4.1 Introduction

4.1.1 Scope

The subject of this chapter is the dynamic analysis of robots modeled using the Newton-Euler method, one of several approaches for deriving the governing equations of motion. It is assumed that the robots are characterized as open kinematic chains with actuators at the joints that can be controlled. The kinematic chain consists of discrete rigid members or links, with each link attached to neighboring links by means of revolute or prismatic joints, i.e., lower-pairs (Denavit and Hartenberg, 1955). One end of the chain is fixed to a base (inertial) reference frame and the other end of the chain, with an attached end-effector, tool, or gripper, can move freely in the robotic workspace and/or be used to apply forces and torques to objects being manipulated to accomplish a wide range of tasks.

Robot dynamics is predicated on an understanding of the associated kinematics, covered in a separate chapter in this handbook. An approach for robot kinematics commonly adopted is that of a $4 \times 4$ homogeneous transformation, sometimes referred to as the Denavit-Hartenberg (D-H) transformation (Denavit and Hartenberg, 1955). The D-H transformations essentially determine the position of the origin and the rotation of one link coordinate frame with respect to another link coordinate frame. The D-H transformations can also be used in deriving the dynamic equations of robots.

4.1.2 Background

The field of robot dynamics has a rich history with many important developments. There is a wide literature base of reported work, with myriad articles in professional journals and established textbooks (Featherstone, 1987; Shahinpoor, 1987; Spong and Vidyasager, 1989; Craig, 1989; Yoshikawa, 1990; McKerrow, 1991;
Sciavicco and Siciliano, 2000; Mason, 2001; Niku, 2001) as well as research monographs (Robinett et al., 2001; Vukobratovic et al., 2003). A review of robot dynamic equations and computational issues is presented in Featherstone and Orin (2000). A recent article (Swain and Morris, 2003) formulates a unified dynamic approach for different robotic systems derived from first principles of mechanics.

The control of robot motions, forces, and torques requires a firm grasp of robot dynamics, and plays an important role given the demand to move robots as fast as possible. Robot dynamic equations of motion can be highly nonlinear due to the configuration of links in the workspace and to the presence of Coriolis and centrifugal acceleration terms. In slow-moving and low-inertia applications, the nonlinear coupling terms are sometimes neglected making the robot joints independent and simplifying the control problem.

Several different approaches are available for the derivation of the governing equations pertaining to robot arm dynamics. These include the Newton-Euler (N-E) method, the Lagrange-Euler (L-E) method, Kane’s method, bond graph modeling, as well as recursive formulations for both Newton-Euler and Lagrange-Euler methods. The N-E formulation is based on Newton’s second law, and investigators have developed various forms of the N-E equations for open kinematic chains (Orin et al., 1979; Luh et al., 1980; Walker and Orin, 1982).

The N-E equations can be applied to a robot link-by-link and joint-by-joint either from the base to the end-effector, called forward recursion, or vice versa, called backward recursion. The forward recursive N-E equations transfer kinematic information, such as the linear and angular velocities and the linear and angular accelerations, as well as the kinetic information of the forces and torques applied to the center of mass of each link, from the base reference frame to the end-effector frame. The backward recursive equations transfer the essential information from the end-effector frame to the base frame. An advantage of the forward and backward recursive equations is that they can be applied to the robot links from one end of the arm to the other providing an efficient means to determine the necessary forces and torques for real-time or near real-time control.

### 4.2 Theoretical Foundations

#### 4.2.1 Newton-Euler Equations

The Newton-Euler (N-E) equations relate forces and torques to the velocities and accelerations of the center of masses of the links. Consider an intermediate, isolated link \( n \) in a multi-body model of a robot, with forces and torques acting on it. Fixed to link \( n \) is a coordinate system with its origin at the center of mass, denoted as centroidal frame \( C_n \), that moves with respect to an inertial reference frame, \( R \), as shown in Figure 4.1. In accordance with Newton’s second law,

\[
\begin{align*}
F_n &= \frac{dP_n}{dt}, & T_n &= \frac{dH_n}{dt} \\
(4.1)
\end{align*}
\]

where \( F_n \) is the net external force acting on link \( n \), \( T_n \) is the net external torque about the center of mass of link \( n \), and \( P_n \) and \( H_n \) are, respectively, the linear and angular momenta of link \( n \),

\[
\begin{align*}
P_n &= m_n \dot{R}v_n, & H_n &= C_n^R \omega_n \\
(4.2)
\end{align*}
\]

In Equation (4.2) \( Rv_n \) is the linear velocity of the center of mass of link \( n \) as seen by an observer in \( R \), \( m_n \) is the mass of link \( n \), \( \dot{R}\omega_n \) is the angular velocity of link \( n \) (or equivalently \( C_n \)) as seen by an observer in \( R \), and \( C_n \) is the mass moment of inertia matrix about the center of mass of link \( n \) with respect to \( C_n \).

Two important equations result from substituting the momentum expressions (4.2) into (4.1) and taking the time derivatives (with respect to \( R \)):

\[
\begin{align*}
\dot{R}F_n &= m_n \dot{R}a_n \\
(4.3)
\end{align*}
\]

\[
\begin{align*}
\dot{R}T_n &= C_n^R \dot{R}\alpha_n + \dot{R} \times (C_n^R \dot{R} \omega_n) \\
(4.4)
\end{align*}
\]
where $\overset{\mathbf{R}}{a}_n$ is the linear acceleration of the center of mass of link $n$ as seen by an observer in $R$, and $\overset{\mathbf{R}}{\alpha}_n$ is the angular acceleration of link $n$ (or equivalently $C_n$) as seen by an observer in $R$. In Equation (4.3) and Equation (4.4) superscript $R$ has been appended to the external force and torque to indicate that these vectors are with respect to $R$.

Equation (4.3) is commonly known as Newton’s equation of motion, or Newton’s second law. It relates the linear acceleration of the link center of mass to the external force acting on the link. Equation (4.4) is the general form of Euler’s equation of motion, and it relates the angular velocity and angular acceleration of the link to the external torque acting on the link. It contains two terms: an inertial torque term due to angular acceleration, and a gyroscopic torque term due to changes in the inertia as the orientation of the link changes. The dynamic equations of a link can be represented by these two equations: Equation (4.3) describing the translational motion of the center of mass and Equation (4.4) describing the rotational motion about the center of mass.

### 4.2.2 Force and Torque Balance on an Isolated Link

Using a “free-body” approach, as depicted in Figure 4.2, a single link in a kinematic chain can be isolated and the effect from its neighboring links can be accounted for by considering the forces and torques they apply. In general, three external forces act on link $n$: (i) link $n - 1$ applies a force through joint $n$, (ii) link

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**FIGURE 4.1** Coordinate systems associated with link $n$.

**FIGURE 4.2** Forces and torques acting on link $n$. 

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\[ n + 1 \text{ applies a force through joint } n + 1, \text{ and } (iii) \text{ the force due to gravity acts through the center of mass.} \]

The dynamic equation of motion for link \( n \) is then

\[ ^R F_n = ^R F_{n-1,n} - ^R F_{n,n+1} + m_n ^R g = m_n ^R a_n \]

(4.5)

where \(^R F_{n-1,n}\) is the force applied to link \( n \) by link \( n - 1 \), \(^R F_{n,n+1}\) is the force applied to link \( n + 1 \) by link \( n \), \(^R g\) is the acceleration due to gravity, and all forces are expressed with respect to the reference frame \( R \).

The negative sign before the second term in Equation (4.5) is used since the interest is in the force exerted on link \( n \) by link \( n + 1 \).

For a torque balance, the forces from the adjacent links result in moments about the center of mass. External torques act at the joints, due to actuation and possibly friction. As the force of gravity acts through the center of mass, it does not contribute a torque effect. The torque equation of motion for link \( n \) is then

\[ ^R T_n = \left( ^R T_{n-1,n} - ^R T_{n,n+1} \right) - ^R d_{n-1,n} \times ^R F_{n-1,n} + ^R d_{n,n} \times ^R F_{n,n+1} + ^C I_n ^R \alpha_n + ^R \omega_n \times \left( ^C I_n ^R \omega_n \right) \]

(4.6)

where \(^R T_{n-1,n}\) is the torque applied to link \( n \) by link \( n - 1 \) as seen by an observer in \( R \), and \(^R d_{n-1,n}\) is the position vector from the origin of the frame \( n - 1 \) at joint \( n \) to the center of mass of link \( n \) as seen by an observer in \( R \).

In dynamic equilibrium, the forces and torques on the link are balanced, giving a zero net force and torque. For example, for a robot moving in free space, a torque balance may be achieved when the actuator torques match the torques due to inertia and gravitation. If an external force or torque is applied to the tip of the robot, changes in the actuator torques may be required to regain torque balance. By rearranging Equation (4.5) and Equation (4.6) it is possible to write expressions for the joint force and torque corresponding to dynamic equilibrium.

\[ ^R F_{n-1,n} = m_n ^R a_n + ^R F_{n,n+1} \]

(4.7)

\[ ^R T_{n-1,n} = ^R T_{n,n+1} + ^R d_{n-1,n} \times ^R F_{n-1,n} - ^R d_{n,n} \times ^R F_{n,n+1} + ^C I_n ^R \alpha_n + ^R \omega_n \times \left( ^C I_n ^R \omega_n \right) \]

(4.8)

Substituting Equation (4.7) into Equation (4.8) gives an equation for the torque at the joint with respect to \( R \) in terms of center of mass velocities and accelerations.

\[ ^R T_{n-1,n} = ^R T_{n,n+1} + ^R d_{n-1,n} \times m_n ^R a_n + ^R p_{n-1,n} \times ^R F_{n,n+1} \]

(4.9)

where \(^R p_{n-1,n}\) is the position vector from the origin of frame \( n - 1 \) at joint \( n \) to the origin of frame \( n \) as seen from by an observer in \( R \). Equation (4.7) and Equation (4.9) represent one form of the Newton-Euler (N-E) equations. They describe the applied force and torque at joint \( n \), supplied for example by an actuator, in terms of other forces and torques, both active and inertial, acting on the link.

### 4.2.3 Two-Link Robot Example

In this example, the dynamics of a two-link robot are derived using the N-E equations. From Newton’s Equation (4.3) the net dynamic forces acting at the center of mass of each link are related to the mass center accelerations,

\[ ^0 F_1 = m_1 ^0 a_1, \quad ^0 F_2 = m_2 ^0 a_2 \]

(4.10)

and from Euler’s Equation (4.4) the net dynamic torques acting around the center of mass of each link are related to the angular velocities and angular accelerations,

\[ ^0 T_1 = I_1 ^0 \alpha_1 + ^0 \omega_1 \times \left( I_1 ^0 \omega_1 \right), \quad ^0 T_2 = I_2 ^0 \alpha_2 + ^0 \omega_2 \times \left( I_2 ^0 \omega_2 \right) \]

(4.11)
where the mass moments of inertia are with respect to each link’s centroidal frame. Superscript 0 is used to denote the inertial reference frame \( R \).

Equation (4.9) can be used to find the torques at the joints, as follows:

\[
{0}T_{1,2} = {0}T_{2,3} + \left( {0}d_{1,2} \times m_2 \, {0}a_2 \right) + \left( {0}p_{1,2} \times {0}F_{2,3} \right) - \left( {0}d_{1,2} \times m_2 \, {0}g + I_2 \, {0}\alpha_2 + {0}\omega_2 \times (I_2 \, {0}\omega_2) \right) \tag{4.12}
\]

\[
{0}T_{0,1} = {0}T_{1,2} + \left( {0}d_{0,1} \times m_1 \, {0}a_1 \right) + \left( {0}p_{0,1} \times {0}F_{1,2} \right) - \left( {0}d_{0,1} \times m_1 \, {0}g + I_1 \, {0}\alpha_1 + {0}\omega_1 \times (I_1 \, {0}\omega_1) \right) \tag{4.13}
\]

An expression for \( {0}F_{1,2} \) can be found from Equation (4.7) and substituted into Equation (4.13) to give

\[
{0}T_{0,1} = {0}T_{1,2} + \left( {0}d_{0,1} \times m_1 \, {0}a_1 \right) + \left( {0}p_{0,1} \times m_2 \, {0}a_2 \right) + \left( {0}p_{0,1} \times {0}F_{2,3} \right) - \left( {0}d_{0,1} \times m_1 \, {0}g + I_1 \, {0}\alpha_1 + {0}\omega_1 \times (I_1 \, {0}\omega_1) \right) \tag{4.14}
\]

For a planar two-link robot with two revolute joints, as shown in Figure 4.3, moving in a horizontal plane, i.e., perpendicular to gravity, several simplifications can be made in Equation (4.11) to Equation (4.14): (i) the gyroscopic terms \( {0}\omega_n \times (C^I_n \, {0}\omega_n) \) for \( n = 1, 2 \) can be eliminated since the mass moment of inertia matrix is a scalar and the cross product of a vector with itself is zero, (ii) the gravity terms \( {0}d_{n-1,n} \times m_n \, {0}g \) disappear since the robot moves in a horizontal plane, and (iii) the torques can be written as scalars and act perpendicular to the plane of motion. Equation (4.12) and Equation (4.14) can then be written in scalar form,

\[
{0}T_{1,2} = {0}T_{2,3} + \left( {0}d_{1,2} \times m_2 \, {0}a_2 \right) \cdot \hat{u}_2 + \left( {0}p_{1,2} \times {0}F_{2,3} \right) \cdot \hat{u}_2 + I_2 \, {0}\alpha_2 \tag{4.15}
\]

\[
{0}T_{0,1} = {0}T_{1,2} + \left( {0}d_{0,1} \times m_1 \, {0}a_1 \right) \cdot \hat{u}_1 + \left( {0}p_{0,1} \times m_2 \, {0}a_2 \right) \cdot \hat{u}_2 + \left( {0}p_{0,1} \times {0}F_{2,3} \right) \cdot \hat{u}_2 + I_1 \, {0}\alpha_1 \tag{4.16}
\]

where the dot product is taken (with the terms in parentheses) with the unit vector \( \hat{u}_z \) perpendicular to the plane of motion.

A further simplification can be introduced by assuming the links have uniform cross-sections and are made of homogeneous (constant density) material. The center of mass then coincides with the geometric center and the distance to the center of mass is half the link length. The position vectors can be

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**FIGURE 4.3** Two-link robot with two revolute joints.

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written as,

\[ \mathbf{p}_{0,1} = 2 \mathbf{d}_{0,1}, \quad \mathbf{p}_{0,2} = 2 \mathbf{d}_{0,2} \] (4.17)

with the components,

\[
\begin{bmatrix}
\mathbf{p}_{0,1x} \\
\mathbf{p}_{0,1y}
\end{bmatrix} = 2 \begin{bmatrix}
\mathbf{d}_{0,1x} \\
\mathbf{d}_{0,1y}
\end{bmatrix} = \begin{bmatrix}
l_1 C_1 \\
l_1 S_1
\end{bmatrix},
\begin{bmatrix}
\mathbf{p}_{0,2x} \\
\mathbf{p}_{0,2y}
\end{bmatrix} = 2 \begin{bmatrix}
l_2 C_{12} \\
l_2 S_{12}
\end{bmatrix} (4.18)
\]

where the notation \( S_1 = \sin \theta_1, C_1 = \cos \theta_1, S_{12} = \sin(\theta_1 + \theta_2); C_{12} = \cos(\theta_1 + \theta_2) \) is introduced; and \( l_1, l_2 \) are the lengths of links 1,2, respectively.

The accelerations of the center of mass of each link are needed in Equation (4.15) and Equation (4.16). These accelerations can be written as,

\[ \mathbf{a}_1 = \mathbf{a}_1 \times \mathbf{d}_{0,1} + \mathbf{\omega}_1 \times (\mathbf{a}_1 \times \mathbf{d}_{0,1}) \] (4.19)

\[ \mathbf{a}_2 = 2 \mathbf{a}_1 \times \mathbf{d}_{0,1} + \mathbf{\omega}_2 \times (\mathbf{a}_2 \times \mathbf{d}_{0,2}) \] (4.20)

Expanding Equation (4.19) and Equation (4.20) gives the acceleration components:

\[
\begin{bmatrix}
\mathbf{a}_{1x} \\
\mathbf{a}_{1y}
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{2} l_1 S_1 \mathbf{a}_1 - \frac{1}{2} l_1 C_1 \mathbf{\omega}_1^2 \\
\frac{1}{2} l_1 C_1 \mathbf{a}_1 - \frac{1}{2} l_1 S_1 \mathbf{\omega}_1^2
\end{bmatrix}
\]

(4.21)

\[
\begin{bmatrix}
\mathbf{a}_{2x} \\
\mathbf{a}_{2y}
\end{bmatrix} = \begin{bmatrix}
-(l_1 S_1 + \frac{1}{2} l_2 S_{12}) \mathbf{a}_1 - \frac{1}{2} l_2 S_{12} \mathbf{\omega}_1 - \frac{1}{2} l_1 C_1 \mathbf{\omega}_1^2 - \frac{1}{2} l_2 C_{12} (\mathbf{a}_1 + \mathbf{\omega}_2)^2 \\
\frac{1}{2} l_1 C_1 \mathbf{a}_1 + \frac{1}{2} l_2 C_{12} \mathbf{\omega}_1 - \frac{1}{2} l_1 S_1 \mathbf{\omega}_1^2 - \frac{1}{2} l_2 S_{12} (\mathbf{a}_1 + \mathbf{\omega}_2)^2
\end{bmatrix}
\]

(4.22)

Equation (4.15) gives the torque at joint 2:

\[ \mathbf{T}_{0,2} = \mathbf{T}_{2,3} + m_2 \left[ (\mathbf{a}_{1,2x} \mathbf{u}_{x0} + \mathbf{a}_{1,2y} \mathbf{u}_{y0}) \times (\mathbf{a}_{2,1x} \mathbf{u}_{x0} + \mathbf{a}_{2,1y} \mathbf{u}_{y0}) \right] \cdot \mathbf{u}_x 
+ \left[ (\mathbf{p}_{0,2x} \mathbf{u}_{x0} + \mathbf{p}_{0,2y} \mathbf{u}_{y0}) \times (\mathbf{F}_{2,3x} \mathbf{u}_{x0} + \mathbf{F}_{2,3y} \mathbf{u}_{y0}) \right] \cdot \mathbf{u}_x + I_2 \mathbf{\alpha}_2 \]

where \( \mathbf{u}_{x0}, \mathbf{u}_{y0} \) are the unit vectors along the \( x_0, y_0 \) axes, respectively. Substituting and evaluating yields,

\[ \mathbf{T}_{0,2} = \mathbf{T}_{2,3} + \left( \frac{1}{2} m_2 l_1 l_2 C_2 + \frac{1}{2} m_2 l_2^2 + l_2 \right) \mathbf{\alpha}_1 - \left( \frac{1}{2} m_2 l_1^2 + l_2 \right) \mathbf{\alpha}_2 + \frac{1}{2} m_2 l_1 l_2 S_1 \mathbf{\alpha}_1^2 
+ I_2 C_{12} F_{2,3y} - l_2 S_{12} B_{2,3x} \]

(4.23)

Similarly, Equation (4.16) gives the torque at joint 1:

\[ \mathbf{T}_{0,1} = \mathbf{T}_{2,3} + m_1 \left[ (\mathbf{a}_{0,1x} \mathbf{u}_{x0} + \mathbf{a}_{0,1y} \mathbf{u}_{y0}) \times (\mathbf{a}_{1,1x} \mathbf{u}_{x0} + \mathbf{a}_{1,1y} \mathbf{u}_{y0}) \right] \cdot \mathbf{u}_x 
+ m_2 \left[ (\mathbf{p}_{0,1x} \mathbf{u}_{x0} + \mathbf{p}_{0,1y} \mathbf{u}_{y0}) \times (\mathbf{a}_{2,1x} \mathbf{u}_{x0} + \mathbf{a}_{2,1y} \mathbf{u}_{y0}) \right] \cdot \mathbf{u}_x 
+ \left[ (\mathbf{p}_{0,1x} \mathbf{u}_{x0} + \mathbf{p}_{0,1y} \mathbf{u}_{y0}) \times (\mathbf{F}_{2,3x} \mathbf{u}_{x0} + \mathbf{F}_{2,3y} \mathbf{u}_{y0}) \right] \cdot \mathbf{u}_x + I_1 \mathbf{\alpha}_1 \]

which yields after substituting,

\[ \mathbf{T}_{0,1} = \mathbf{T}_{2,3} + \left( \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 + l_1 \right) \mathbf{\alpha}_1 - \left( \frac{1}{2} m_2 l_1 l_2 C_2 \right) \mathbf{\alpha}_2 
- \frac{1}{2} m_2 l_1 l_2 S_2 \mathbf{\alpha}_1^2 - \frac{1}{2} m_2 l_1 l_2 S_2 \mathbf{\alpha}_1^2 - m_2 l_1 l_2 S_1 \mathbf{\alpha}_1 \mathbf{\alpha}_2 
+ I_1 C_{12} F_{2,3y} - l_2 S_{12} B_{2,3x} \]

(4.24)
The first term on the right hand side of Equation (4.24) has been found in Equation (4.23) as the torque at joint 2. Substituting this torque into Equation (4.24) gives:

\[
{^0T_{11}} = {^0T_{23}} + \left( \frac{1}{4}m_1l_1^2 + \frac{1}{4}m_2l_2^2 + m_2l_1^2 + m_2l_1l_2C_2 + I_1 + I_2 \right) \dot{\theta}_1 \\
+ \left( \frac{1}{4}m_2l_2^2 + \frac{1}{4}m_3l_1l_2C_2 + I_2 \right) \dot{\theta}_2 - \frac{1}{2}m_2l_1l_2S_2\ddot{\theta}_2 \\
- m_2l_1l_2\dot{\theta}_1\dot{\theta}_2 + (l_1C_1 + l_2C_{12}){^0F_{2,3F}} - (l_1S_1 + l_2S_{12}){^0F_{2,3S}}
\]  

(4.25)

### 4.2.4 Closed-Form Equations

The N-E equations, Equation (4.7) and Equation (4.9), can be expressed in an alternative form more suitable for use in controlling the motion of a robot. In this recast form they represent input-output relationships in terms of independent variables, sometimes referred to as the generalized coordinates, such as the joint position variables. For direct dynamics, the inputs can be taken as the joint torques and the outputs as the joint position variables, i.e., joint angles. For inverse dynamics, the inputs are the joint position variables and the outputs are the joint torques. The inverse dynamics form is well suited for robot control and programming, since it can be used to determine the appropriate inputs necessary to achieve the desired outputs.

The N-E equations can be written in scalar closed-form,

\[
T_{n-1,n} = \sum_{j=1}^{N} J_n\dot{\theta}_j + \sum_{j=1}^{N} \sum_{k=1}^{N} D_{njk}\dot{\theta}_j\dot{\theta}_k + T_{\text{ext},n}, \quad n = 1, \ldots, N
\]  

(4.26)

where \( J_n \) is the mass moment of inertia of link \( j \) reflected to joint \( n \) assuming \( N \) total links, \( D_{njk} \) is a coefficient representing the centrifugal effect when \( j = k \) and the Coriolis effect when \( j \neq k \) at joint \( n \), and \( T_{\text{ext},n} \) is the torque arising from external forces and torques at joint \( n \), including the effect of gravity and end-effector forces and torques.

For the two-link planar robot example, Equation (4.26) expands to:

\[
T_{0,1} = J_{11}\dot{\theta}_1 + J_{12}\dot{\theta}_2 + (D_{112} + D_{122})\dot{\theta}_1\dot{\theta}_2 + D_{122}\dot{\theta}_2^2 + T_{\text{ext},1}
\]  

(4.27)

\[
T_{1,2} = J_{21}\dot{\theta}_1 + J_{22}\dot{\theta}_2 + D_{211}\dot{\theta}_1^2 + T_{\text{ext},2}
\]  

(4.28)

Expressions for the coefficients in Equation (4.27) and Equation (4.28) can be found by comparison to Equation (4.23) and Equation (4.25). For example,

\[
J_{11} = \frac{1}{4}m_1l_1^2 + \frac{1}{4}m_2l_2^2 + m_2l_1^2 + m_2l_1l_2C_2 + I_1 + I_2
\]  

(4.29)

is the total effective moment of inertia of both links reflected to the axis of the first joint, where, from the parallel axis theorem, the inertia of link 1 about joint 1 is \( \frac{1}{4}m_1l_1^2 + I_1 \) and the inertia of link 2 about joint 1 is \( m_2(l_1^2 + \frac{1}{2}l_2^2 + l_1l_2C_2) + I_2 \). Note that the inertia reflected from link 2 to joint 1 is a function of the configuration, being greatest when the arm is straight (when the cosine of \( \theta_2 \) is 1).

The first term in Equation (4.27) is the torque due to the angular acceleration of link 1, whereas the second term is the torque at joint 1 due to the angular acceleration of link 2. The latter arises from the contribution of the coupling force and torque across joint 2 on link 1. By comparison, the total effective moment of inertia of link 2 reflected to the first joint is

\[
J_{12} = \frac{1}{4}m_2l_2^2 + m_2l_1l_2C_2 + I_2
\]

which is again configuration dependent.
The third term in Equation (4.27) is the torque due to the Coriolis force,

\[ T_{0,1,\text{Coriolis}} = (D_{112} + D_{212}) \dot{\theta}_1 \dot{\theta}_2 = -m_2 l_1 S_2 \dot{\theta}_1 \dot{\theta}_2 \]

resulting from the interaction of the two rotating frames. The fourth term in Equation (4.27) is the torque due to the centrifugal effect on joint 1 of link 2 rotating at angular velocity \( \dot{\theta}_2 \),

\[ T_{0,1,\text{centrifugal}} = D_{122} \dot{\theta}_2^2 = -\frac{1}{2} m_2 l_1 S_2 \dot{\theta}_1 \dot{\theta}_2^2 \]

The Coriolis and centrifugal torque terms are also configuration dependent and vanish when the links are co-linear. For moving links, these terms arise from Coriolis and centrifugal forces that can be viewed as acting through the center of mass of the second link, producing a torque around the second joint which is reflected to the first joint.

Similarly, physical meaning can be associated with the terms in Equation (4.28) for the torque at the second joint.

### 4.3 Additional Considerations

#### 4.3.1 Computational Issues

The form of Equation (4.26) matches the inverse dynamics representation, with the inputs being the desired trajectories, typically described as time functions of \( \theta_1(t) \) through \( \theta_N(t) \), and known external loading. The outputs are the joint torques applied by the actuators needed to follow the specific trajectories. These joint torques are found by evaluating Equation (4.26) at each instant after computing the joint angular velocities and angular accelerations from the given time functions. In the inverse dynamics approach, the joint positions, velocities, and accelerations are assumed known, and the joint torques required to cause these known time-dependent motions are sought.

The computation can be intensive, posing a challenge for real time implementation. Nonetheless, the inverse dynamics approach is particularly important for robot control, since it provides a means to account for the highly coupled, nonlinear dynamics of the links. To alleviate the computational burden, efficient inverse dynamic algorithms have been developed that formulate the dynamic equations in recursive forms allowing the computation to be accomplished sequentially from one link to another.

#### 4.3.2 Moment of Inertia

The mass distribution properties of a link with respect to rotations about the center of mass are described by its mass moment of inertia tensor. The diagonal components in the inertia tensor are the moments of inertia, and the off-diagonal components are the products of inertia with respect to the link axes. If these link axes are chosen along the principal axes, for which the link is symmetrical with respect to each axis, the moments of inertia are called the principal moments of inertia and the products of inertia vanish. Expressions for the moments of inertia can be derived and those for common shapes are readily available in tabulated form. For the planar robot example above, if each link is a thin-walled cylinder of radius \( r \) and length \( l \) the moment of inertia about its centroidal axis perpendicular to the plane is \( I = \frac{1}{2} mr^2 + \frac{1}{12} ml^2 \).

Other geometries can be evaluated. If each link is a solid cylinder of radius \( r \) and length \( l \) the moment of inertia about its centroidal axis perpendicular to the plane is \( I = \frac{1}{2} mr^2 + \frac{1}{12} ml^2 \). If each link is a solid rectangle of width \( w \) and length \( l \), the moment of inertia is \( I = \frac{1}{12} mw^2 + \frac{1}{12} ml^2 \).

#### 4.3.3 Spatial Dynamics

Despite their seeming complexity, the equations of motion for the two-link planar robot in the example (Equation (4.23) and Equation (4.25)) are relatively simple due to the assumptions, in particular, restricting
the motion to two-dimensions. The expansion of the torque balance equation for robot motions in three-dimensions results in expressions far more involved.

In the example, gravity is neglected since the motion is assumed in a horizontal plane. The Euler equation is reduced to a scalar as all rotations are about axes perpendicular to the plane. In two-dimensions, the inertia tensor is reduced to a scalar moment of inertia, and the gyroscopic torque is zero since it involves the cross product of a vector with itself.

In the three-dimensional case, the inertia tensor has six independent components in general. The vectors for translational displacement, velocity, and acceleration each have three components, in comparison with two for the planar case, adding to the complexity of Newton's equations. The angular displacement, velocity, and acceleration of each link are no longer scalars but have three components. Transformation of these angular vectors from frame to frame requires the use of rotation transforms, as joint axes may not be parallel. Angular displacements in three dimensions are not commutative, so the order of the rotation transforms is critical. Gyroscopic torque terms are present, as are terms accounting for the effect of gravity that must be considered in the general case. For these reasons, determining the spatial equations of motion for a general robot configuration can be a challenging and cumbersome task, with considerable complexity in the derivation.

4.4 Closing

Robot dynamics is the study of the relation between the forces and motions in a robotic system. The chapter explores one specific method of derivation, namely the Newton-Euler approach, for rigid open kinematic chain configurations. Many topics in robot dynamics are not covered in this chapter. These include algorithmic and implementation issues (recursive formulations), the dynamics of robotic systems with closed chains and with different joints (e.g., higher-pairs), as well as the dynamics of flexible robots where structural deformations must be taken into account. Although some forms of non-rigid behavior, such as compliance in the joint bearings, are relatively straightforward to incorporate into a rigid body model, the inclusion of dynamic deformation due to link (arm) flexibility greatly complicates the dynamic equations of motion.

References