Abstract.
In this paper we introduce a new graph based bidding language for combinatorial auctions. In our language, each bidder submits to the arbitrator a generalized flow network (netbid) representing her bids. The interpretation of the winner determination problem is that of an aggregation of individual preferences represented as flowbids allows building an aggregate netbid for its representation. Labelling the nodes with appropriate procedural functions considerably improves upon the expressivity of previous bidding languages.

1 Introduction
A Combinatorial Auction (CA) is an abstraction of a marked-based centralized distributed system for the determination of welfare allocations of heterogeneous indivisible resources. In such a Resource Allocation (RA) system, there is a central node $a$, the auctioneer, and a set of $n$ nodes, $I = \{1, \ldots , n\}$, the bidders, which concurrently demands bundles of resources from a common set of available resources, $R = \{r_1, \ldots , r_m\}$, held by the auctioneer. The auctioneer broadcasts $R$ to all $n$ bidders, asking them to submit in a specified common language, the bidding language, their R-valuations over bundles of resources. Bidder’s $i$ R-valuation, $v_i$, is a non-negative real function on $P(R)$, expressing for each bundle $S \subseteq R$ the individual interest (value), $v_i(S)$, of bidder $i$ in obtaining $S$. It is assumed that $v_i(\emptyset) = 0$, and $v_i(S) \leq v_i(T)$ whenever $S \subseteq T$. No bidder $i$ knows the value of any other $n-1$ bidders, but all the participants in the system agreed on a welfare outcome: Based on bidders R-valuations, the auctioneer will determine a resources allocation $O = (O_1, \ldots , O_n)$, specifying for each bidder $i$ her obtained bundle $O_i$. $O$ is a (weak) n-partition of $R$, that is $O_i \cap O_j = \emptyset$, for any different bidders $i$ and $j$, and $\cup_{i=1,n} O_i = R$. Furthermore, the global (social) value of the outcome $v(O) = \sum_{j=1,n} v_j(O_j)$ is a maximum value allocation, that is $v(O) = \max \{v(O') \mid O' \text{ is a n-partition of } R\}$.

The task of the auctioneer to find the maximum value allocation for a given set of bidder valuations is called the Winner Determination Problem (WDP). This is a NP-hard problem, being equivalent to weighted set-packing. WDP is expressed as an integer linear program and solved using standard methods. WDP can be parameterized by the set $R$ of resources, considering a fixed set $I$ of bidders and bidders R-valuations $\{v_i \mid i \in I\}$. Therefore we can write $WDP(R)$ and its corresponding maximum value $v_o(R)$. With these notations, $WDP(S)$ and $v_o(S)$ are well defined for each subset $S \subseteq R$ (by considering the restriction of $v_i$ to $P(S)$). We have obtained a global R-valuation $v_o$ assigning to each bundle $S \subseteq R$ the maximum value of an $S$-allocation to the bidders from $I$. Therefore WDP can be viewed as the problem of constructing a social aggregation of R-valuations of bidders.

If we denote by $V(R)$ the set of all R-valuations, it is natural to consider in our RA system the set of superadditive $R$-valuations due to the synergies among the resources: $SV(R) = \{v \in V(R) \mid v(A_1 \cup A_2) \geq v(A_1) + v(A_2) \text{ for all } A_1, A_2 \subseteq R, A_1 \cap A_2 = \emptyset\}$.

It is easy to see that if all $v_i \in I$ are superadditive then $v_o$ is superadditive and the following theorem holds:

**Theorem 1** If all bidders $R$-valuations are superadditive, then the aggregate R-valuation $v_o$ satisfies:

$$v_o(A) = \max_{B \subseteq A} [v_o(B) + v_o(A - B)] \text{ for all } A \subseteq R.$$ 

Let $v \in V(R)$. A $v$-basis is any $B \subseteq P(R)$ such that for each $A \subseteq R$ we have $v(A) = \max_{B \subseteq A} [v_o(B) + v(A - B)]$. In other words, if $B$ is a $v$-basis, then the value of $v(A)$ is uniquely determined by the values of $v$ on the elements of the basis contained in $A$, for each $A \subseteq R$. The elements of a $v$-basis, $B \in E$, are called bundles and the pairs $(B, \{v(B)\}) \in B$ are called bids. It is not difficult to prove that a $v$-valuation $v \in V(R)$ has a $v$-basis iff $v \in SV(R)$ and furthermore, the following representational theorem holds:

**Theorem 2** If in a RA system the bidder superadditive $R$-valuations $v_i$ are represented using $v_i$-basis $B_i$, for each $i \in I$, then the aggregate $R$-valuation $v_o$ is represented by the $v_o$-basis $B_o = \cup_{i \in I} B_i$, by taking $v_o(B) = \max \{v_i(B) \mid i \in I \text{ and } B \in B_i\}$, for all $B \in B_o$.

2 Approach
In the new language, each bidder submits to the arbitrator a generalized flow network called NETBID, which represents the valuation of the bidder, by specifying a basis for it. More precisely, if the set of resources is $R = \{r_1, r_2, \ldots , r_m\}$, then in the NETBID of each agent there is a special starting node $s$ connected to all nodes $r_j$ by directed edges with capacity 1. An integer flow in NETBID will represent an assignment of resources to the agent by considering the set of resources $r_j$ with flow value 1 on the directed edge $(s, r_j)$. The node $r_j$ is an usual node, i.e. it satisfies the conservation law: the total (sum) of incoming flows equals the total flow of outgoing flows.

In the network there are also bundle nodes which doesn’t satisfy the conservation law: the total (sum) of incoming flows equals the total flow of outgoing flows. The combination is conducted by the (integer) directed edges flows together with appropriate lower and capacity bounds. Once the NETBID constructed, any maximum value flow (in the sense described below) will represent the valuation function of the agent. For example, the NETBID in Figure 1 expresses that the bidder is interested in a bundle consisting in two or three resources of type $E$, together with the resource $M$ which adds 10 to the values sum of particular resources of type $E$. Formally a NETBID, the bidflows and their values are defined as follows:

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Definition 1 A R-NETBID is a tuple $\mathcal{N} = (D, s, t, c, l, \lambda)$, where:

1. $D = (V, E)$ is a digraph with two distinguished nodes $s, t \in V$; the other nodes, $V \setminus \{s, t\}$, are partitioned $R \cup B \cup I$: $R$ is the set of resources nodes, $B$ is the set of bundles nodes and $I$ is the set of interior nodes. There is a directed edge $(s, r) \in E$ for each $r \in R$, and also $(b, t) \in E, \forall b \in B$. There are no other directed edges entering in a resource node or leaving a bundle node.

2. $c, l$ are nonnegative integer partial functions defined on the set of edges of $D$: if $(i, j) \in E$ and $c$ is defined on $(i, j)$ then $c((i, j)) \in \mathbb{Z}_+$, denoted $c_{ij}$, is the capacity of edge $(i, j)$; $l((i, j)) \in \mathbb{Z}_+$ if defined, is the lower bound on the edge $(i, j)$ and is denoted $l_{ij}$; if $(i, j)$ has assigned a capacity and a lower bound then $l_{ij} \leq c_{ij}$. All edges $(s, r)$ have $c_{sr} = 1$ and $l_{sr} = 0$. No edge $(b, t)$ has capacity and lower bound.

3. $\lambda$ is a labelling function on $V - \{s, t\}$ which assign to a vertex $v$ a pair of rates $(\lambda_1(v), \lambda_2(v))$ (described in the next definitions).

Definition 2 Let $\mathcal{N} = (D, s, t, c, l, \lambda)$ be a R-NETBID. A bidflow in $\mathcal{N}$ is a function $f : E \rightarrow \mathbb{Z}_+$ such that $(f_{ij}$ denotes $f((i, j))$):

1. For each directed edge $(i, j) \in E$: if $f_{ij} > 0$ and $c_{ij}$ is defined, then $f_{ij} \leq c_{ij}$; if $f_{ij} > 0$ and $l_{ij}$ is defined, then $f_{ij} \geq l_{ij}$.

2. If $v \in V - \{s, t\}$ has $\lambda_1(v) = \text{conservation}$ then $\sum_{(i,v) \in E(D)} f_{iv} = \sum_{(v,j) \in E(D)} f_{vj}$.

3. For each $v \in B$, $f_{sv} \in \{0, 1\}$; $f_{st} = 1$ if and only if for each $w \in R \cup I$, such that $(w, v) \in E$, we have $f_{wv} > 0$.

The set of all bidflows in $\mathcal{N}$ is denoted by $\mathcal{F}^{\mathcal{N}}$.

In order to simplify our presentation we have considered here that for each $v \in V - \{s, t\}$, $\lambda_1(v) \in \{\text{conservation, bundle}\}$ giving rise to the flow rules in 2 and 3 above. In the figure considered above, the function $\lambda_1(v)$ is illustrated by the color of the node $v$: a gray node is a bundle node and a white node is a conservation node.

Definition 3 Let $f$ be a bidflow in the R-NETBID $\mathcal{N} = (D, s, t, c, l, \lambda)$. The value of $f$, $v(f)$, is defined as $v(f) = \sum_{b \in B} \sum_{v : b \rightarrow v} \text{val}(b) f_{sv}$, where $\text{val}(v)$ is 0 if $v = s$ and $\text{val}(v) = \lambda_2(D^{-1}(v))$ if $v \neq s, t$. $D^{-1}(v)$ is the set of all vertices $w \in V(D)$ such that $(w, v) \in E(D)$ and $f_{vw} > 0$. $\lambda_2(D^{-1}(v))$ is the rule (specified by the second label associated to vertex $v$) of computing $\text{val}(v)$ from the values of its predecessors which already flows into $v$.

Definition 4 Let $\mathcal{N} = (D, s, t, c, l, \lambda)$. The R-valuation designated by $\mathcal{N}$ is the function $v^{\mathcal{N}} : \mathcal{P}(R) \rightarrow \mathbb{R}_+$, where for each $S \subseteq R$, $v^{\mathcal{N}}(S) = \max\{v(f) \mid f \in \mathcal{F}^{\mathcal{N}}, f_{sv} = 0 \forall v \in R - S\}$.

By the above two definitions, the value associated by $\mathcal{N}$ to a set $S$ of resources is the maximum sum of the (disjoint) bundles which are contained in the set (assignment) $S$. This is in concordance with the definition of a v-basis given section [1] for a superadditive valuation $v$. However, the NETBID structure defined above is more flexible in order to express any valuation. If the bidder desires to express that at most $k$ bundles from some set of bundle nodes must be considered, then these nodes are connected to a new interior node and this last node linked to a new superbundle node by a directed edge having as lower bound 1 and capacity $k$. Clearly, any valuation represented in a XOR language can be obtained in such way and any $R$-valuation can be represented [5][8].

The NETBID submitted by the bidders are merged by the arbitrator in a common NETBID sharing only the nodes corresponding to $s$ and $R$, and also a common $t$ node in which are projected the corresponding $t$ nodes of the individual NETBIDS. This common NETBID, $\mathcal{N}_a$, is a symbolic representation of the aggregate valuation of the society and is illustrated in Figure 2 below.

REFERENCES


