there exists a hamiltonian cycle consisting of two monochromatic paths. Hence, there exists a monochromatic cycle C, of size at least two, and a monochromatic path P with different colors partitioning the vertex set. The induction in our proof of Theorem 38 runs on the size of C: at each step, either we can find the two desired cycles, or we increase the length of C.

There exist many other interesting questions dealing with substructures in colored graphs. I list below some of them, which are either generalizations of Lehel's Conjecture, or famous questions raised in that field. The first natural extension to this problem is to increase the number of colors of the background structure. It was considered by P. Erdős, A. Gyárfás and L. Pyber in their seminal paper, where there raised the following conjecture, still open for $k \geq 3$.

Conjecture 39 (P. Erdős, A. Gyárfás and L. Pyber, 1991, [60])**.** For every coloring of the edges of the complete graph K_n with k colors, there exists a partition of the vertex set of K_n into r monochromatic cycles.

Nothing is said on requirement for cycles with different colors. Maybe, some connectivity conditions could be asked for each graph induced by the edges with same color. In particular, the following question is interesting.

Problem 40. Let be a coloring of the edges of the complete graph K_n with 3 colors such that each graph induced by the edges with the same color has vertex-connectivity at least 2. Is it possible to partition the vertex set of K_n into three monochromatic cycles, one of each color?

Another kind of questions arises when we change the background graph. For instance, partitioning the complete balanced bipartite graph $K_{n,n}$ with colored edges has been studied by P. Haxell [79]. She proved that for every k, there exists an integer c_k such that, for every coloring of the edges of $K_{n,n}$ with k colors, their exists a partition of the vertex set of $K_{n,n}$ into c_k monochromatic cycles. In the paper [79], there is no precise mention to the case $k = 2$. The bound computed on c_k gives $c_2 \geq 64$, but we can conjecture that the real value of c_2 is really less than 64.

Problem 41. If n is large enough, for every coloring of the edges of $K_{n,n}$ with 2 colors, does there exist a partition of the vertex set of $K_{n,n}$ into two monochromatic cycles?

Remark that for $n = 3$, it is possible to color the edges of $K_{3,3}$ with two colors such that each color class induced a forest of $K_{3,3}$. Then, in this case, there is no hope to have a partition of the vertex set of $K_{3,3}$ into two monochromatic cycles. That is why we ask for n to be large enough, to have enough space in the edge set to form cycles.

A third generalization of Lehel's Conjecture could be obtained in considering objects of higher dimensions. On a ground set V of n points, we color into two colors, say red and blue, all the d-subsets of V. We call a *d-dimensional cycle* the set of all the facets (faces of dimension d) of a polytope (a bounded intersection of half planes) of the $d+1$-dimensional space which vertices are seen as elements of V. The natural extension to Lehel's question is the following.

Conjecture 42 (S. Bessy, S. Thomassé, 2007)**.** For every coloring of the d-subsets of a n-set V, there exists a partition of V into 2 parts, each of them being covered by a monochromatic d-dimensional cycle, one red and one blue.

For $d = 2$, we obtain the statement of Lehel's Conjecture, which is then true. However, for $d = 3$, the question remains open, and we obtained the following short statement.

Conjecture 43 (S. Bessy, S. Thomassé, 2007)**.** For every coloring of the triples of a n-set V, there exists a partition of V into 2 planar triangulations, one red and one blue.

Finally, the last kind of problems I am interested in that field concerns digraphs. If we want a direct translation of Lehel's problem on the complete digraph, we need to ensure that the considered colorings do not induce big acyclic part in each color. For instance, if we consider an enumeration of the vertices of the complete digraph and color all forward arcs in blue and all backward arcs in red, there is no hope to find a vertex partition into two monochromatic directed cycles. To raise a possible conjecture, we ask that for each color, the digraph induced by the arcs of this color must be strongly connected.

Conjecture 44 (S. Bessy, S. Thomassé, 2007)**.** For every coloring of the arcs of the complete digraph $K_n^\star$ into two colors such that each color induces a strongly connected digraph, there exists a vertex partition of $K_n^\star$ into two directed cycles, one of each color.

Remark that there exists an oriented version of the starting point of the proof of Theorem 38: the existence of a hamiltonian cycle consisting of two monochromatic paths in every edge colored complete graph. Indeed, in 1973, H. Raynaud [111] proved that every arc coloring of the complete digraph contains a hamiltonian (oriented) cycle consisting of two monochromatic (oriented) paths. This should be worth to exploit this result in order to tackle Conjecture 44.

To conclude, let us mention a long standing open problem, initially stated by P. Erdős (cited in [116]) and concerning arc-colored tournaments. This problem is not related to monochromatic cycles but it is quite natural in that field and deals with unavoidable structures in colored directed graphs. In a colored digraph D, a set of vertices S is a *set of monochromatic sources* if from every vertex x of D, there exists a monochromatic path from a vertex of S to x. This problem first appeared in print in a paper of B. Sands, N. Sauer, and R. Woodrow [116], where they proved that every tournament with arcs colored with two colors has a vertex which is a monochromatic source.

Problem 45 (P. Erdős, 1982)**.** For every k, is there an integer $f(k)$ such that every tournament with arcs colored with k colors has set of monochromatic sources with cardinality at most $f(k)$.

So, according to this formalism, B. Sands, N. Sauer, and R. Woodrow proved that $f(2) = 1$. However, for $k \geq 3$, the value $f(k)$ is not known and even not known to exist. More precisely, for $k = 3$ the last authors conjectured the following

Problem 46 (B. Sands, N. Sauer, and R. Woodrow, 1982, [116])**.** We have $f(3) = 3$, that is, every tournament with arcs colored with 3 colors has set of monochromatic sources with cardinality at most 3.

4 Algorithmic problems on graphs

This last section deals with some algorithmic problems on graphs and exact solutions for these problems. The first subsection is an overview of the work I have done with A. Perez (and other co-authors) while he prepared his PhD Thesis which I co-supervised with C. Paul. We looked at some parameterized problems and presented kernelization algorithms for these problems. The second subsection presents a join work with F. Havet on a problem consisting in counting the number of edge-colorings of a regular graph. We gave upper bound for this number and yielded to exponential algorithms, with lowest as possible exponential basis, to enumerate all this colorings.

4.1 Kernelization for some editing problems

With A. Perez, we worked on some modification problems for graphs and digraphs. Given a class Π of graphs (or digraphs), generally defined by some properties or a set of forbidden induced subgraphs, the generic modification problem is the following.

PARAMETERIZED Π-MODIFICATION PROBLEM
Input: A graph (or a directed graph) $G = (V, E)$.

Parameter: An integer $k \geqslant 0$.
Question: Is there a subset $F \subseteq V \times V$ with $|F| \leqslant k$ such that the graph $G + F = (V, E \vartriangle F)$ belongs to the class Π?

Graph modification problems cover a broad range of *NP*-Complete problems and have been extensively studied in the literature [100, 119, 120]. Well-known examples include the Vertex Cover [46], Feedback-Vertex-Set [126], or Cluster Editing [41] problems. These problems find applications in various domains, such as computational biology [83, 120], image processing [119] or relational databases [124]. Precisely, for a given graph $G = (V, E)$, in a *completion problem*, the set F of modified edges is constrained to be disjoint from E, whereas in an *edge deletion problem* F has to be a subset of E. If no restriction applies to F, then we obtain an *edition problem*. Though most of the edge-modification problems turn out to be *NP*-hard problems, in some cases, efficient algorithms can be obtained to solve the natural parameterized version of some of them. The goal is to obtained a classification in the context of parameterized complexity (polynomial kernel, *FPT* without polynomial kernel or not *FPT*, for instance) of the Parameterized Π-modification Problems according to the class Π. Very few general results are known in this problematic. For instance, a graph modification problem is *FPT* whenever Π can be characterized by a finite set of forbidden induced subgraphs [34]. But, even in this simple case, the existence of a polynomial kernel is not ensure. We will discuss later this question more precisely, but it motivated our work on graph modification problems. Thus, in order to find polynomial kernel for some Parameterized Π-modification Problems, we focused on very structured class of graphs Π (classes having a tree-like decomposition and tournaments). More precisely, we found three polynomial kernelizations: for the 3-leaf power editing problem (join work with C. Paul and A. Perez), for the proper interval completion problem (join work with A. Perez) and for the feedback-arc-set in tournament problem (join work with F.V. Fomin, S. Gaspers, C. Paul, A. Perez, S. Saurabh and S. Thomassé). We used a very similar approach for the two first problems, which are closed. For the third problem, which is on tournaments, the techniques are quite different but the general scheme of the algorithm design is also similar.
Very classically, given a Parameterized Π-modification Problem, the general process used to find a small kernel is the following. For an instance (G, k) of this problem, we apply on G a set of *rules* to obtain a graph G' equivalent to G, and we show that if G is a positive instance of the considered problem, then, the size of G' is bounded by a polynomial in k. Some of the rules we used are quite generic. I list them bellow.

- If a connected component of G is already a graph belonging to Π, then we can remove it, under the condition that Π is closed under disjoint union. Provided that the class Π has a polynomial algorithm of recognition, this rule can be applied in polynomial time.

- If G has a big set of vertices which have the same behavior, then we can edit this set in a same way. Precisely, we proved in [19] the following. If Π is closed under true twin addition and induced subgraphs then, from every set T of true twins in G with $|T| > k$, we can remove $|T| - (k+1)$ arbitrary vertices from T. Moreover, this rule can be applied in polynomial time using a modular decomposition algorithm or more easily, partition refinement (see [78] for example).

- If G has an edge or a non-edge e contained in more than k obstructions of the class Π, which are elsewhere disjoint, then we can edit e and reduce the parameter consequently. Classically, this rule is call a *sunflower rule*, and is usually applied for the finite obstructions of the class Π. Thus, it can be computed in polynomial time.

After that, we tried to generalize the first of these rules by reducing parts of the graph G which are already 'clean', but not necessarily form a connected component. We call such a part of G, a *branch* of G. The exact definition has to be adapted for each singular case, but broadly speaking, a subgraph H of G is a branch if H belongs to Π and has some special properties of adjacency with the remaining of G. The *'branches rule'* consists then in localizing big branches in the graph and reducing them ('cutting the branches'). It is possible if we can show that for the considered Parameterized Π-modification Problem, the relevant information contained in a branch lie in its border. This 'concept' of branch is a natural idea, and as been used before for kernelization algorithms (see [83], for instance). Finally, we have to prove that if G is a positive instance of the problem, then, after applying these rules, the size of the obtained graph G' will be small, i.e. polynomial in k. This will be possible, as if G is a positive instance, up to few edges, it looks like a graph of Π, and then has big parts, branches, behaving like subgraphs of a graph of Π, and will be reduced by the 'branches rule'.

4.1.1 A polynomial kernel for the 3-leaf power editing problem[7]

In this subsection, I present a joint work with C. Paul and A. Perez, which consists in finding a polynomial kernel for a Parameterized Π-modification Problem, where Π is the class of *3-leaf powers*, graphs arising from a phylogenetic reconstruction context [86, 87, 106]. Briefly these graphs come from the following problem. We want to extract, from a threshold graph G on a set S of species, a tree T, whose leaf set is S and such that the distance between two species is at most p in T if, and only if, they are adjacent in G (p being the value used to extract G from dissimilarity information). If such a tree T exists, then G is a *p-leaf power* and T is its *p-leaf root*. Here, we are dealing with 3-leaf power which have several nice characterizations (see [32] and [52]). The *critical graph* of a graph G is obtained by contracting all the set of pairwise true twins of the graphs G. Then, a graph is a 3-leaf power if its critical graph is a tree. Equivalently, 3-leaf powers are the chordal graphs without induced *bull* (a 3-cycle with two pending vertices), *dart* (build from a path of length 2 and an isolated vertex, both dominated by a fifth vertex) and *gem* (a path of length three with a dominating vertex).
Following theoretical motivations, we looked for a polynomial kernel for the 3-leaf power editing problem, answering to an open question of M. Dom, J. Guo, F. Hüffner and R. Niedermeier [53, 51].

Concerning algorithmic on p-leaf power, the following is known. For $p \leqslant 5$, the p-leaf power recognition is polynomial time solvable [33, 36], whereas the question is still open for p strictly larger than 5. Parameterized p-leaf power edge modification problems have been studied so far for $p \leqslant 4$. The edition problem for $p = 2$ is known as the classical Cluster Editing problem for which the kernel size bound has been successively improved in a series of papers [62, 71, 109, 74], culminating in 2010 [41] with a kernel with $2k$ vertices. For larger values of p, the edition problem is known as the closest p-leaf power problem. For $p = 3$ and 4, the closest p-leaf power problem is known to be *FPT* [52, 51], while its fixed-parameterized tractability is still open for larger values of p. However, the existence of a polynomial kernel for $p > 2$ remained an open question [50, 53]. Moreover, though the completion and edge-deletion problems also are *FPT* for $p \leqslant 4$ [51, 53], no polynomial kernel was known for $p \neq 2$ [74]. In this context, we focused on the case $p = 3$ and obtained the following.

Theorem 47 (S. Bessy, C. Paul, A. Perez, 2010, [19])**.** *The* closest 3-leaf power*, the* 3-leaf power completion *and the* 3-leaf power edge-deletion *admit a kernel with $O(k^3)$ vertices.*

To obtain this kernel, we followed the general scheme explained previously. In this context, for an instance graph G, a branch of G is a subgraph which forms a sub-tree of the critical graph of G containing at most two vertices with neighbors outside of this sub-tree in the critical graph of G. Then, the analysis of the rules yielded to the desired kernel.

4.1.2 A polynomial kernel for the proper interval completion problem[8]

In this second subsection, I present a join work with A. Perez, related to a Parameterized Π-modification Problem, where the allowed modifications are only edge completions and the class Π is the class of *proper interval graphs*, which are the intersection graphs of finite sets of unit length intervals on a line. Thus, we studied the proper interval completion problem and found a kernelization algorithm for this problem. The class of proper interval graphs is a well-studied class of graphs, and several characterizations are known to exist. In particular, there exists an set of forbidden induced subgraphs that characterizes proper interval graphs [133]: all the k-cycles with $k \geq 4$, the *claw*, which the complete bipartite graph $K_{1,3}$, the *net*, a 3-cycle with three pending vertices, and the *3-sun*, which is the complementary of the net. The proper

[7]This subsection is linked with the paper: S. Bessy, C. Paul, and A. Perez, Polynomial kernels for 3-leaf power graph modification problems, *Discrete Applied Mathematics*, 158(16):1732–1744, 2010.

[8]This subsection is linked with the paper: S. Bessy and A. Perez. Polynomial kernels for proper interval completion and related problems. In *FCT*, volume 6914 of *LNCS*, pages 1732–1744, 2011.

interval graphs are also characterized by having an *umbrella ordering* [97]. An umbrella ordering of a graph G is an ordering $v_1, \ldots, v_n$ of its vertices such that for every edge v_iv_j of G with $i < j$, the set $\{v_i, \ldots, v_j\}$ is a clique of G (it corresponds to the order of the first extremity of each interval in an interval representation of G).
Interval completion problems find applications in molecular biology and genomic research [80, 83], and in particular in *physical mapping* of DNA. This motivation was cited in the first papers dealing with PROPER INTERVAL COMPLETION PROBLEM (see [83] for instance). This problem is known to be *NP*-Complete for a long time [70], but fixed-parameter tractable due to a result of H. Kaplan, R. Shamir and R.E. Tarjan in FOCS '94 [83, 84]. Nevertheless, it was not known whether this problem admit a polynomial kernel or not. We settled this question by proving the following.

Theorem 48 (S. Bessy, A. Perez, 2011, [20])**.** *The* PROPER INTERVAL COMPLETION *problem admits a kernel with at most* $O(k^3)$ *vertices.*

Remark that this problem is quite similar to the 3-LEAF POWER COMPLETION: this is an edge completion problem to a class of chordal graph defined by a finite set of obstructions. The proof also follows the previous general scheme, but this time the proof for the 'branches rules' is really more technical. A branch for this problem is a subgraph H of our instance graph G which induces a proper interval graph and such that the edges standing across the partition $(H, G \setminus H)$ form at most two *generalized join*. Such a join is a set of edges which contains no induced $2K_2$ (two disjoint edges). It corresponds to the edges across a partition $(\{v_1, \ldots, v_k\}, \{v_{k+1}, \ldots v_n\})$ of an umbrella ordering of a proper interval graph. Detecting and reducing the branches produced the cubic kernel.

Moreover, we applied our techniques to the so-called BIPARTITE CHAIN DELETION problem, closely related to the PROPER INTERVAL COMPLETION problem where one is given a graph $G = (V, E)$ and seeks a set of at most k edges whose deletion from E result in a bipartite chain graph (a graph that can be partitioned into two independent sets connected by a generalized join). For this problem, we obtained a quadratic kernel.

Theorem 49 (S. Bessy, A. Perez, 2011, [20])**.** *The problem* BIPARTITE CHAIN DELETION *admits a kernel with at most* $O(k^2)$ *vertices.*

This result completes a previous result of Guo [74] who proved that the BIPARTITE CHAIN DELETION WITH FIXED BIPARTITION problem admits a kernel with $O(k^2)$ vertices.

To conclude these two parts, we return on a more general framework for the PARAMETERIZED Π-MODIFICATION PROBLEMS. As previously mentioned, it is known that a graph modification problem is *FPT* whenever Π can be characterized by a finite set of forbidden induced subgraphs [34]. However, recent results proved that several graph modification problems do not admit a polynomial kernel even for such classes Π [73, 94]. For instance, an impressive result is that if Π is the class of the graphs without induced $2K_2$, then the PARAMETERIZED Π-COMPLETION PROBLEM has no polynomial kernel (personal communication of F. Havet, C. Paul, A. Perez and S. Guillemot on a work in progress). So, in this field, the following question is a central one.

Problem 50. Is it possible to characterize the class Π of graphs such that the PARAMETERIZED Π-MODIFICATION PROBLEM is *FPT* or admits a polynomial kernel?

More precisely, with A. Perez, focusing on completion problems, we tried to generalize the notion of branches and apply it to the PARAMETERIZED Π-COMPLETION PROBLEM. In the two examples presented, the fact that the classes of graphs (3-leaf power and proper interval) are chordal seems very useful to obtain polynomial kernels. So, we asked the following question.

Conjecture 51 (S. Bessy, A. Perez, 2011 [20])**.** If Π is a class of chordal graphs defined by a finite set of obstructions, then the PARAMETERIZED Π-COMPLETION PROBLEM admits a polynomial kernel.

As mentioned in [20], it is easy to see that such problems are *FPT*. Moreover, another clue for this conjecture is that when Π is simply the class of chordal graphs, H. Kaplan, R. Shamir and R.E. Tarjan have shown in 1994 [83] that the Parameterized Π-completion problem (also called the Minimum Fill-In problem) admits a cubic kernel.

4.1.3 A polynomial kernel for the feedback-arc-set in tournament problem[9]

The last problem I looked at in the context of parameterized complexity deals with feedback-arc-set in tournaments. It is (very) collective work done in 2009 and join with F.V. Fomin, S. Gaspers, C. Paul, A. Perez, S. Saurabh and S. Thomassé and it is not very far from the problems presented in the two first subsection. Given a directed graph $G = (V, A)$ on n vertices and an integer parameter k, the Feedback-Arc-Set problem asks whether the given digraph has a set of k arcs whose removal results in an acyclic directed graph. It is a Parameterized Π-arc-deletion problem, where Π stands for the class of acyclic digraphs. We considered this problem in a the class of tournaments. More precisely, the problem is the following.

Feedback-Arc-Set in Tournaments (FAST):
Input: A tournament $T = (V, A)$ and a positive integer k.
Parameter: k.
Question: Is there a subset $F \subseteq A$ of at most k arcs whose removal makes T acyclic?

Feedback-arc-sets in tournaments are well studied from the combinatorial [61, 82, 113, 118, 122, 135], statistical [121] and algorithmic [1, 4, 44, 90, 129, 130] points of view. The problem FAST has some nice applications, as for instance, in *rank aggregation*, where we are given several rankings of a set of objects, and we wish to produce a single ranking that on average is as consistent as possible with the given ones, according to some chosen measure of consistency. This problem has been studied in the context of voting [31, 39, 43], machine learning [42], and search engine ranking [56, 57]. A natural consistency measure for rank aggregation is the number of pairs that occur in a different order in the two rankings. This leads to *Kemeny rank aggregation* [88, 89], a special case of a weighted version of FAST.

However, we were mainly motivated by theoretical aspects of the problem, on which the following is known. The FAST problem is *NP*-complete by recent results of N. Alon [4] and P. Charbit et al. [38]. From an approximation perspective, FAST admits an approximation algorithm, found in 2007 by C. Kenyon-Mathieu and W. Schudy [90] and which is used it in our kernelization process. The problem is also well studied in parameterized complexity. V. Raman and S. Saurabh [110] showed that FAST is *FPT* and a kernel on $O(k^2)$ vertices is known for this problem, a result from N. Alon et al. [5] and M. Dom et al. [54]. We improved these last results by providing a *linear vertex* kernel for FAST.

Theorem 52 (S. Bessy, F.V. Fomin, S. Gaspers, C. Paul, A. Perez, S. Saurabh and S. Thomassé., 2009, [15])**.** *Given any fixed $\epsilon > 0$,* FAST *admits a kernel with a most $(2+\epsilon)k$ vertices.*

For that, given an instance (T, k) of FAST, we start by computing a feedback-arc-set with at most $(1+\frac{\epsilon}{2})k$ arcs by using the approximation algorithm of C. Kenyon-Mathieu and W. Schudy [90] (if we do not success, then we answer 'NO' for the instance (T, k)). Then, if T is large enough, with more than$(2+\epsilon)k$ vertices, we find a partition of T with few arcs going across the partition. Using the following useful lemma, we can remove these arcs and decrease k accordingly.

Lemma 53 (S. Bessy, F.V. Fomin, S. Gaspers, C. Paul, A. Perez, S. Saurabh and S. Thomassé., 2009, [15])**.** *Let $E = v_1, \ldots, v_n$ be an enumeration of a tournament T with p backward arcs (i.e. arcs v_iv_j with $i > j$). If every interval $v_i, \ldots, v_j$ of E with $i < j$ contains at most $\frac{j-i}{2}$ backward arcs of T, then, T contains exactly p arc-disjoint 3-cycles.*

After that, we remove from T the vertices not contained in any cycle and repeat the process until we obtain the kernel with the desired size (or answer 'NO' if we have to reduce k below 0).

[9]This subsection is linked with the paper: S. Bessy, F. V. Fomin, S. Gaspers, C. Paul, A. Perez, S. Saurabh, and S. Thomassé, Kernels for feedback arc set in tournaments, In *FSTTCS*, volume 4 of *LIPIcs*, pages 37–47, 2009.

Remark that the complexity needed to compute the kernel depends on the complexity of the feedback-arc-set approximation of [90], and thus is in time $O(n^{O(\epsilon^{-12})})$. However, C. Paul, A. Perez and S. Thomassé gave in 2011 [107] a simpler kernelization process for FAST, providing a $4k$ kernel in quadratic time.
To conclude, there is an interesting open question related to the FAST problem. Indeed, if we are interested now in computing a feedback-vertex-set of size less than k in a tournament, there exists a kernel on $O(k^2)$ vertices for the parametrized version of this problem, see [126] for instance. But the existence of a linear kernel for this problem is still open.

Problem 54. Is there a kernel on $O(k)$ vertices for the PARAMETRIZED FEEDBACK-VERTEX-SET IN TOURNAMENT problem?

4.2 Counting edge-colorings of regular graphs[10]

In this last subsection, I present a work done with F. Havet on the number of edge-colorings of regular graphs. This work was initially motivated by results and questions from P.A. Golovach, D. Kratsch and J.F. Couturier. Indeed, in [69] there were interested in enumerating all the edge-colorings of a regular graph and provided exponential exact algorithms for this problem. However, they asked if the exponential bases for their algorithms could be improved, specially in the case of edge-coloring of cubic graphs. By using some structural tools to enumerate the edge-colorings, we settled the question and improved their results.

Algorithmic for graph coloring is a very large field of research and a lot of results are known in this area. Very classically, every kind of usual chromatic number (related to vertex-coloring, edge-coloring...) is *NP*-complete to compute (see [68, 81, 115] for instance). So, many exact algorithms with exponential time concerning these problems have been published in the last decade. One of the major results is the $O^*(2^n)$ inclusion-exclusion algorithm to compute the chromatic number of a graph found independently by A. Björklund, T. Husfeldt [24] and M. Koivisto [92]. This approach may also be used to establish a $O^*(2^n)$ algorithm to count the k-colorings and to compute the chromatic polynomial of a graph. It also implies a $O^*(2^m)$ algorithm to count the k-edge-colorings. Since edge-coloring is a particular case of vertex-coloring, a natural question is to ask if faster algorithms than the general one may be designed in these cases. For instance, very recently A. Björklund et al. [25] showed how to detect whether a k-regular graph admits a k-edge-coloring in time $O^*(2^{(k-1)n/2})$.
The existential problem, asking whether a graph has a coloring with a fixed and small number k of colors, has also attracted a lot of attention. For vertex-colorability the fastest algorithm for $k = 3$ has running time $O^*(1.3289^n)$ and was proposed by R. Beigel and D. Eppstein [12], and the fastest algorithm for $k = 4$ has running time $O^*(1.7272^n)$ and was given by F. Fomin et al. [64]. They also established algorithms for counting k-colorings for $k = 3$ and 4. The existence problem for a 3-edge-coloring is considered in [12, 93, 69]. L. Kowalik [93] gave an algorithm deciding if a graph is 3-edge-colorable in time $O^*(1.344^n)$ and polynomial space and P.A. Golovach et al. [69] presented an algorithm counting the number of 3-edge-colorings of a graph in time $O^*(3^{n/6}) = O^*(1.201^n)$ and exponential space. They also showed a branching algorithm to enumerate all the 3-edge-colorings of a connected cubic graph of running time $O^*(25^{n/8}) = O^*(1.5423^n)$ using polynomial space. In particular, this implies that every connected cubic graph of order n has at most $O(1.5423^n)$ 3-edge-colorings. Moreover,they gave an example of a connected cubic graph of order n having $\Omega(1.2820^n)$ 3-edge-colorings.
We filled the gap between these two bounds and improved their results by proving the following.

Theorem 55 (S. Bessy, F. Havet, 2011, [16])**.** *In every connected cubic multi-graph of order n, the number of 3-edge-colorings is at most $3 \cdot 2^{n/2}$. Furthermore, they can be all enumerated in time $O^*(2^{n/2}) = O^*(1.4143^n)$ using polynomial space by a branching algorithm.*

Moreover, we gave an example of connected cubic multi-graph achieving this bound. To compute efficiently the number of edge-colorings of a cubic graph G, we used an special enumeration of G, defined

[10]This subsection is linked with the paper: S. Bessy and F. Havet. Enumerating the edge-colourings and total colourings of a regular graph. *accepted in Journal of Combinatorial Optimization*, 2011.

by A. Lempel et al. in 1967 and called a *st*-ordering of G [95]. In such an ordering, every of vertex of G (excepted from the first and the last) has degree at least one on its left and also degree at least one on its right. More precisely, if we orient every edge of G from the left to the right in an *st*-ordering, we can see that the first vertex has out-degree 3, the last vertex as out-degree 0 and $\frac{n}{2}-1$ vertices of G have out-degree 1 and $\frac{n}{2}-1$ vertices of G have out-degree 2. So, given an *st*-ordering E of G, we sort the edges of G according to their left extremity in E. Then, we greedily enumerate all the edge-colorings of G by coloring as many ways as possible the edges of G according to this sorting. As, the only choices for colorings appear when a vertex has two neighbors in its right in E, we obtained the announced bound.

For simple graphs, we tried to sharpen the previous bound and showed the following.

Theorem 56 (S. Bessy, F. Havet, 2011, [16])**.** *In every connected cubic simple graph of order n, the number of 3-edge-colorings is at most $\frac{9}{4}\cdot 2^{n/2}$.*

However, we did not find an example of graph having this number of 3-edge-colorings, and we believe that $\frac{9}{4}$ is not the optimal value in the previous statement. Precisely, guided by some examples with high number of 3-edge-colorings, we conjectured the following.

Conjecture 57 (S. Bessy, F. Havet, 2011, [16])**.** Up to an additive constant, in every connected cubic simple graph of order n the number of 3-edge-colorings is at most $2^{n/2}$.

To conclude, let me mention that we extended our results to k-regular connected multi-graph and obtained the following statement.

Theorem 58 (S. Bessy, F. Havet, 2011, [16])**.** *In every connected k-regular multi-graph of order n, the number of k-edge-colorings is at most $k\cdot((k-1)!)^{n/2}$. Furthermore, they can be all enumerated in time $O^*(((k-1)!)^{n/2})$ using polynomial space by a branching algorithm.*

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