

Well-quasi-order of relabel functions

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Abstract

We define $\text{NLC}_k^{\mathcal{F}}$ to be the restriction of the class of graphs NLC_k , where relabelling functions are exclusively taken from a set \mathcal{F} of functions from $\{1, \dots, k\}$ into $\{1, \dots, k\}$. We characterize the sets of functions \mathcal{F} for which $\text{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered by the induced subgraph relation \leq_i . Precisely, these sets \mathcal{F} are those which satisfy that for every $f, g \in \mathcal{F}$, we have $\text{Im}(f \circ g) = \text{Im}(f)$ or $\text{Im}(g \circ f) = \text{Im}(g)$. To obtain this, we show that words (or trees) on \mathcal{F} are well-quasi-ordered by a relation slightly more constrained than the usual subword (or subtree) relation.

A class of graphs is *n-well-quasi-ordered* if the collection of its vertex-labellings into n colors forms a well-quasi-order under \leq_i , where \leq_i respects labels. Pouzet [16] conjectured that a 2-well-quasi-ordered class closed under induced subgraph is in fact *n-well-quasi-ordered* for every n . A possible approach would be to characterize the 2-well-quasi-ordered classes of graphs. In this respect, we conjecture that such a class is always included in some well-quasi-ordered $\text{NLC}_k^{\mathcal{F}}$ for some family \mathcal{F} . This would imply Pouzet's conjecture.

1 Introduction

Let S be a set and \leq be a quasi-order on S , i.e. a reflexive and transitive relation. Given a sequence $(x_i)_{i \in \omega}$ of elements of S , a *good pair* consists of two elements $x_i \leq x_j$, with $i < j$. A sequence with no good pair is called a *bad sequence* of (S, \leq) . A quasi-order with no bad sequence is a *well-quasi-order*.

There are other equivalent presentations of the notion of well-quasi-ordering (see for instance [12]). A quasi-order is a well-quasi-order if and only if it

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has no infinite antichain and no infinite strictly decreasing sequence. Also, a quasi-order is a well-quasi-order if and only if every infinite sequence has an infinite increasing subsequence. Equivalently, a quasi-order is a well-quasi-order if and only if every nonempty subset of S has a nonempty finite set of minimal elements (up to equivalence).

The theory of well-quasi-ordering has been flourishing. Higman's theorem states that the set of words over a well-quasi-ordered set is well-quasi-ordered by the subword relation [7], and this has been extended by Kruskal's to trees [11]. Robertson and Seymour's celebrated graph minor theorem [17] asserts that the minor relation is a well-quasi-order on the set of finite graphs. It implies that every graph class closed under minor (or minor *ideal*) can be characterized by a finite list of excluded minors. This in turn implies that every minor ideal can be recognized in polynomial time.

The class of finite graphs is not well-quasi-ordered by the induced subgraph relation since the cycles form an antichain. The good properties of the minor ideals ensured by the minor theorem do not hold for induced subgraph ideals (for instance, the set of paths, which is well-quasi-ordered, does not have a finite set of forbidden induced subgraphs). This is a motivation to the stronger notion of 2-well-quasi-ordering.

In the following, we will be exclusively interested in the induced subgraph relation. Throughout this paper, we will abbreviate "well-quasi-ordered by the induced subgraph relation" in *well-quasi-ordered*, being understood that we are dealing with the induced subgraph relation.

An extension of the notion of well-quasi-order is the notion of n -well-quasi-order (see Kriz and Thomas [9] for a more general discussion in terms of QO-category). A set S of graphs is *n -well-quasi-ordered* if the class \hat{S} consisting of all vertex n -colorings of graphs in S , is well-quasi-ordered by the colored induced subgraph relation \leq_1 , where $G \leq_1 G'$ if there is an injection from $V(G)$ to $V(G')$ preserving adjacency and color. The set S is *∞ -well-quasi-ordered* if S is n -well-quasi-ordered, for any $n \geq 1$. Being 2-well-quasi-ordered is a stronger property than being well-quasi-ordered, for instance the set of paths is not 2-well-quasi-ordered.

The notion of 2-well-quasi-ordering is especially interesting in view of algorithmic properties, as induced subgraph ideals which are 2-well-quasi-ordered can be characterized by a finite list of forbidden induced subgraphs, and thus are polynomially recognizable. Our ultimate aim would be to characterize the 2-well-quasi-ordered ideals of graphs, in order to prove the following conjecture of Pouzet [16], also appearing in Fraïssé [4]:

Conjecture 1 *An induced subgraph ideal is 2-well-quasi-ordered if and only if it is ∞ -well-quasi-ordered.*

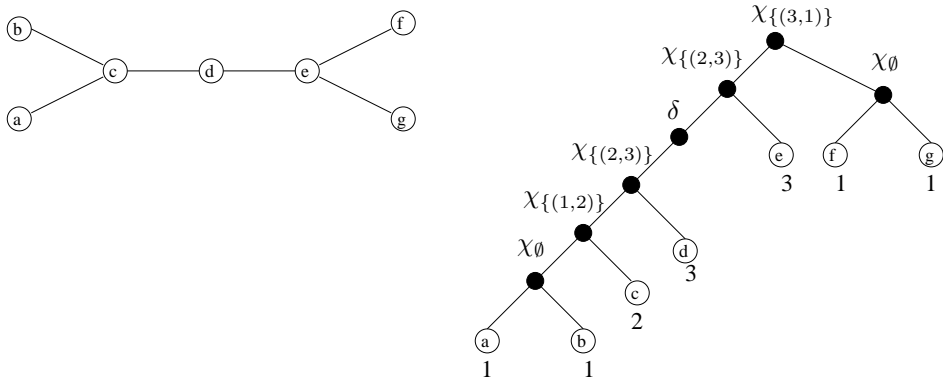


Fig. 1. The graph G_3 and an associated NLC-expression, on one relabelling function, $\delta: 3 \rightarrow 2 \rightarrow 1$.

We will come back to this topic in Section 5. Our main purpose here is to study a restriction of the hierarchy of graph classes NLC.

The class NLC_k of k -node labelled controlled graphs was introduced in [19]. Let \mathcal{F} be a set of functions from $\{1, \dots, k\}$ into $\{1, \dots, k\}$. The class $\text{NLC}_k^{\mathcal{F}}$ is defined recursively by using k node-labels and three operators: \bullet_i , \circ_f and χ_S . For $i \in \{1, \dots, k\}$, the operator \bullet_i creates a single vertex labelled by i . The operator \circ_f , with $f \in \mathcal{F}$, applied to a labelled graph replaces every label i with $f(i)$. The operator χ_S , with $S \subseteq \{1, \dots, k\} \times \{1, \dots, k\}$, applied to two labelled graphs G and H in this order, creates the disjoint union of graphs G and H , and for all $(i, j) \in S$ adds edges between every vertex of label i in G and every vertex of label j in H . The class NLC_k is defined to be NLC_k^{Φ} where Φ is the set of all functions from $\{1, \dots, k\}$ into $\{1, \dots, k\}$. The *NLC-width* of G is the minimum k for which some labelling of G is in NLC_k .

See Fig. 1 for an exemple of an NLC_3 expression of a small graph.

It is not known whether there exists a polynomial time algorithm computing a NLC decomposition using k colors for graphs in NLC_k . Only the cases $k = 1$ (which corresponds to cographs) and $k = 2$ (see [8]) have been solved so far. Computing the NLC-width is NP-hard (see [6]).

The NLC-width has a strong link with another well-known parameter: the clique-width, introduced by Courcelle *et al.* (see [1]). NLC-width and clique-width indeed differ by a factor at most 2 (more precisely, the clique-width of a graph is bounded below by its NLC-width, and above by twice its NLC-width). Moreover, transformations respecting these bounds between decompositions of the two types can be done in linear time.

The class of graphs NLC_1 (cographs) is well-quasi-ordered, see [2] and [18] for the countable case. The class of graphs NLC_2 is well-quasi-ordered (and even ∞ -well-quasi-ordered), this easily follows from the results in [8]. Indeed, the NLC_2 prime graphs for the modular decomposition are constructible in NLC_2

without relabelling, and thus form a well-quasi-ordered family by Kruskal's tree theorem. However, the class NLC_3 is not well-quasi-ordered, as it contains for every i the graph G_i (a path of length i with two pending vertices added to each extremity) depicted in Fig. 1. These graphs indeed do not form a well-quasi-ordered family. Allowing all relabelling operators \circ_f is too much to construct a well-quasi-ordered class of graphs if we have at least 3 colors. This is why we define a restriction of NLC , using only relabelling operators from a specified set of functions \mathcal{F} . Our main purpose is to characterize the sets \mathcal{F} such that $\text{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered. We will see that $\text{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered (equivalently ∞ -well-quasi-ordered) if and only if it does not contain arbitrarily large paths.

In Section 2, we introduce a binary relation \preceq on a set of functions. In Section 3 we introduce a subword order \leq on words labelled with a set of functions which is more constrained than Higman's order. In Section 4 we extend \leq to trees, with the purpose of applications to $\text{NLC}_k^{\mathcal{F}}$ expressions. In Section 5, we characterize the sets \mathcal{F} for which $\text{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered. In the final section, we discuss Pouzet's conjecture on n -well-quasi-ordering, and further well-quasi-ordering problems.

Throughout this paper, we will obtain the following equivalent characterizations of \mathcal{F} :

- The set \mathcal{F} is totally quasi-ordered by \preceq .
- The set of words on \mathcal{F} is well-quasi-ordered by \leq .
- The set of trees on \mathcal{F} is well-quasi-ordered by \leq .
- The set of graphs $\text{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered.
- The set of graphs $\text{NLC}_k^{\mathcal{F}}$ is ∞ -well-quasi-ordered.
- The set of graphs $\text{NLC}_k^{\mathcal{F}}$ does not contain arbitrarily large paths.

2 Totally ordered sets of functions

Let \mathcal{F} be a set of functions from $\{1, \dots, k\}$ into $\{1, \dots, k\}$ closed under composition (with the convention that the identity function ε belongs to \mathcal{F}). The key-definition of this section is the following. Let us say that $f \preceq g$ whenever $\text{Im}(f \circ g) = \text{Im}(f)$.

Assume that \preceq is total on \mathcal{F} , i.e. for every f, g in \mathcal{F} , at least one of $f \preceq g$ and $g \preceq f$ holds. This implies in particular that $\text{Im}(f^2) = \text{Im}(f)$ for all $f \in \mathcal{F}$. Observe that $f \preceq g$ implies that $|\text{Im}(f)| \leq |\text{Im}(g)|$.

Lemma 1 *If \preceq is total on \mathcal{F} , then \preceq is transitive.*

Proof: Assume by way of contradiction that $Im(f \circ g) = Im(f)$, $Im(g \circ h) = Im(g)$ and $f \not\preceq h$, i.e. $|Im(f \circ h)| < |Im(f)|$. Since \preceq is total, we must have $h \preceq f$, and then $Im(h \circ f) = Im(h)$. Hence $f \preceq g \preceq h \preceq f$, and thus $|Im(f)| = |Im(h)|$. As $h \circ f \preceq h \circ f$ holds, we have $|Im(h \circ f \circ h \circ f)| = |Im(h \circ f)| = |Im(h)|$. Moreover $|Im(h \circ (f \circ h) \circ f)| \leq |Im(f \circ h)| < |Im(f)| = |Im(h)|$, which is a contradiction. \square

Thus \preceq is a reflexive and transitive relation, in other words \preceq is a total quasi-order on \mathcal{F} . This is equivalent to the existence of a partition of \mathcal{F} into t equivalence classes F_1, \dots, F_t such that $f \in F_i$ and $g \in F_j$ verify $f \preceq g$ if and only if $i \leq j$.

Lemma 2 *When \mathcal{F} is totally quasi-ordered by \preceq , the equivalence classes F_1, \dots, F_t are exactly the classes of functions having an image of the same size, in increasing order of the image size.*

Proof: If $|Im(f)| < |Im(g)|$ then $g \not\preceq f$, so we must have $f \preceq g$. Suppose now by contradiction that $|Im(f)| = |Im(g)|$, $f \preceq g$ and $g \not\preceq f$. This means that $|Im(f \circ (g \circ f) \circ g)| \leq |Im(g \circ f)| < |Im(g)|$. We also have that $|Im(f \circ g)| = |Im(f)| = |Im(g)|$, thus $|Im(f \circ g \circ f \circ g)| < |Im(f \circ g)|$. This contradicts $f \circ g \preceq f \circ g$. \square

Observe that the top class F_t contains ε , and contains only permutations.

Lemma 3 *For all i , F_i and $\cup_{k \geq i} F_k$ are closed under composition.*

Proof: The first part the statement follows by Lemma 2. To prove that $\cup_{k \geq i} F_k$ is closed under composition, consider $f \in F_i$ and $g \in F_j$ with $i < j$. Since $f \preceq g$, $f \circ g$ is in F_i . Assume now for contradiction that $g \circ f \in F_p$, with $p < i$. Then $|Im(f \circ g)| = |Im(f)| > |Im(g \circ f)|$. Moreover $|Im(g \circ f)| \geq |Im(f \circ (g \circ f) \circ g)|$. Thus $|Im(f \circ g)| \neq |Im((f \circ g)^2)|$, contradicting $f \circ g \preceq f \circ g$. \square

Lemma 4 *The functions of the bottom class F_1 verify a "left-cancellation" identity:*

$$\forall f \in F_1, \forall h, h' \in \mathcal{F}, \text{ if } h \circ f \circ h' = h \circ f \text{ then } f \circ h' = f \quad (1)$$

Proof: This identity actually holds whenever $f \preceq h$. Indeed, by Lemma 3, $h \circ f$ and f belong to the same class. Assume $h \circ f \circ h' = h \circ f$. If there is an $x \in \{1, \dots, k\}$ such that $f \circ h'(x) \neq f(x)$, then these two distinct elements belong to the image of f and have the same image by h . This means that $|Im(h \circ f)| < |Im(f)|$, and contradicts the fact that $f \preceq h \circ f$. \square

Here is an example of a set of functions which is totally ordered by \preceq . An (i, j) -cast, with $i \leq j$, is a function f from $\{1, \dots, k\}$ into itself such that $f(l) = l$ for all $l < i$ and $f(l) = j$ whenever $i \leq l$. It is routine to check

that the set of casts is indeed totally ordered by \preceq . We feel that the following problem would give some insight on the well-quasi-ordered $NLC_k^{\mathcal{F}}$ classes:

Problem 2 *Find a generic class of functions \mathcal{G} (like casts for instance) such that for every totally ordered \mathcal{F} and k , there exists some k' for which $NLC_k^{\mathcal{F}}$ is included in $NLC_{k'}^{\mathcal{G}}$*

Such a class of function would describe much more precisely how to construct the well-quasi-ordered classes $NLC_k^{\mathcal{F}}$.

3 Words on functions

An \mathcal{F} -word is a finite word on the alphabet \mathcal{F} , i.e. a finite sequence f_1, \dots, f_l of elements of \mathcal{F} . Let $\mathcal{W}^{\mathcal{F}}$ be the set of \mathcal{F} -words. Let $M = f_1, \dots, f_l$ and $M' = f'_1, \dots, f'_{l'}$ be two \mathcal{F} -words. The word M is a *subword* of M' if there exists an increasing injection ϕ from $\{1, \dots, l\}$ into $\{1, \dots, l'\}$ such that $f_i = f'_{\phi(i)}$. Higman's theorem asserts that the subword partial order is a well-quasi-order when the alphabet is finite. In our more constrained partial order on \mathcal{F} -words, we have $M \leq M'$ if two conditions are satisfied:

- There is a function ϕ for which M is a subword of M' .
- For all $1 \leq i < l$, we have $f_i = f'_{\phi(i)} \circ f'_{\phi(i)+1} \circ \dots \circ f'_{\phi(i+1)-1}$.

Thus, when $M \leq M'$ and $i < j$, the composition of functions $f_i \circ f_{i+1} \circ \dots \circ f_{j-1}$ is equal to the function $f'_{\phi(i)} \circ f'_{\phi(i)+1} \circ \dots \circ f'_{\phi(j)-1}$. And since $f_j = f'_{\phi(j)}$, we also have $f_i \circ f_{i+1} \circ \dots \circ f_j = f'_{\phi(i)} \circ f'_{\phi(i)+1} \circ \dots \circ f'_{\phi(j)}$.

Our goal is to prove here that $\mathcal{W}^{\mathcal{F}}$ is well-quasi-ordered by \leq if and only if \preceq is total on \mathcal{F} . For this, we have to be a little bit more general and need to consider $\mathcal{W}_Q^{\mathcal{F}}$, the set of words on the set $\mathcal{F} \times Q$, where Q is a set endowed by a well-quasi-ordered \leq_Q .

We naturally extend the partial order \leq on $\mathcal{W}_Q^{\mathcal{F}}$. For w in $\mathcal{W}_Q^{\mathcal{F}}$ and $1 \leq x \leq |w|$ we denote by (f_x^w, q_x^w) the x^{th} letter of w . For any couple of indices a, b , with $1 \leq a < b \leq |w|$, we define $L^w(a, b)$ to be the composition $f_a^w \circ f_{a+1}^w \circ \dots \circ f_{b-1}^w$. When $a = b$, we set $L^w(a, b) = \epsilon$. Let ϕ be an increasing injection from $\{1, \dots, |w|\}$ into $\{1, \dots, |w'|\}$. We say that ϕ is *compatible with labels* if $f_x^w = f'_{\phi(x)}$ and $q_x^w \leq_Q q'_{\phi(x)}$. We say that ϕ *preserves path-composition* if for every $x < |w|$, we have that $L^w(x, x+1) = L^{w'}(\phi(x), \phi(x+1))$ (observe that by definition we have $L^w(x, x+1) = f_x^w$). We write $w \leq w'$ if there exists an increasing injection ϕ from $\{1, \dots, |w|\}$ into $\{1, \dots, |w'|\}$ compatible with labels and preserving path-composition. When ϕ is only compatible with labels, we simply say that w is a *subword* of w' and write $w \leq_0 w'$.

Theorem 1 *The set of words $\mathcal{W}_Q^{\mathcal{F}}$ is well-quasi-ordered by \leq if and only if \preceq is a total quasi-order on \mathcal{F} .*

Proof: Assume first that \preceq is not a total quasi-order on \mathcal{F} , i.e. there exist two incomparable functions $f, g \in \mathcal{F}$. Depending if $f = g$ or not, let us show in both cases that $\mathcal{W}^{\mathcal{F}}$ is not well-quasi-ordered by \leq .

- If $f = g$, we claim that $S = (w_k)_{k \geq 0}$, where $w = \epsilon(f)^k \epsilon$, is a bad sequence. Indeed, if $w_i \leq w_j$ with $i < j$, the identity functions must be mapped to identity functions. Moreover, to preserve path-composition, the first and the last f of w_i must be mapped to the first and the last f of w_j . Hence, two consecutive f in w_i must be mapped to non consecutive f in w_j , which contradicts path-composition since $f \neq f^l$ for every $l > 1$.
- If $f \neq g$, we claim that $S = (w_k)_{k \geq 0}$, where $w = \epsilon(fg)^k \epsilon$, is a bad sequence. Again, if $w_i \leq w_j$ with $i < j$, the identity functions must be mapped to identity functions. Moreover, to preserve path-composition, the first f and the last g of w_i must be mapped to the first f and to the last g of w_j . Hence, two consecutive f and g in w_i must be mapped to non consecutive f and g in w_j , which contradicts path-composition.

Assume now that \preceq is a total quasi-order on \mathcal{F} . Our goal is to prove that $\mathcal{W}_Q^{\mathcal{F}}$ forms a well-quasi-order.

To achieve this, we need to enrich a little bit our structure, and consider words on $\mathcal{F} \times \mathcal{F} \times Q$ instead of $\mathcal{F} \times Q$. We just label every letter of our word by another function of \mathcal{F} . Precisely, we add to every letter x of our word w the function consisting of the composition of all functions of the prefix of w before x . The reason for that is to keep track, in every letter of the word, of some information preceding this letter. Technically, this bit of information allows us to use a Nash-Williams' minimum bad sequence argument. The idea is simply to cut every word in a prefix-suffix way and apply the minimality to prove well-quasi-order. The key-idea is to cut the words on some letter which belongs to the bottom class of \mathcal{F} , since left-cancellation (thanks to the extra label) enables to glue back the prefix and the suffix.

Let us turn to technicalities. Instead of dealing with words $w = ((f_x^w, q_x^w))_{x=1 \dots |w|}$ on $\mathcal{F} \times Q$, we transform w into the word $\tilde{w} = ((f_x^w, L^w(1, x), q_x^w))_{x=1 \dots |w|}$ on $\mathcal{F} \times \mathcal{F} \times Q$. To simplify notations, we still call this word w .

A word $((f_x, h_x, q_x))_{x=1 \dots k}$ on $\mathcal{F} \times \mathcal{F} \times Q$ is *coherent* if for every $1 \leq x < k$, we have $h_{x+1} = h_x \circ f_x$. Observe that w is coherent by construction, and so is every factor of w . We denote by $\widetilde{\mathcal{W}}_{\mathcal{F} \times Q}^{\mathcal{F}}$ the set of coherent words on $\mathcal{F} \times \mathcal{F} \times Q$. The set $\mathcal{F} \times Q$ is equipped with the well-quasi-order $(f, q) \leq_{\mathcal{F} \times Q} (f', q')$ if $f = f'$ and $q \leq_Q q'$. Theorem 1 follows from the following result.

Lemma 5 *For any well-quasi-ordered set Q , the set $\widetilde{\mathcal{W}}_{\mathcal{F} \times Q}^{\mathcal{F}}$ is well-quasi-*

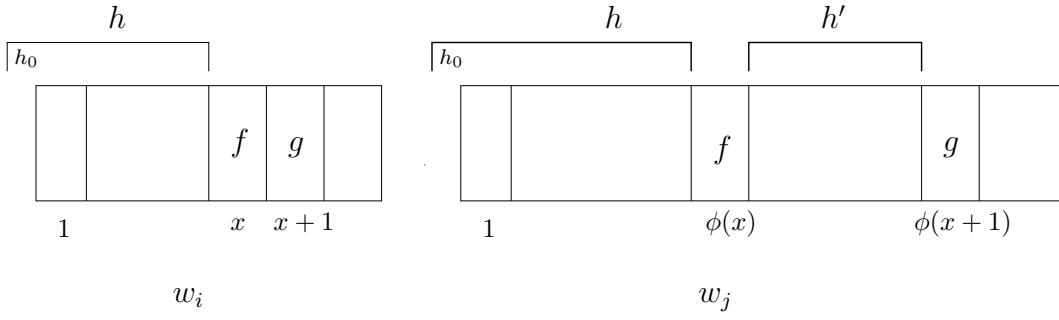


Fig. 2. The words w and w'

ordered by \leq .

Proof: By induction on t , the number of equivalence classes of \mathcal{F} .

Assume first that $t = 1$. Observe that $\{\varepsilon\} \subseteq \mathcal{F} \subseteq \sigma_n$, where σ_n is the set of permutations of $\{1, \dots, n\}$. By Higman's theorem, $\widetilde{\mathcal{W}}_{\mathcal{F} \times Q}^{\mathcal{F}}$ is well-quasi-ordered by \leq_0 . Let us prove that \leq and \leq_0 coincide in this case. Suppose that $w = ((f_x^w, h_x^w, q_x^w))_{x=1 \dots |w|}$ and $w' = ((f_x^{w'}, h_x^{w'}, q_x^{w'}))_{x=1 \dots |w'|}$ belong to $\widetilde{\mathcal{W}}_{\mathcal{F} \times Q}^{\mathcal{F}}$ and satisfy $w \leq_0 w'$. There exists an increasing injection ϕ from $\{1, \dots, |w|\}$ into $\{1, \dots, |w'|\}$ which is compatible with labels. Consider any letter $x < |w|$. Since ϕ preserves labels, we have $f_x^w = f_{\phi(x)}^{w'}$ (we call this function f) and $h_x^w = h_{\phi(x)}^{w'}$ (we call this function h). To prove that \leq_0 preserves path-composition, we just have to check that $f_x^w = f_{\phi(x)}^{w'} \circ f_{\phi(x)+1}^{w'} \circ \dots \circ f_{\phi(x+1)-1}^{w'}$. Equivalently, by letting $h' = f_{\phi(x)+1}^{w'} \circ \dots \circ f_{\phi(x+1)-1}^{w'}$ (the empty product being ε), we want to prove that $f = f \circ h'$. Since w is coherent, we have $h_{x+1}^w = h_x^w \circ f_x^w = h \circ f$. Since w' is coherent, we have $h_{\phi(x+1)}^{w'} = h_{\phi(x)}^{w'} \circ f_{\phi(x)}^{w'} \circ f_{\phi(x)+1}^{w'} \circ \dots \circ f_{\phi(x+1)-1}^{w'}$, hence $h_{\phi(x+1)}^{w'} = h \circ f \circ h'$. Since ϕ preserves labels, we have $h \circ f = h_{x+1}^w = h_{\phi(x+1)}^{w'} = h \circ f \circ h'$. We can then deduce that $f = f \circ h'$ by left-cancellation of the permutation h .

Let us assume that the induction hypothesis holds for $t-1$. Consider \mathcal{F} with t equivalence classes. We will prove that $\widetilde{\mathcal{W}}_{\mathcal{F} \times Q}^{\mathcal{F}}$ is well-quasi-ordered by \leq using Nash-Williams' minimal bad sequence argument (see [14]). By contradiction, let $S = (w_i)_{i \in \omega}$ be an infinite bad sequence of $\widetilde{\mathcal{W}}_{\mathcal{F} \times Q}^{\mathcal{F}}$, minimal in the sense that for every $i \geq 1$, the i^{th} word w_i of this sequence is defined to be a word of minimal size for which there exists a bad sequence (with respect to \leq) starting with w_1, \dots, w_{i-1}, w_i .

Set $w_i = ((f_x^{w_i}, (h_x^{w_i}, q_x^{w_i})))_{x=1 \dots |w_i|}$, and consider the minimal p_i such that $f_{p_i}^{w_i} \in F_1$, if such a p_i exists. The subword $w'_i = ((f_x^{w_i}, (h_x^{w_i}, q_x^{w_i})))_{x=1 \dots p_i}$, or $w'_i = w_i$ if p_i is undefined, is called the *beginning* of w_i . Likewise, the *end* of w_i is the subword $w''_i = ((f_x^{w_i}, (h_x^{w_i}, q_x^{w_i})))_{x=p_i+1 \dots |w_i|}$, or the empty word if p_i is undefined.

Let X be the set of all ends of words in S . We now prove that X is well-quasi-ordered by \leq . Let σ be an infinite sequence of X . Let $\alpha = (z_i)_{i \in \omega}$ be an infinite extracted subsequence of σ such that for all i , if z_i is the end of w_j and z_{i+1} is the end of w_k , then $j \leq k$. Let w_n be the word of which z_1 is the end. The sequence $(w_1, \dots, w_{n-1}, z_1, z_2, \dots)$ is good by the minimality of S . Since S is bad, one cannot have $w_a \leq z_b$, or $w_i \leq w_j$ with $i < j$. Hence there exists a good pair $z_i \leq z_j$ with $i < j$, thus X is well-quasi-ordered by \leq .

By Lemma 3, we know that $\mathcal{F} - F_1$ is closed under composition. Let $R = (F_1 \times X) \cup \{\Delta\}$ with Δ a new marker. The set R is ordered by \leq_R as follows: $\Delta \leq_R \Delta$, and $(f, x) \leq_R (g, x')$ if $f = g$ and $x \leq x'$. Since X is well-quasi-ordered, we have that R , and hence $Q' = Q \times R$, forms a well-quasi-order. By our induction hypothesis, $\widetilde{\mathcal{W}}_{(\mathcal{F}-F_1) \times Q'}^{\mathcal{F}-F_1}$ is well-quasi-ordered by \leq , where the order $\leq_{Q'}$ on Q' is defined by: $(q, r) \leq_{Q'} (q', r')$ if $q \leq_Q q'$ and $r \leq_R r'$.

Denote by w_i''' the word of $\widetilde{\mathcal{W}}_{(\mathcal{F}-F_1) \times Q'}^{\mathcal{F}-F_1}$ which is the concatenation of the word $((f_x^{w_i}, (h_x^{w_i}, (q_x^{w_i}, \Delta))))_{x=1 \dots p_i-1}$ with the extra letter $(\varepsilon, (h_{p_i}^{w_i}, (q_{p_i}^{w_i}, (f_{p_i}^{w_i}, w_i''))))$ if p_i exists, or $((f_x^{w_i}, (h_x^{w_i}, (q_x^{w_i}, \Delta))))_{x=1 \dots p_i-1}$ otherwise. Note that w_i''' is coherent since w_i is coherent.

As $\widetilde{\mathcal{W}}_{(\mathcal{F}-F_1) \times Q'}^{\mathcal{F}-F_1}$ is well-quasi-ordered by \leq , there exist $i < j$ such that $w_i''' \leq w_j'''$. We denote by ϕ the mapping from w_i''' into w_j''' . Observe that if w_i''' does not have an extra letter, we would directly have that $w_i \leq w_j$ which is impossible since S is a bad sequence. Hence p_i and p_j exist. Let us now exhibit a mapping Φ from w_i into w_j which preserves labels and path-composition.

First, we set Φ to be the restriction of the function ϕ from $\{1, \dots, p_i\}$ into $\{1, \dots, p_j\}$. Observe that $\phi(p_i) = p_j$ since extra letters do not carry markers. Moreover, the extra letter $(\varepsilon, (h_{p_i}^{w_i}, (q_{p_i}^{w_i}, (f_{p_i}^{w_i}, w_i''))))$ is mapped to the extra letter $(\varepsilon, (h_{p_j}^{w_j}, (q_{p_j}^{w_j}, (f_{p_j}^{w_j}, w_j''))))$. In particular, we have $w_i'' \leq w_j''$. The mapping ϕ'' from $w_i'' \leq w_j''$ which realizes $w_i'' \leq w_j''$ is our extension of Φ from $\{p_i + 1, \dots, |w_i|\}$ into $\{p_j + 1, \dots, |w_j|\}$.

Now Φ is completely defined. Moreover, it preserves labels by definition. We then just have to show that it also preserves path-composition, and more precisely path-composition exactly after the letter p_i , since the other cases are already taken into account by ϕ or ϕ' .

We then have to show that $f_{p_i}^{w_i} = f_{\Phi(p_i)}^{w_j} \circ \dots \circ f_{\Phi(p_i+1)-1}^{w_j}$ or equivalently $f_{p_i}^{w_i} = f_{p_j}^{w_j} \circ f_{p_j+1}^{w_j} \dots \circ f_{\phi''(p_i+1)-1}^{w_j}$. Observe that this condition holds vacuously when $p_i = |w_i|$. Since the extra letter of w_i''' is mapped to the extra letter of w_j''' , we have $f_{p_i}^{w_i} = f_{p_j}^{w_j}$ (we call this function f) and $h_{p_i}^{w_i} = h_{p_j}^{w_j}$ (we call this function h). We now let $h' = f_{p_j+1}^{w_j} \dots \circ f_{\phi''(p_i+1)-1}^{w_j}$ (the empty product being ϵ). Since the word w_i is coherent, we have $h_{p_i+1}^{w_i} = h_{p_j}^{w_j} \circ f_{p_i}^{w_i} = h \circ f$. Since w_j is coherent, we have $h_{\phi''(p_i+1)}^{w_j} = h_{p_j}^{w_j} \circ f_{p_j}^{w_j} \circ f_{p_j+1}^{w_j} \dots \circ f_{\phi''(p_i+1)-1}^{w_j}$. Hence $h_{\phi''(p_i+1)}^{w_j} = h \circ f \circ h'$.

Since ϕ'' preserves labels, we have $h \circ f = h_{p_{i+1}}^{w_i} = h_{\phi''(p_{i+1})}^{w_j} = h \circ f \circ h'$. We now conclude by Lemma 4 that $f = f \circ h'$. This concludes the proof of Lemma 5 and Theorem 1. \square

4 Trees on functions

We extend in this section our results to trees. However, since the arguments are similar to the previous section, we will not give the same level of details, especially concerning the verification of path-composition.

A *structured tree* is a finite tree where the childs of a node are ordered from left to right. Our trees have their nodes labelled by a well-quasi-ordered set Q . We denote by $\mathcal{T}_Q^{\mathcal{F}}$ the set of structured rooted trees with nodes labelled by $\mathcal{F} \times Q$, where \mathcal{F} is as usual a set of functions. A node x is then labelled by a pair $l(x) = (f(x), q(x))$. We simply write $\mathcal{T}^{\mathcal{F}}$ when there is no additional label Q . The set of nodes of T is denoted by $V(T)$. We write $x \wedge y$ for the least common ancestor of x and y . We say that (x, y) is an *arc* of T when x is the father of y . A sequence of nodes z_0, z_1, \dots, z_n is a *downward path* in T if (z_i, z_{i+1}) is an arc, for every $i = 0, \dots, n-1$. For such a downward path z_0, z_1, \dots, z_n , we denote by $L(z_0, z_n)$ the composition $f(z_0) \circ f(z_1) \circ \dots \circ f(z_{n-1})$.

Let us define a partial order \leq on $\mathcal{T}_Q^{\mathcal{F}}$ which extends the order \leq on words. Precisely, let us write that $T \leq T'$ if there exists an injection ϕ from $V(T)$ into $V(T')$ such that:

- ϕ preserves descendance.
- ϕ preserves least common ancestors, i.e. $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$.
- ϕ preserves *left/right order*, i.e. if x and y are not in descendance relation, and the branch of $x \wedge y$ containing x is to the left of the one containing y , the same holds for the branches of $\phi(x) \wedge \phi(y)$ containing $\phi(x)$ and $\phi(y)$.
- ϕ preserves labels, i.e. $f(x) = f(\phi(x))$ and $q(x) \leq_Q q(\phi(x))$.
- ϕ preserves *path-composition*, i.e. for any arc (x, y) in T , we have that $L(x, y) = L(\phi(x), \phi(y))$, i.e. $f(x) = L(\phi(x), \phi(y))$.

When ϕ satisfies all these properties except possibly path-composition, we simply write $T \leq_0 T'$. Kruskal's tree theorem asserts that \leq_0 is a well-quasi-order on the set of trees.

This more constrained order relation \leq presents some analogies with the so-called *gap-condition embedding* studied by Kriz in [10]. For instance, when the class of functions \mathcal{F} is totally ordered, and hence partitioned into F_1, \dots, F_t , the path-composition property implies that if y is a child of x and $f(x)$ belongs to F_i , then every fonction of the product $L(\phi(x), \phi(y))$ belong to classes with

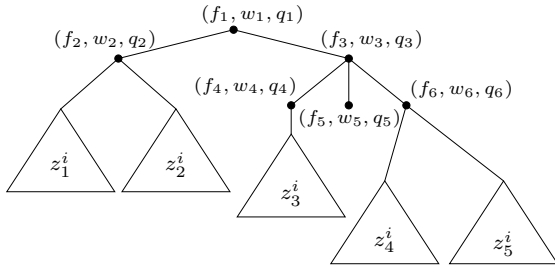


Fig. 3. The tree T_i (f_2, f_4, f_5 and f_6 are in F_1 , while f_1 and f_3 are not).

height at least i . It could be interesting to state a common generalization of these results, possibly involving ordinal functions.

Theorem 2 $\mathcal{T}_Q^{\mathcal{F}}$ is well-quasi-ordered by \leq if and only if \preceq is total on \mathcal{F} .

Proof: If \preceq is not total on \mathcal{F} , the set of words, hence of trees, is not well-quasi-ordered as we have seen in the previous section.

Assume now that \preceq is total on \mathcal{F} . As for words, we transform a tree T in $\mathcal{T}_Q^{\mathcal{F}}$ into a tree in $\tilde{\mathcal{T}}_{\mathcal{F} \times Q}^{\mathcal{F}}$. To every vertex x of T distinct from the root r , we give the extra label $L(r, x)$. The extra label of the root being ϵ . More generally, we say that a rooted tree T with nodes labelled by $\mathcal{F} \times \mathcal{F} \times Q$ is *coherent* if for every arc $(x, y) \in T$, with $l(x) = (f, h, q)$ and $l(y) = (g, h', q')$, we have $h' = h \circ f$. We denote by $\tilde{\mathcal{T}}_{\mathcal{F} \times Q}^{\mathcal{F}}$ the set of coherent structured rooted trees with nodes labelled by $\mathcal{F} \times \mathcal{F} \times Q$. Theorem 2 is a consequence of the following result.

Lemma 6 If Q is well-quasi-ordered, the set $\tilde{\mathcal{T}}_{\mathcal{F} \times Q}^{\mathcal{F}}$ is well-quasi-ordered by \leq .

Let us prove this by induction on t , the number of equivalence classes of \mathcal{F} .

When $t = 1$, the proof goes accordingly to the case of words: by Kruskal's theorem, $\tilde{\mathcal{T}}_{\mathcal{F} \times Q}^{\mathcal{F}}$ is well-quasi-ordered by \leq_0 , which again coincides with \leq .

Let us assume that the induction hypothesis holds for $t - 1$. Consider \mathcal{F} with t equivalence classes. We prove that $\tilde{\mathcal{T}}_{\mathcal{F} \times Q}^{\mathcal{F}}$ is well-quasi-ordered by \leq using Nash-Williams' minimal bad sequence argument. By contradiction, let $S = (T_i)_{i \in \omega}$ be an infinite bad sequence of $\tilde{\mathcal{T}}_{\mathcal{F} \times Q}^{\mathcal{F}}$, minimal in the sense that for $i \geq 1$, the i^{th} tree T_i of this sequence is defined to be a tree of minimal size for which there exists a bad sequence starting with T_1, \dots, T_{i-1}, T_i .

A *branching vertex* is a node x labelled by (f, w, q) , with $f \in F_1$, and with no ancestor having a label (g, w', q') with g in F_1 . A *branch* is a subtree which is rooted in any child of a branching vertex. We denote by T'_i the subtree of T_i obtained by deleting all the branches of T (see Fig. 3 and Fig. 4). Let X be the set of all branches of the trees in S . As in the previous section, the minimality of S ensures that X is well-quasi-ordered by \leq .

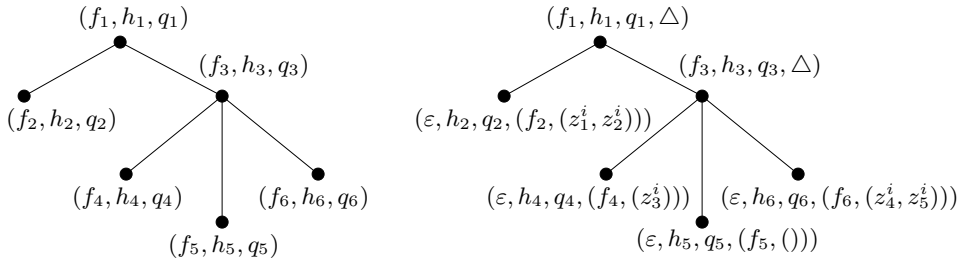


Fig. 4. The trees T'_i (left) and T''_i (right)

Recall that $\mathcal{F} - F_1$ is closed under composition. We denote by $Seq(X)$ the set of sequences of X . The usual order \leq_{seq} on $Seq(X)$ is defined as follows: for any two sequences (x_1, \dots, x_m) and (x'_1, \dots, x'_n) of X , we have $(x_1, \dots, x_m) \leq_{seq} (x'_1, \dots, x'_n)$ if there exists an increasing injection σ from $[1, m]$ to $[1, n]$ such that $\forall i \in [1, m]$ $x_i \leq x'_{\sigma(i)}$. Let $R = ((F_1 \times Seq(X)) \cup \{\Delta\})$ with Δ a new marker. Let \leq_R be the following order on R : $\Delta \leq_R \Delta$, and $(f, x) \leq_R (g, x')$ if $f = g$ and $x \leq_{seq} x'$. By our induction hypothesis $\tilde{\mathcal{T}}_{(\mathcal{F}-F_1) \times Q'}^{\mathcal{F}-F_1}$, with $Q' = Q \times R$, is well-quasi-ordered by \leq .

Denote by T''_i the tree obtained from T'_i by modifying the labels of its nodes as follows:

- Labels of internal nodes are modified from (f, h, q) into $(f, h, (q, \Delta))$.
- Labels of leaves x are modified from (f, h, q) into $(\epsilon, h, (q, s))$, where s is the sequence of branches of the branching vertex x (see Fig. 4).

Observe that the trees T''_i are coherent, and since $\tilde{\mathcal{T}}_{(\mathcal{F}-F_1) \times Q'}^{\mathcal{F}-F_1}$ is well-quasi-ordered by \leq , there exist $i < j$ such that $T''_i \leq T''_j$. The node injection ϕ from T''_i into T''_j can be extended, via the labels s , to the vertices of the branches of T_i and T_j . By construction, ϕ directly gives $T_i \leq_0 T_j$, and coherence property, as for words, gives path-composition. Thus $T_i \leq T_j$, contradicting the badness of S . \square

5 NLC with restricted relabelling functions

We can see $NLC_k^{\mathcal{F}}$ expressions as binary trees, where the leaves are labelled by \bullet_i , the nodes of degree 1 by \circ_f , and the nodes of degree 2 by χ_S . To fit in the framework of the previous section, we add an extra label to every node of such an $NLC_k^{\mathcal{F}}$ construction tree, to see χ_S and \bullet_i as identity relabelling functions. For this, replace \bullet_i with (ϵ, \bullet_i) , \circ_f with (f, \circ_f) and χ_S with (ϵ, χ_S) . Such a tree is a *construction tree* for the graph corresponding to this $NLC_k^{\mathcal{F}}$ expression. Let T_G be a construction tree for a graph G . Let x be a vertex of G which corresponds to the leaf x' of T_G of label \bullet_i and y be an ancestor of x' in T_G . When we apply the operation corresponding to the node y of T_G to

the vertex x , the color of x , denoted by $c_x(y)$ is exactly $L(y, x')(i)$.

Lemma 7 *Let G and H be two NLC_k graphs together with their $NLC_k^{\mathcal{F}}$ construction trees T_G and T_H . If $T_G \leq T_H$, then $G \leq_i H$.*

Proof: Let ϕ be an injection from $V(T_G)$ into $V(T_H)$. The restriction of ϕ on the leaves of T_G can be seen as an injection from $V(G)$ into $V(H)$. Let x, y be two vertices of G , with x on the left of y in T . Then x and y are neighbours in G if and only if their least common ancestor in $V(T_G)$ is a node labelled by χ_S with $(c_x(x \wedge y), c_y(x \wedge y)) \in S$. As ϕ preserves labels, path composition and right/left order, this is the case if and only if $\phi(x)$ and $\phi(y)$ are neighbours in H . So $G \leq_i H$. \square

Theorem 2 immediately gives:

Corollary 1 *If \preceq is total on \mathcal{F} , then $NLC_k^{\mathcal{F}}$ is well-quasi-ordered by \leq_i .*

Moreover, since we can always add some extra vertex-labels, we obtain that $NLC_k^{\mathcal{F}}$ is ∞ -well-quasi-ordered when \preceq is total on \mathcal{F} . The converse of the previous corollary actually holds: if there exist two incomparable functions f and g for the relation \preceq in \mathcal{F} , then for any n , the graph G_n depicted in Fig. 1 is in $NLC_k^{\{f,g\}}$. The set $\{G_n | n \in \omega\}$ is clearly not well-quasi-ordered.

Theorem 3 *$NLC_k^{\mathcal{F}}$ is well-quasi-ordered by \leq_i if and only if \preceq is total on \mathcal{F} .*

Proof: Assume that \preceq is not total on \mathcal{F} , and let (f, g) be an incomparable pair for the relation \preceq . Let us show that for any $n \geq 1$, the graph G_n depicted in Fig. 1 is in $NLC_k^{\mathcal{F}}$. The set $\{G_n | n \in \omega\}$ is clearly not well-quasi-ordered.

Assume first that $f = g$, that is $|Im(f^2)| < |Im(f)|$. Hence there exist $x, y \notin Im(f^2)$, such that $f(x) = y$. To construct G_n , start from two vertices labelled by y and one vertex labelled by x , and apply $\chi_{\{(x,y)\}}$ to form a path of length 2. Relabel by f . Observe that the two extremities of this path will never be again labelled by x or y since their labels will stay within $Im(f^2)$. Add a vertex labelled by x , apply again $\chi_{\{(x,y)\}}$. This adds an edge between the middle vertex of the path and the new one. Then relabel by f , and keep on building the path up to the desired length. The point is that after any step, the extremity of the path is distinguished by its label from the other vertices. When the last vertex of the path has been added (with label x as usual), add two isolated vertices with label x for instance, and apply $\chi_{\{(x,x)\}}$, completing the graph G_n .

We can generalize this when f and g are distinct. An f -class is a subset S of $\{1, \dots, k\}$ such that $|f(S)| = 1$ and which is maximum with respect to inclusion. Since $f \not\preceq g$, there exists an f -class disjoint from $Im(g)$. Let x be one of its elements. Similarly, let y be in a g -class disjoint from $Im(f)$. Let us

prove by induction that for every n , we can build paths of length $2n$ where the last vertex is labelled by y and the other vertices are labelled in the set $Im(f)$. We will therefore be able to build graphs G_n for arbitrarily large n then, adding two pending nodes on each extremity as in the previous case.

To start with, take a vertex $z \in Im(f)$, add a vertex y , and apply $\chi_{\{(z,y)\}}$. Now assume that we have a path of length $2n$ where the last vertex is labelled by y and the other vertices by some elements of $Im(f)$. Relabel by g . Observe that the last vertex is still distinguished from the rest. Add a vertex x . At this point, no other vertex has label x , since x is not in $Im(g)$. Apply $\chi_{\{(x,y)\}}$. This constructs a path of length $2n+1$. Now relabel by f , add a vertex y and apply $\chi_{\{(y,x)\}}$ in order to get a path of length $2(n+1)$ which satisfies the induction hypothesis. \square

To sum-up the different results of this section, let us mention the following equivalent statements:

- $NLC_k^{\mathcal{F}}$ is well-quasi-ordered by \leq_i
- $NLC_k^{\mathcal{F}}$ is ∞ -well-quasi-ordered by \leq_i
- \preceq is total on \mathcal{F}
- $NLC_k^{\mathcal{F}}$ does not contain arbitrarily large paths.

6 Further well-quasi-ordering questions

As we have mentioned before, one important motivation for the notion of 2-well-quasi-ordered class is that it can be described by a finite set of bounds.

Proposition 3 *Let I be a 2-well-quasi-ordered induced subgraph ideal. There are finitely many graphs in the complement \bar{I} of I which are minimal with respect to the induced subgraph relation.*

Proof: By contradiction, we assume that the set B of minimal graphs in \bar{I} is infinite. For every graph G in B , choose a vertex, color its neighbours red and its non-neighbours black, and delete it. Call the resulting bicolored graph G' . The set $B' = \{G' | G \in B\}$ is infinite, and consists of graphs which underlying graphs are in I , by minimality of the graphs in B . As I is 2-well-quasi-ordered, there exist two graphs G'_1 and G'_2 in B' , such that G'_1 is a colored induced subgraph of G'_2 . Hence G_1 is an induced subgraph of G_2 , contradicting the fact that G_2 is in B . \square

We call these graphs the *forbidden graphs of I* , and denote by $F(I)$ the set of these graphs. The same argument shows that $I \cup F(I)$ is well-quasi-ordered when I is 2-well-quasi-ordered.

Proposition 3 implies that any 2-well-quasi-ordered induced subgraph ideal is polynomially recognizable. This means in particular that for a set \mathcal{F} totally quasi-ordered by \preceq , the class $NLC_k^{\mathcal{F}}$ is polynomially recognizable.

The following question would give an answer to Pouzet's conjecture.

Conjecture 4 *If \mathcal{G} is a 2-well-quasi-ordered induced subgraph ideal, there exists a well-quasi-ordered set $NLC_k^{\mathcal{F}}$ which contains \mathcal{G} .*

This problem seems hard. One first step would be to show the following:

Conjecture 5 *No class of graphs closed under induced subgraph, and of unbounded clique-width, is 2-well-quasi-ordered.*

The next step would be to show that if indeed a subclass of NLC_k is 2-well-quasi-ordered, then it is contained in some well-quasi-ordered set $NLC_{k'}^{\mathcal{F}}$.

Recall that clique-width and NLC-width are equivalent from the point of view of boundedness. We present the conjectures of this final section in terms of clique-width, the more usual of the two, while we used NLC-width in our proofs for technical convenience.

Concerning classes of unbounded clique-width, the following problem (which generalizes Conjecture 5) is of independent interest:

Problem 6 *Is it true that every class of graphs closed under induced subgraph, and of unbounded clique-width, is not well-quasi-ordered by the induced subgraph relation ?*

Let us say that an ideal $I' \subseteq I$ is a *sub-ideal* of an ideal I , and a *strict sub-ideal* if $I' \subsetneq I$. To prove Conjecture 5, one can actually restrict to minimal such classes:

Proposition 7 *Let I be a 2-well-quasi-ordered induced subgraph ideal of unbounded clique-width. There exists a sub-ideal I' of I of unbounded clique-width such that every strict sub-ideal of I' is of bounded clique-width.*

Proof: By contradiction, assume this is not the case. There must exist an infinite sequence $(I_n)_{n \in \omega}$ such that $I_0 = I$, I_k is a strict sub-ideal of I_{k-1} for every k , and I_k has unbounded clique-width for every k . As I is 2-well-quasi-ordered, each set I_k has a finite number of forbidden graphs (see Proposition 3). Let us denote the set of these graphs by $S_k = \{G_1^k, \dots, G_{n_k}^k\}$. These graphs are in $I \cup F(I)$, which is a well-quasi-ordered set, thus these finite sets S_k are well-quasi-ordered by the natural extension of the induced subgraph relation to finite sets. Let $S' = (S_{\sigma(k)})_{k \in \omega}$ be an extracted increasing subsequence of $(S_k)_{k \in \omega}$. If $j < k$ then the graph $G_i^{\sigma(j)}$ does not belong to $I_{\sigma(k)}$, thus the

induced subgraph mapping from $G_i^{\sigma(j)}$ into a graph of $S_{\sigma(k)}$ is actually an equality. This means that $S_{\sigma(j)}$ is a (strict) subset of $S_{\sigma(k)}$ whenever $j < k$. Thus S' is a sequence of inclusionwise strictly increasing sets, each of which is an antichain. Taking the limit gives an infinite antichain of the well-quasi-ordered set $I \cup F(I)$, a contradiction. \square

Thus the following conjecture implies Conjecture 5:

Conjecture 8 *No ideal \mathcal{G} verifies all of the following properties:*

- \mathcal{G} has unbounded clique-width
- \mathcal{G} has a finite number of forbidden graphs
- Every strict sub-ideal of \mathcal{G} has bounded clique-width

For further discussion on minimal ideals of unbounded clique-width, see [13] for instance.

Finally, let us mention a question which would push Pouzet's conjecture to a full extent. The answer for cographs can be found in [18].

A quasi-order Q is a *better-quasi-order* if the class of countable ordinals labelled by Q is a wqo.

Conjecture 9 *Let \mathcal{G} be a class of countable graphs. If the class of finite induced subgraphs \mathcal{G}_F of \mathcal{G} is 2-well-quasi-ordered, then \mathcal{G} is better-quasi-ordered for every better-quasi-ordered vertex-label.*

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