Quasiperiods in Infinite Words

Guilhem Gamard, Gwenaël Richomme

November 16, 2016
Plan

Introduction

Determining sets of quasiperiods

Going through some examples

Expressive power of quasiperiodicity

Conclusion
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Conclusion
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(Cf. Apostolico, Ehrenfeucht 1993 and Marcus 2004)
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Initial motivation

Quasiperiodic words are regular, but not too much.

Remark
If $w$ is finite and $q$-quasiperiodic, it contains at least $|w|_q - 1$ squares.

A square is a factor written $u \cdot u$. 

\[ q \quad q \quad q \quad q \quad q \]
Initial motivation

Quasiperiodic words are regular, but not too much.

Remark

If $w$ is finite and $q$-quasiperiodic, it contains at least $|w|_q - 1$ squares.

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Initial motivation

Quasiperiodic words are regular, but not too much.

Remark
If $w$ is finite and $q$-quasiperiodic, it contains at least $|w|_q - 1$ squares.

A square is a factor written $u \cdot u$.

This was the initial motivation: interesting (counter-)examples for algorithms.
My background: combinatorics on infinite words

Definition
The factor complexity $P_w(n)$ of $w$ is $\#\{ \text{factors of length } n \text{ in } w \}$. A classical measure of "regularity" of a word.

Theorem (Morse, Hedlund, 1938)
The word $w$ is eventually periodic iff $P_w(n) \leq n$ for some $n$.

Definition
A word $w$ is Sturmian iff $P_w(n) = n + 1$ for all $n$. The "least complex" aperiodic words.

\[ \text{abaababaabaababaababaababaababaababaababaab}\ldots \]
Motivations: quasiperiodicity for infinite words

Theorem (Levé, Richomme, 2004)

Almost all Sturmian words are quasiperiodic.
All except two in the shift space.

Theorem (Marcus, Monteil, 2006)

There are quasiperiodic words with exponential complexities.
Quasiperiodicity: regular, but not too much...

Where does quasiperiodicity lie in the zoo of infinite words?
A normal form for quasieperiodic words

Remark
An infinite word $w$ is $aba$-coverable iff $w \in \{ab, aba\}^\omega$.

In each $aba$, either the last $a$ overlaps with the next, or not.

$$ab aba ab ab aba ab ab ab aba ab \ldots$$
A normal form for quasiperiodic words

Remark
An infinite word $w$ is $aba$-coverable iff $w \in \{ab, aba\}^\omega$.
In each $aba$, either the last $a$ overlaps with the next, or not.

$$ab\ aba\ ab\ aba\ ab\ aba\ ab\ ab\ aba\ ab\ . . .$$

Theorem (Mouchard, 2000)
Let $q \in \Sigma^*$ and $(r_i), (l_i)$ and $(b_i)$ be all the words such that

$$q = l_i b_i = b_i r_i.$$

Then $w$ is $q$-coverable iff $w \in \{l_1, \ldots, l_k\}^\omega$. 
A Quasiperiodic word with high complexity

Pick a word \( w \) with complexity \( P_w(n) = 2^n \).

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a b a a a b b a b b a a a b a a b a b b a a \ldots
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Consider its image by the morphism:

\[
\varphi(a) = aba \quad \varphi(b) = ab
\]

This means: in \( w \), replace each \( a \) with \( aba \) and each \( b \) with \( ab \) to get \( \varphi(w) \).
A Quasiperiodic word with high complexity

Pick a word $w$ with complexity $P_w(n) = 2^n$.

$$a \ b \ aa \ ab \ ba \ bb \ aaa \ aab \ aba \ abb \ baa \ldots$$

Consider its image by the morphism:

$$\varphi(a) = aba \quad \varphi(b) = ab$$

This means: in $w$, replace each $a$ with $aba$ and each $b$ with $ab$ to get $\varphi(w)$.

Then $\varphi(w)$ is quasiperiodic with exponential complexity.

$$aba \ ab \ abaaba \ abaab \ ababa \ abab \ abaabaaba \ abaabaab \ abaababa \ldots$$
Quasiperiodicity is not a good notion of symmetry

\[ \varphi(a) = aba \quad \varphi(b) = ab \]

We could use \( \varphi \) to transfer many properties:

- Factor complexity
- Non-minimality
- Number of ergodic measures
- Turing degree
- ...


Therefore we get pretty “ugly” quasiperiodic words.
Quasiperiodicity is not a good notion of symmetry

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- Factor complexity
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- ...


Therefore we get pretty “ugly” quasiperiodic words.

**Quasiperiodicity does not imply well-known symmetries.**

Not surprising: it is a local notion!
A stronger notion...

Definition (Marcus, Monteil, 2006)

A word is **multi-scale quasiperiodic** if it has infinitely many quasiperiods.

Examples:

- Periodic words
- Fixed-points of \( \varphi(a) = aba, \varphi(b) = ab \) and the like
- All quasiperiodic Sturmian words (Cf. Levé, Richomme 2004)

\[
abaababaabaababaababaababaababaababaababaabababaabababaababaababaababaababaab... 
\]
... with better dynamical properties

**Theorem (Marcus, Monteil, 2006)**

Let $w$ be a multi-scale word. Then $w$ has $O(n^2)$ factor complexity, is minimal and uniquely ergodic.

I also worked on generalizations to $2D$ words…

**Theorem (G, Richomme, 2015)**

Let $w$ be a $\mathbb{Z}_2$-word. If $w$ is multi-scale, then $w$ has uniform factor frequencies and $0$ topological entropy.

...but that's another story.
... with better dynamical properties

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Quasiperiodic

Multi-scale quasiperiodic
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Recent contributions

There is interest around words with infinitely many quasiperiods.

Problem
Given a multi-scale quasiperiodic word $w$, how do we determine the set of its quasiperiods?
Surprisingly, this have not been studied.

Remark
A quasiperiod of $w$ is always a prefix, thus we only need to study the *lengths* of quasiperiods.
Proposition

Let $w \in \Sigma^\mathbb{N}$ and set $p_n$ the prefix of length $n$ of $w$.
Suppose $p_i$ is a cover of $w$.
Then $p_{i+1}$ is a cover iff $p_i$ is not right special.

(A factor $u$ is right-special if $u \cdot a$ and $u \cdot b$ occur in $w$.)
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(A factor $u$ is right-special if $u \cdot a$ and $u \cdot b$ occur in $w$.)

Proof.

If $p_i$ is not right special, any occurrence of $p_i$ extends to $p_{i+1}$.

Conversely, suppose $p_{i+1} = p_i \cdot a$ quasip. and $p_i \cdot b$ factor of $w$.

\[\begin{array}{cccccccccc}
- - - - - - - - & p_i & a & - - - \\
- - - - - - - - & p_i & a & - - - - - - - \\
- - - - - - - - & p_i & b & - - - - - - - \\
\end{array}\]
Proposition

Let \( w \in \Sigma^\mathbb{N} \) and set \( p_n \) the prefix of length \( n \) of \( w \). Suppose \( p_i \) is a cover of \( w \). Then \( p_{i+1} \) is a cover iff \( p_i \) is not right special.

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Conversely, suppose \( p_{i+1} = p_i \cdot a \) a quasip. and \( p_i \cdot b \) factor of \( w \).

Counting the number of \( a \)'s and \( b \)'s yields \( a = b \).
Proposition

Let $w \in \Sigma^\mathbb{N}$ and set $p_n$ the prefix of length $n$ of $w$. Suppose $p_i$ is a cover of $w$. Then $p_{i-1}$ is not a cover iff $(p_i)^2$ is a factor of $w$ and $p_{i-1}$ is not an internal factor of $p_i \cdot p_{i-1}$.
Proposition

Let $w \in \Sigma^\mathbb{N}$ and set $p_n$ the prefix of length $n$ of $w$. Suppose $p_i$ is a cover of $w$. Then $p_{i-1}$ is not a cover iff $(p_i)^2$ is a factor of $w$ and $p_{i-1}$ is not an internal factor of $p_i \cdot p_{i-1}$.

Proof.
The only situation when $p_{i-1}$ is not a cover while $p_i = p_{i-1}a$ is:

```
- - - | p_{i-1} | a | p_{i-1} | a | - - -
```

(where there are no other occurrences of $p_{i-1}$)
Quasiperiods are always prefixes. (We only need to study their lengths.)

Given $w$ and a q.p. $p_i$, we know whether $p_{i+1}$ and $p_{i-1}$ are q.p.'s.

Check whether $p_i$ is right-special and whether $(p_i)^2$ is a factor of $w$. 

We get simple proofs for characterizations of the quasiperiods in various words. For instance, we simplify an existing proof for the Fibonacci word. (cf. Christou, Crochemore, Iliopoulos, 2002 and Levé, Richomme, 2004 and Mousavi, Schaeffer, Shallit, 2015.)
Recap

Quasiperiods are always prefixes. (We only need to study their lengths.)
Given $w$ and a q.p. $p_i$, we know whether $p_{i+1}$ and $p_{i-1}$ are q.p.'s.
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We get simple proofs for characterizations of the quasiperiods in various words.

For instance, we simplify an existing proof for the Fibonacci word.
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Example: quasiperiodic quasiperiods

The following morphism

\[ h(a) = abaaba \quad h(b) = bababa, \]

has a fixed point

\[ h^\omega(a) = abaababababaabaababaabaababababababababa \ldots \]
Example: quasiperiodic quasiperiods

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Its set of quasiperiods is \( \{ h^n(aba), \ n \in \mathbb{N} \} \).
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Its set of quasiperiods is \( \{ h^n(aba), \, n \in \mathbb{N} \} \).

1. The \( h^n(aba) \) are quasiperiods of \( h^\omega(a) \).
2. The factors \( h^n(aba) \) are right special.
3. The prefixes \( p \) such that \( pp \) is a factor are \( h^n(a), h^n(ab), h^n(aba) \).
4. The \( h^n(a) \) and \( h^n(ab) \) are not quasiperiods of \( h^\omega(a) \).
\[ h(a) = abaaba \quad h(b) = bababa \]
\[ h^\omega(a) = abaababababaabaabaabaabababababababa \ldots \]

**Step 1**

The words \( h^n(aba) \) are quasiperiods of \( h^\omega(a) \).

- The image of anything by \( h \) is \( aba \)-quasiperiodic.
- If \( q \) is a quasiperiod of \( w \), then \( h(q) \) is a q.p. of \( h(w) \).
- Thus this step holds by recurrence.
\[ h(a) = abaaba \quad h(b) = bababa \]
\[ h^\omega(a) = abaababababaabaababaababababababababa \ldots \]

**Step 2**
The factors \( h^n(aba) \) are right special.

- By direct observation, \( aba \) is right special
- By recurrence, \( h^n(aba) \) is right special as well.
\[ h(a) = abaaba \quad h(b) = bababa \]
\[ h^\omega(a) = abaababababaabaabaabaabababababababababababababa \ldots \]

**Step 3**

The prefixes \( p \) such that \( pp \) is a factor are \( h^n(a) \), \( h^n(ab) \), \( h^n(aba) \).

- For \( |p| \leq 6 \), check this directly.
- Otherwise \( p \) starts with \( abaabab \), which “synchronizes” \( h \).
- \( \exists p' = h^{-1}(p) \) such that \( p'p' \) is also a prefix of \( h^\omega(a) \).
- Recurrence on the length of \( p \)…
\[ h(a) = abaaba \quad h(b) = bababa \]
\[ h^\omega(a) = abaababababaabaabaabaababababababababababa \ldots \]

**Step 4**

The words \( h^n(a) \) and \( h^n(ab) \) are not quasiperiods of \( h^\omega(a) \).

- Check directly that the statement holds for \( n = 0, 1 \).
- Observe \( h^n(a) \) starts with \( abaabab \) for \( n \geq 2 \).
- If \( h^n(a) \) were a quasiperiod, so would \( h^{n-1}(a) \).

Same idea for \( h^n(ab) \).
Example: quasiperiodic quasiperiods

\[ h(a) = abaaba \quad h(b) = bababa \]
\[ h^\omega(a) = abababababaabaabaabaabababaabababa \ldots \]

We have proved that:

- \( h^n(aba) \) are quasiperiods;
- \( h^n(aba) \) are right-special and squares of \( h^\omega(a) \);
- if \( h^\omega(a) \) had another q.p., then it would be of the form \( h^n(a) \) or \( h^n(ab) \);
- \( h^n(a) \) and \( h^n(ab) \) are not quasiperiods of \( h^\omega(a) \).
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- \( h^n(aba) \) are quasiperiods;
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- if \( h^\omega(a) \) had another q.p., then it would be of the form \( h^n(a) \) or \( h^n(ab) \);
- \( h^n(a) \) and \( h^n(ab) \) are not quasiperiods of \( h^\omega(a) \).

Therefore the only quasiperiods are \( h^n(aba) \).
Example: quasiperiodic quasiperiods

\[ h(a) = abaaba \quad h(b) = bababa \]
\[ h^\omega(a) = abaababababaabaabaabaabaababababababababababa \ldots \]

The set of quasiperiods of \( h^\omega(a) \) is \( \{h^n(aba), n \in \mathbb{N}\} \).

Remark
All quasiperiods of \( h^\omega(a) \) are themselves quasiperiodic, except \( aba \).

Can we do the opposite?
Other example: no quasiperiodic quasiperiod

Let \( q = abbababba \) and

\[
\psi(a) = (abbab)^7 \quad \psi(b) = bababba(q)^2(abbab)^2
\]

(We have \( |\psi(a)| = |\psi(b)| = 35 \), to simplify boring technical arguments.)

This morphism has a fixed point \( \psi^\omega(a) \).
Other example: no quasiperiodic quasiperiod

Let \( q = abbababba \) and

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This morphism has a fixed point \( \psi^\omega(a) \).

Similar arguments yield its set of quasiperiods: \( \{\psi^n(q), n \in \mathbb{N}\} \).
Other example: no quasiperiodic quasiperiod

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This morphism has a fixed point \( \psi^\omega(a) \).

Similar arguments yield its set of quasiperiods: \( \{\psi^n(q), n \in \mathbb{N}\} \).

A boring recurrence proves that each \( \psi^n(q) \) is \textbf{not} quasiperiodic.

We have a word with \textit{no quasiperiodic quasiperiod}!
The “is a quasiperiod of” relation

Inside \( h^\omega(a) \), the “is a quasiperiod of” relation is a total order.

Inside \( \psi^\omega(a) \), that relation is empty.

We feel that we could design words with pretty much any order as a “is a quasiperiod of” relation, but we are not there yet.
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Recap

Quasiperiods are always prefixes. (We only need to study their lengths.)

Given $w$ and a q.p. $p_i$, we know whether $p_{i+1}$ and $p_{i-1}$ are q.p.’s.

Check whether $p_i$ is right-special and whether $(p_i)^2$ is a factor of $w$. 

Theorem

A word $w$ is periodic iff $\exists k$ such that $\forall n \leq k$, the prefix $p_n$ is a quasiperiod.

(No right special prefixes after some length.)
Recap

Quasiperiods are always prefixes. (We only need to study their lengths.) Given \( w \) and a q.p. \( p_i \), we know whether \( p_{i+1} \) and \( p_{i-1} \) are q.p.’s. Check whether \( p_i \) is right-special and whether \( (p_i)^2 \) is a factor of \( w \).

**Theorem**

A word \( w \) is periodic iff

\[ \exists k \text{ such that } \forall n \geq k, \text{ the prefix } p_n \text{ is a quasiperiod.} \]

(No right special prefixes after some length.)
Expressive power

**Theorem**

A word $w$ is periodic iff

$\exists k$ such that $\forall n \geq k$, the prefix $p_n$ is a quasiperiod.

(No right special prefixes after some length.)

**What is the “maximum” set of quasiperiods possible for aperiodic words?**

Could we find an aperiodic $w$ whose all prefixes, except continuations of right special ones, are quasiperiods?
And a surprise!

Theorem

A word \( w \) is standard sturmian iff all its prefixes are quasiperiods, except the continuation of right special prefixes.

(Recall that Sturmian words are defined by \( P_w(n) = n + 1 \) for all \( n \).)

(A word is standard if all its prefixes are left special. This is just a technicality.)
And a surprise!

Theorem

A word $w$ is **standard sturmian** iff all its prefixes are quasiperiods, except the continuation of right special prefixes.

(Recall that **Sturmian** words are defined by $P_w(n) = n + 1$ for all $n$.)

(A word is **standard** if all its prefixes are left special. This is just a technicality.)

We get:

1. a characterization of quasiperiods for all Sturmian words;
2. a definition of (standard) Sturmian words in terms of quasiperiods.
Characterization of Sturmian words’ quasiperiods

Theorem (Part I)

Let \( w \) denote a standard Sturmian word. Then \( p_i \) is a quasiperiod if and only if \( p_{i-1} \) is right special.

Since \( P_w(n) = n + 1 \) for all \( n \), the word \( w \) has one right and one left special factor of length \( n \).

The graph of factors has a very specific shape, from which the result is deduced.

(It’s easier to do on a blackboard.)
Characterization of Sturmian words by their quasiperiods

Theorem (Part II)

If \( w \) is a word whose set of quasiperiods is
\[ \{ p_i, p_{i-1} \text{ is not right special} \} \], then this word is Sturmian.
Characterization of Sturmian words by their quasiperiods

Theorem (Part II)
If $w$ is a word whose set of quasiperiods is 
\{\{p_i, p_{i-1} \text{ is not right special}\}, \text{ then this word is Sturmian.}\}

The proof is a bit long for a talk with slides. Here are the main steps.
Define the following morphism.

\[
L_a(a) = a \quad L_a(b) = ab
\]

1. The word $w$ is on a binary alphabet.
2. There exists a word $L_a^{-1}(w)$ (up to swapping letters).
3. The smallest quasiperiod of $w$ is $a^kba$ for some $k \in \mathbb{N}$.
4. The word $L_a^{-1}(w)$ also satisfies the theorem’s condition.
This is sufficient to conclude that $w$ is Sturmian.
Periodic words are the simplest ones.  
Periodic words are the words with the most quasiperiods.

Sturmian words are the simplest aperiodic words.  
Sturmian words are the aperiodic words with the most quasiperiods.
Summary

Periodic words are the simplest ones.
Periodic words are the words with the most quasiperiods.

Sturmian words are the simplest aperiodic words.
Sturmian words are the aperiodic words with the most quasiperiods.

We have a parallel between “simplicity” and “# of quasiperiods”.

In the end, quasiperiodicity is a good notion of symmetry...
The number of quasiperiods is a good measure of symmetry.
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Conclusion

- $q$-Quasiperiodic: covered with occurrences of $q$
- does not imply any global properties
- multi-scale: having infinitely many quasiperiods
- a method to determine sets of quasiperiods
- building extremal examples of quasiperiodic words
- the more quasiperiods you have, the simpler you are
- measuring symmetry in “number” of quasiperiods

Thank you for your attention!