A Additionnal proof

Observation 1 *Let G a complete graph, let v be one of its vertices, let $\sigma_{\mathrm{p}} \geq 1$ and let $|V(G)| \geq 2\sigma_{\mathrm{p}} + 4\sigma_{\mathrm{c}}$. Then, there exists an optimal solution S for G such that v has degree one in* $\mathrm{Gr}(S \cup M^*)$.

Proof (of Lemma 1). $\boxed{\Rightarrow}$ We suppose that the problem $(\sigma_{\mathrm{p}}, \sigma_{\mathrm{c}})$ SP admits a feasible solution in G. We denote this solution by $\mathcal{S} = \{P_1, \ldots, P_{\sigma_{\mathrm{p}}}, C_1, \ldots, C_{\sigma_{\mathrm{c}}}\}$.
$(\sigma_{\mathrm{p}}, \sigma_{\mathrm{c}}) = (1, 0)$: There is only one path P in $\mathcal{S}$. Since P covers G, it crosses at least once between G_1 and G_2, thus $B \neq \varnothing$.
$(\sigma_{\mathrm{p}}, \sigma_{\mathrm{c}}) = (0, 1)$: There is only one cycle C in $\mathcal{S}$. Let $B' = C \cap B$. Since C covers G, it crosses at least twice between G_1 and G_2. Thus, $B' \neq \varnothing$. Since the solution contains every edge of M^*, we have necessarily $B \cap M^* \subseteq B'$. Finally, each time the cycle crosses between G_1 and G_2, it has to turn back, either by a round trip, or by another distinct bridge. Thus there is an even number of distinct bridges, and eventually round trips, in B'.
$(\sigma_{\mathrm{p}}, \sigma_{\mathrm{c}}) = (0, \geq 2)$: The solution $\mathcal{S}$ is a set of cycles $\{C_1, \ldots, C_{\sigma_{\mathrm{c}}}\}$. Let $B' = (C_1 \cup \ldots C_{\sigma_{\mathrm{c}}}) \cap B$. Again, $B \cap M^* \subseteq B'$. The above discussion about the even number of distinct bridges is still true for each cycles, thus the total number of distinct bridges in B' is even, and B' contains eventually round trips.

$\boxed{\Leftarrow}$ We suppose now that the conditions on the structure of the graph hold. We want to construct a feasible solution. We suppose that there are n' bridges being in M^*. We denote them by $e_1, \ldots, e_{n'}$.
$(\sigma_{\mathrm{p}} \geq 1, \sigma_{\mathrm{c}})$ with $\sigma_{\mathrm{p}} + \sigma_{\mathrm{c}} \geq 2$: If $n' = 0$: we construct an alternating path covering G_1, and all the other cycles and paths in G_2, which is possible since $n_2 \geq 2\sigma_{\mathrm{p}} + 4\sigma_{\mathrm{c}}$. If $n' = 1$: we construct an alternating path covering G_1, and ending with the bridge, and all the other cycles and paths in G_2, which is possible since the number of vertices available in G_2 is $\geq 2(\sigma_{\mathrm{p}} - 1) + 4\sigma_{\mathrm{c}}$. If $n' > 1$: we construct the maximum of alternating cycles of size four using the bridges in M^*, in the limit of σ_{c}. Then we construct the maximum number of paths of size one using the bridges in M^*, in the limit of $\sigma_{\mathrm{p}} - 2$. Then, we construct a path with all eventually remaining bridges in M^* and covering the remaining vertices of G_1. Finally, we construct the other paths and cycles in G_2.
$(\sigma_{\mathrm{p}}, \sigma_{\mathrm{c}}) = (0, \geq 2)$: If $n' = 0$: We construct one alternating cycle covering G_1, and the others in G_2. If $n' \neq 0$: Then, let B' be a (thus non empty) set constituted by an even number of edges defining distinct bridges, and eventually round trips, and containing $B \cap M^*$. We construct the maximum of alternating cycles of size four, in the limit of $\sigma_{\mathrm{c}} - 1$, using the bridges in M^*. If there remains cycles to be build in order to reach the number of $\sigma_{\mathrm{c}} - 1$, we construct them in G_1. Therefore, a new cycle is constructed using all remained uncovered $G_1 \cup G_2$ vertices, all the remaining bridges in $B \cap M^*$ and the eventually remaining round trip with an edge in $B \cap M^*$. This new cycle is clearly feasible.
$(\sigma_{\mathrm{p}}, \sigma_{\mathrm{c}}) = (0, 1)$ (see **??**): Let B' be a non empty set constituted by an even num ber of edges defining distinct bridges, and eventually round trips, and containing $B \cap M^*$. We begin to construct an alternating path p_1 covering G_1 without the bridges and round trips defined by B', and an alternating path p_2 without the bridges and round trips defined by B'. We glue p_1 to the eventual remaining

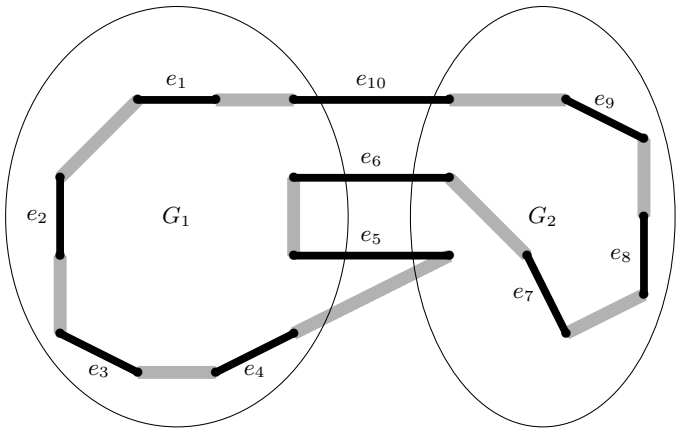

Fig. 4: Example for the case $(\sigma_{\mathrm{p}}, \sigma_{\mathrm{c}}) = (0, 1)$. Edges which complete G_1 and G_2, and other edges in B, are not represented. The path p_1 is the path between edges e_1 and e_3, p_2 is the path between e_7 and e_9, edges e_6 and e_{10} are two bridges in M^*, and the path between e_4 and e_5 is a round trip.

round trip starting and ending in G_1, and then to one of the bridges of B', then continue with the eventual remaining round trip starting and ending in G_2, then p_2, and thereafter we link consecutively the bridges and buckle the cycle. This is possible since there is an even number of bridges in B'.
$(\sigma_{\mathrm{p}}, \sigma_{\mathrm{c}}) = (1, 0)$: We do the same construction as previously, except that B' does not need to contain an even number of distinct bridges. Thus, if $n' = 0$, we just need one bridge, which exists since $B \neq \varnothing$, and if $n' > 0$, we take $B' = B \cap M^*$.