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Stabilization of an Inertia Wheel Inverted Pendulum using Model Based Predictive Control

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Abstract— Model Predictive Control (MPC) refers to a specific procedure in controller design, explicitly using the plant model to predict the future output, the control law is computed by solving an optimization problem and receding horizon control is used for control implementation. This paper studies the problem of stabilization of a non-minimum phase inertia wheel inverted pendulum (IWP) having one actuator and two degrees of freedom using Model based predictive control strategy. Two different schemes of MPC are proposed, to begin with MPC based on Generalized Predictive Control (GPC) is used and then GPC is extended to MPC based on Laguerre functions to reduce the number of terms used to solve the optimization problem. It is shown using simulations that both the control schemes stabilize the IWP system around an unstable equilibrium point and effectively maintain this state. In the basic MPC approach approximation of control signal required a large number of parameters i.e. a high control horizon resulting in increased computational load. Furthermore, the MPC based on Laguerre functions is able to achieve stabilization by using a fraction of terms as compared to GPC. Parametric uncertainties are considered to demonstrate the robustness of MPC control based on Laguerre Functions.

Keywords— Model Predictive Control, optimization problem, inertia wheel inverted pendulum, Generalized Predictive Control, Laguerre functions

I. INTRODUCTION

Inertia Wheel Pendulum (IWP) is classified as a benchmark under-actuated mechanical system in the class of nonlinear non-minimum phase systems. The dynamic system consists of numerous complexly coupled elements which makes the stabilization problem of such a system relatively difficult and generally requires a special designed controller to handle the system. Furthermore, the unstable zero dynamics of the system need to be catered to in the design process as well using output feedback or predictive control. The system consists of a simple pendulum with a wheel attached to its independent and floating end, whereas the other pendulum end is connected to an unactuated rotating joint. The wheel is driven by a “high torque to weight ratio” DC motor that can spin the wheel in both clockwise and anticlockwise directions. When the wheel is rotated at a fairly high speed, angular acceleration is produced and resultant torque is used as an input for controlling the pendulum angle. IWP, thus, falls into the category of under-actuated mechanical systems, since one end of the pendulum remains unactuated, and IWP is controlled by a single actuator.

Prominent work reported for control of such under-actuated mechanical systems includes [1], [2], [3], [4], [5] and [6] which classify the system into eight different categories depending on certain system properties. According to the classification given in [1], the IWP falls together with Translational Oscillator with Rotating Actuator (TORA) due to similar mathematical model in which the dynamic equations appear, in strict feedback normal form. While the above mentioned work effectively addresses issues using dynamic state feedback, many researchers investigated stabilization and control of nonlinear systems using various nonlinear design tools, e.g. sliding mode control, Lyapunov redesign, Back-stepping for synthesis of suitable state feedback laws and then extending these to output feedback using a high-gain observer [5], [7].

This work takes into consideration the contributions [1], [2], [4], and [7] and proposes a predictive controller for the nonlinear system. To begin with Model Predictive Control (MPC) based on Generalized Predictive Control (GPC) [8] is used and then GPC is extended to MPC based on Laguerre functions [9] and [10] to reduce the number of terms used to solve the optimization problem. It is shown using simulations that both the control schemes stabilize the IWP system and effectively maintain this state in the presence of parametric uncertainties. The simulation results included depict the effectiveness of the proposed MPC controller.

II. PROBLEM FORMULATION

The nonlinear dynamic model of IWP is expressed as follows [11] and the mechanical model of IWP is shown in Fig.1 [7]

\[
\begin{bmatrix}
I + i_2 & i_2 \\
i_2 & i_2
\end{bmatrix} \begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix} + \begin{bmatrix}
-m^1 g \sin \theta_1 \\
0
\end{bmatrix} = \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
\]

(1)

where \(\theta_1\) and \(\theta_2\) are respectively, the angular positions of the pendulum body and the inertia wheel, \(\dot{\theta}_1\) and \(\dot{\theta}_2\) represent their corresponding accelerations, \(c_1\) is the external disturbing torque applied on the pendulum, \(c_2\) is the torque
generated by the system’s actuator, \( i_1 \) and \( i_2 \) are respectively the moments of inertia of the pendulum body and the wheel, \( l = m_1 l_1^2 + m_2 l_2^2 + i_1 \) with \( m_1, m_2 \) being the mass of pendulum and the inertia wheel, \( l_1, l_2 \) are the distance from the origin to the gravity centers of the pendulum and the rotating mass (respectively), \( \overline{m}l = m_1 l_1 + m_2 l_2 \).

**Fig. 1.** Inertia Wheel Pendulum (IWP)

### A. State Space Representation

Through linearization around the unstable equilibrium point, the state space representation of the inertia wheel inverted pendulum is obtained for the nonlinear dynamic model. This representation is vital for developing the MPC Controller. To start with, let the state vector be defined as

\[
\begin{bmatrix}
\theta_1(t) \\
\theta_2(t)
\end{bmatrix}
\]

and \( u = c_c \). The unstable equilibrium point is taken as \( x_0^* = [0 \ 0 \ 0] \), \( u^* = 0 \). The linear state space representation of the IWP results in

\[
\begin{align*}
\dot{x}_c &= \begin{bmatrix} \theta_1 \ 
\dot{\theta}_1 \ 
\theta_2 \ 
\dot{\theta}_2
\end{bmatrix}
u \\
\dot{y}_c &= C_c x_c + D_c u
\end{align*}
\]

where

\[
A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\
-\frac{1}{l} & 0 & 0 & 0 \\
\frac{m g}{l} & -\frac{1}{l} & 0 & 0 \\
\frac{m g}{l} & 0 & 0 & 0
\end{bmatrix}, \quad
B_c = \begin{bmatrix} 0 \\
0 \\
i_2 + i_1 \\
i_2 + i_1
\end{bmatrix}, \quad
C_c = \begin{bmatrix} 1 & 0 & 0 & 0
\end{bmatrix}, \quad
D_c = 0
\]

### III. CONTROL DESIGN

The objective of the control design is to robustly stabilize the inertia wheel inverted pendulum around its unstable equilibrium state in the presence of external disturbances and parametric uncertainties.

#### A. Model representation for Predictive Control

To start with, the MPC design requires the dynamical model of the system to be represented in discrete-time. To that end let’s consider the state vector \( x_d \) defined by:

\[
x_d = [\theta_1(k) \ \theta_1(k) \ \theta_2(k)]^T
\]

The discretization of the state space model leads to following discrete time representation:

\[
\begin{align*}
x_d(k+1) &= A_d x_d(k) + B_d u(k) \\
y_d(k) &= C_d x_d(k) + D_d u(k)
\end{align*}
\]

where

\[
A_d = e^{A_c T}, \quad B_d = \int_0^T e^{A_c (T - t)} B_c \tau_d C_d = C_c, \quad D_d = D_c.
\]

Where \( A_c, B_c, C_c \) and \( D_c \) are the matrices of the continuous-time state space system and \( T \) is the sampling period.

The control input at instant \( k \) is represented as an incremental vector: \( \Delta u(k) = u(k) - u(k-1) \) and the augmented state vector \( x_e = [x_d(k+1) \ u(k)]^T \), the system dynamics can therefore be written as

\[
\begin{align*}
x_e(k+1) &= A_e x_e(k) + B_e \Delta u(k) \\
y_e(k) &= C_e x_e(k) + D_e \Delta u(k)
\end{align*}
\]

where

\[
A_e = \begin{bmatrix} A_d & B_d \\
0 & 1
\end{bmatrix}, \quad
B_e = \begin{bmatrix} B_d \\
1
\end{bmatrix}, \quad
C_e = \begin{bmatrix} C_d & 0
\end{bmatrix}, \quad
D_e = D_d.
\]

This extended model shall be used to implement the proposed control schemes.

#### B. Generalized Predictive Control (GPC)

Generalized Predictive Control (GPC) was introduced by D.W. Clarke [8] and is considered the most popular method of MPC to date. The variance between several existing methods of MPC is primarily the approach in which the control problem is formulated. The notion behind GPC is to calculate the future control signals while principally minimizing a cost function over the prediction horizon. The controller is designed to minimize the difference between predicted output and the desired control objective i.e. the required unstable set-point.

GPC can deal with non-minimum phase systems [10], open-loop unstable systems [11], systems having variable or unknown dead-time [8] and systems with unknown order [6].

For the problem in hand the objective is to impose

\[
\theta_1(t_f) = \theta_1(t_f) = \theta_2(t_f) = 0 \quad \text{where} \quad t_f \text{ defines the end of the prediction horizon at each sampling instant.}
\]

With this formulation, the cost function to be minimized is as follows:

\[
J = \sum_{j=N_1}^{N_p} (y(k+j) - w(k+j))^T Q (y(k+j) - w(k+j)) + (x(k + N_c) - w_x(k + N_p))^T R (x(k + N_c) - w_x(k + N_p)) + \sum_{j=1}^{N_c} \lambda(j) \Delta u(k+j-1)^T \Delta u(k+j-1))
\]

where: \( N_1 \) and \( N_p \) are the the beginning and end of prediction horizons, \( N_c \) is the length of the control horizon which dictates the number of parameters used to capture the future control trajectory, \( Q \) is the weight on the system states, \( \lambda \) is the weight on control and \( w_x \) is the desired final state of the system.

The calculation of the control input requires the prediction of future outputs \( y(k+j), j = 1 \ldots N_p \) at each sampling instant \( k \). From the state space representation of the system, we have:

\[
x(k+1) = Ax(k) + B \Delta u(k) \\
x(k+2) = Ax(k+1) + B \Delta u(k+1) = A^2 x(k) + B u(k+1)
\]
Therefore, the predicted output of the system at time $k+j$ is written as:

$$y(k+j) = CA^j y(k) + \sum_{i=0}^{j-1} CA^{j-i-1} B \Delta u(k+i)$$  \hspace{1cm} (7)

From (8) it can be seen that these predictions are formulated in terms of current state information and the future control increments. It is clear that the length of control horizon is shorter than the length of the optimization windows i.e. $N_p \geq N_c \forall i \geq N_c, u(k+i) = 0$, and take the structure

$$y = \begin{bmatrix} y(k+1) \\ y(k+2) \\ \vdots \\ y(k+N_i) \\ \vdots \\ y(k+N_p) \end{bmatrix}, \quad \Delta u = \begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+N_i) \\ \vdots \\ \Delta u(k+N_c) \end{bmatrix} \hspace{1cm} (9)$$

At this point (8) and (9) can be represented as:

$$y = G \Delta u + F$$  \hspace{1cm} (10)

and $F$ are given as:

$$G = \begin{bmatrix} CB & 0 & 0 & \ldots & 0 \\ CAB & CB & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{N_i-1}B & CA^{N_i-2}B & CA^{N_i-3}B & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{N_p-1}B & CA^{N_p-2}B & CA^{N_p-3}B & \ldots & CA^{N_p-N_c}B \end{bmatrix}$$

The condition for minimum cost is $\frac{\partial J}{\partial \Delta u} = 0$. The optimal control can thus be found by simplifying (13).

The obtained optimal solution $\Delta u$ is then written as:

$$\Delta u = -K x(k)$$  \hspace{1cm} (14)

where

$$K = (G^T G + \bar{C}^T \bar{C})^{-1}(G^T L + \bar{C}^T Q A^p)$$

The Receding Horizon Principle is now applied and only the first sample of the control sequence is applied to the system and the rest is discarded. The process is then repeated on the next sample where the window is receded.

C. MPC using Laguerre Functions

MPC approach using Laguerre functions uses the same methodology as GPC, which utilizes the forward operator to expand the incremental control signal and resulting in high computational load. Laguerre functions are orthonormal functions and can be used to approximate the increments of control signals. Taking their advantage, in this scheme Laguerre functions are utilized in the GPC approach. The z-transform of discrete-time Laguerre functions is given as:

$$\Gamma_N(z) = \frac{1-a^{-1}}{1-a^{-1} z^{-1}}$$  \hspace{1cm} (15)

where, $0 \leq a < 1$ is the scaling factor and $N=1, 2...$ is the number of Laguerre terms.

Discrete-time Laguerre functions can be formulated from (15) using the following relation:

$$\Gamma_N(z) = \Gamma_{N-1}(z) \frac{z^{-1} - a}{1 - a z^{-1}}$$  \hspace{1cm} (16)

Let the partial inverses in vector form be represented as $L(k)$. Using (16) and considering initial condition $L(0)$, the discrete-time Laguerre functions can be represented in the form of difference equation:

$$L(k+1) = A_l L(k)$$  \hspace{1cm} (17)

where

$$A_l = \begin{bmatrix} a & 0 & 0 & \ldots \\ \beta & a & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ (-1)^{N-2} a^{-N+2} & (-1)^{N-3} a^{-N+3} & \ldots & (-1)^{N-1} a^{-N+1} \\ \beta & a & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ (-1)^{N-2} a^{-N+2} & (-1)^{N-3} a^{-N+3} & \ldots & (-1)^{N-1} a^{-N+1} \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

Thus at time $k$ discrete Laguerre functions can be used to characterize the control trajectory:

$$\Delta u(k_i + m) = \sum_{i=1}^{N} C_i(k_i) \eta_i = L(m)^T \eta$$  \hspace{1cm} (18)
where, \( m=0,1,2 \ldots N_p \), \( \eta = [c_1, c_2, \ldots] \).

The value of \( x(k_i + m) \) for single-input can be shown as follows:

\[
x(k_i + m | k_i) = A^m x(k_i) + \sum_{i=0}^{m-1} A^{m-i-1} B L(i)^T \eta
\]  

(19)

Extending the system from single-input to multi-input and the control variable is formulated as follows:

\[
\Delta u(k) = [\Delta u_1(k) \Delta u_2(k) \ldots]^T
\]  

(20)

where, \( p \) is the number of inputs and \( \Delta u_i(k) = L_i(k)\eta_i \).

The prediction of future state at time \( m \) becomes:

\[
x(k_i + m | k_i) = A^m x(k_i) + \phi(m)^T \eta
\]  

(21)

where,

\[
\phi(m) = \sum_{j=0}^{m-1} A^{m-j-1} [B_1 L_1(j)^T B_2 L_2(j)^T \ldots (j)^T]
\]  

(22)

and the control law can be realized as follows:

\[
\Delta u(k) = \left( \begin{array}{cccc}
L_1(k)^T & 0 & \ldots & 0 \\
0 & L_2(k)^T & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & L_p(k)^T
\end{array} \right) \eta
\]  

(23)

where, \( \eta = [\eta_1^T \eta_2^T \ldots]^T \).

The cost function (13) in terms of Laguerre parameter \( \eta \) is given by,

\[
J = \eta^T \Omega \eta + 2 \eta^T \Psi x(k_i) + \sum_{m=1}^{N_c} x(k_i)^T (A^T)^m Q A^m x(k_i)
\]  

(24)

where,

\[
\Omega = \sum_{m=1}^{N_c} \phi(m)Q\phi(m)^T + R_L
\]

\[
\Psi = \sum_{m=1}^{N_c} \phi(m)Q A^m
\]

After optimal value of \( \eta \) is calculated the receding horizon control is realized.

\[
\Delta u(k) = \left( \begin{array}{cccc}
L_1(0)^T & 0 & \ldots & 0 \\
0 & L_2(0)^T & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & L_p(0)^T
\end{array} \right) \eta
\]  

(25)

IV. RESULTS

This section presents and discusses the Simulation results, obtained using Matlab environment. To demonstrate the feasibility and effectiveness of the proposed control scheme two scenarios are considered. The first scenario is when the system is considered in the nominal case without any external disturbances and the objective of the second scenario is to show the robust behavior of the controller against parameter’s uncertainties.

Considering the nominal scenario with no external disturbances the simulation results of both GPC and MPC approach based on Laguerre functions when used for stabilizing the IWP are presented.

In the second scenario, parameter uncertainty is considered on the parameter I i.e. inertia. Results demonstrate the robustness of the Laguerre functions based MPC scheme to overcome uncertainties in system parameters.

Considering the model (1) the system parameters are given in Table 1. These parameters are of a real prototype system [11].

### TABLE I. PARAMETERS OF THE INERTIA WHEEL PENDULUM

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_1)</td>
<td>Body Mass</td>
<td>3.30810 Kg</td>
</tr>
<tr>
<td>(m_2)</td>
<td>Wheel Mass</td>
<td>3.33081 Kg</td>
</tr>
<tr>
<td>(l_1)</td>
<td>Body Center of Mass Position</td>
<td>0.06 m</td>
</tr>
<tr>
<td>(l_2)</td>
<td>Wheel Center of Mass Position</td>
<td>0.044 m</td>
</tr>
<tr>
<td>(i_1)</td>
<td>Body Inertia</td>
<td>0.0314683 Kg m²</td>
</tr>
<tr>
<td>(i_2)</td>
<td>Wheel Inertia</td>
<td>0.0004176 Kg m²</td>
</tr>
</tbody>
</table>

The objective of the controller is the stabilization of the inertia wheel inverted pendulum around its unstable equilibrium point i.e. to impose \( \theta_1(t_f) = \theta_2(t_f) = 0 \) where \( t_f \) defines the end of the optimization windows at each sampling instant. The initial condition considered is \( x_0 = [\theta_1 = 18^\circ \ \dot{\theta}_1 = 0 \ \dot{\theta}_2 = 0]^T \)

A. Scenario 1: Generalized Predictive Control

The simulation results obtained by implementing GPC on the inertia wheel inverted pendulum are shown in (Fig.2). The results show that the GPC controller stabilizes the IWP system around the unstable equilibrium point and maintains the angular position of the pendulum.
The controller parameters for this simulation are summarized in Table 2.

### TABLE II. GPC CONTROL PARAMETERS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_p$</td>
<td>Maximum Prediction Horizon</td>
<td>40</td>
</tr>
<tr>
<td>$N_c$</td>
<td>Length of Control Horizon</td>
<td>40</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Weight on control</td>
<td>40</td>
</tr>
<tr>
<td>$Q$</td>
<td>Weight on states</td>
<td>Identity Matrix(3,3)</td>
</tr>
</tbody>
</table>

B. Scenario 1: MPC based on Laguerre Functions

The results of obtained by implementation of MPC approach based on Laguerre Functions are shown in (Fig.3). It can be observed that the control stabilizes the system around its equilibrium point and keep it at this angular position.

MPC based on Laguerre functions has tuning parameters $\alpha$ i.e scaling factor of the Laguerre network and $N$ i.e the number of Laguerre functions used to generate the control trajectory. The scaling factor is always in the range $0<\alpha<1$. Laguerre functions are actually exponential functions with a decay factor $\alpha$. Hence by using the Laguerre functions the projected control signal is forced to decay exponentially.

The controller parameters for this simulation are summarized in Table 3.

### TABLE III. PARAMETERS FOR MPC BASED ON LAGUERRE FUNCTIONS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>Scaling Factor</td>
<td>0.4</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of Laguerre Terms</td>
<td>4</td>
</tr>
</tbody>
</table>

Comparing results in (Fig.2) and (Fig.3), it is evident that the MPC control based on Laguerre Functions is more aggressive and stabilizes at the equilibrium point faster than GPC. Also, the MPC based on Laguerre Function Proves to be more feasible as the number of terms used in the optimization procedure i.e N=4 is far less in comparison to terms used in GPC i.e N=40.
C. Scenario 2: Robustness Analysis with Parametric Uncertainty

Let’s now consider the scenario in which parameter uncertainty exists on the parameter Inertia I that is:

\[ I' = I + \Delta I \times I \]  \hspace{1cm} (26)

Three cases of uncertainty in parameter I are considered: The nominal case \( \Delta I = 0\% \), the second case with \( \Delta I = 10\% \) and the third case of \( \Delta I = 45\% \).

(Fig.4) shows the simulation results when the MPC based on Laguerre Functions is used in all three cases of uncertainty in parameter I. The results clearly show that the controller can compensate the parametric uncertainty. The system performance is slightly reduced as the uncertainty increases.

![Fig. 4. Robustness Analysis of MPC based on Laguerre Functions](image)

V. CONCLUSION AND FUTURE WORK

This paper studies the problem of stabilization of a non minimum phase Inertia Wheel Pendulum having one actuator and two degrees of freedom using Model based predictive Control methodology. Two different schemes of MPC are proposed, to begin with MPC based on Generalized Predictive Control is used and then GPC is extended to MPC based on Laguerre functions. Simulation results show that both the control schemes stabilize the IWP system around an unstable equilibrium point and effectively maintain this state. MPC based on Laguerre functions performs better and is able to achieve stabilization by using a fraction of terms as compared to GPC. Furthermore, the MPC approach based on Laguerre Functions is able to compensate parametric uncertainties. Our future work emphasis will be on incorporating constraints in the MPC scheme to control IWP.

REFERENCES


