Logical Axiomatizations of Pseudoline Arrangements and of Graph Drawings with Edge Crossings.

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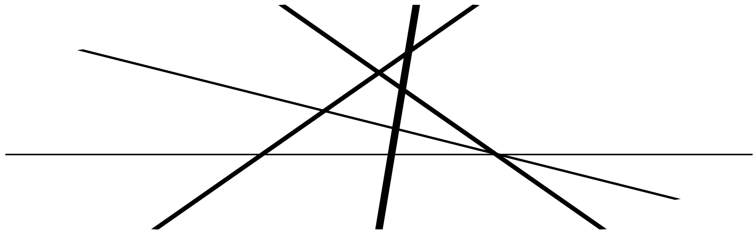
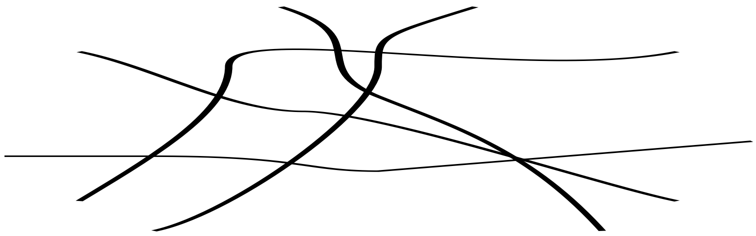
Line and pseudoline arrangements

• *Line arrangement*:
finite set of lines in the real affine plane such that any line intersects every other line



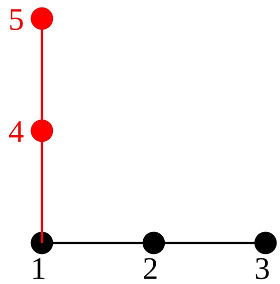
• *Pseudoline arrangement*:
finite set of curves in the real affine plane such that
- any pseudoline is homeomorphic to a line
- any pseudoline intersects every other line exactly once, at a point where they cross.



Rks.
- We consider these structures **up to an orientation preserving homeomorphism**.
(the two above drawings are homeomorphic)
- There is a "symmetry at infinity".

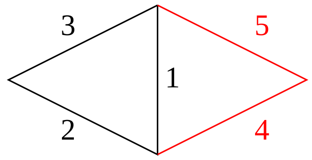
From point configurations to line arrangements

• Configuration of points



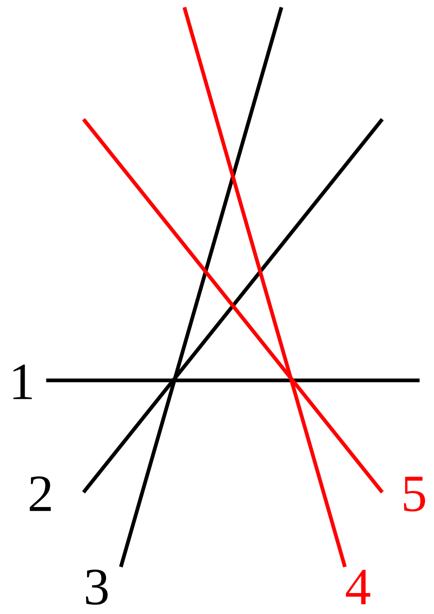
circits of the matroid (minimal affine dependencies): 123, 145, ...
bases of the matroid (affine bases): 124, 125, 134, 135, 234, 235, 245, 345

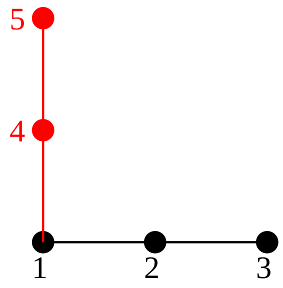
Rk. This example represents the cycle matroid of the graph



• Line arrangement

Dual of a rank-3 point configuration in the linear algebra sense.





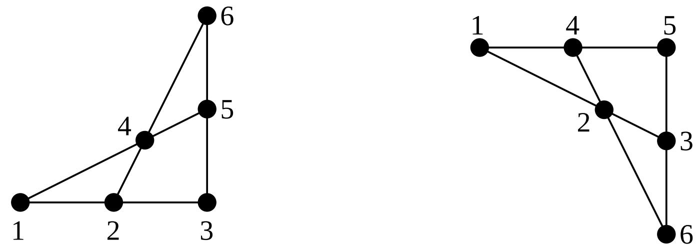
The **matroid** encodes the incidence relations between the lines
(it does not determine the drawing)

The **oriented matroid** encodes moreover the relative positions of the lines
(it determines the drawing up to homeomorphism assuming the directions at infinity are known)

From matroids to oriented matroids

bases of an oriented matroid = bases of a matroid together with a sign rule for base signs
base sign (chirotope) $\sim$ orientation of the simplex
sign rule $\sim$ Grassman-Plücker relations

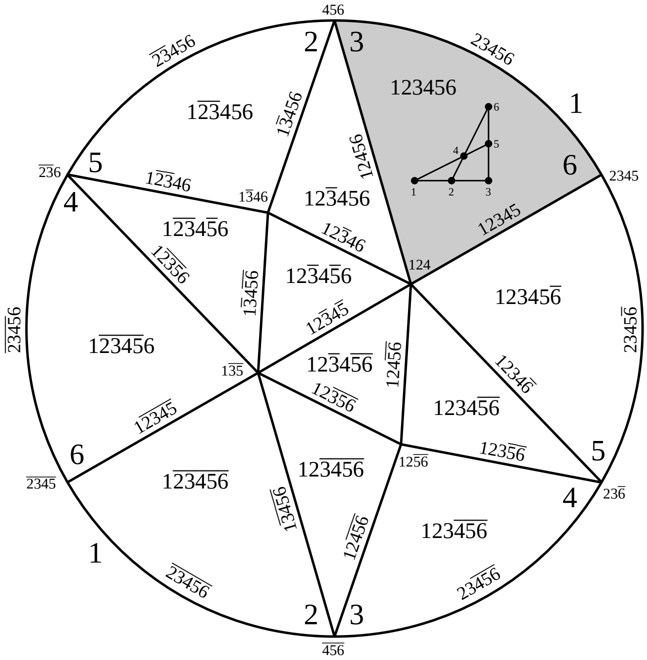
circuits of an o. m. = circuits of a matroid with a sign for each element in each circuit
signs of elements $\sim$ signs in a dependency relation



These two configurations have same matroid but not same oriented matroid.
(here the element 6 has been reoriented, it is a projective transformation)

Rk. Testing orientability is NP-complete [Richter-Gebert 99]

Oriented matroids and line arrangements



Drawing of circles on a sphere: each face is associated with a signed subset (covector)

the zero signs are encoded with the matroid

the non-zero signs are encoded with the oriented matroid

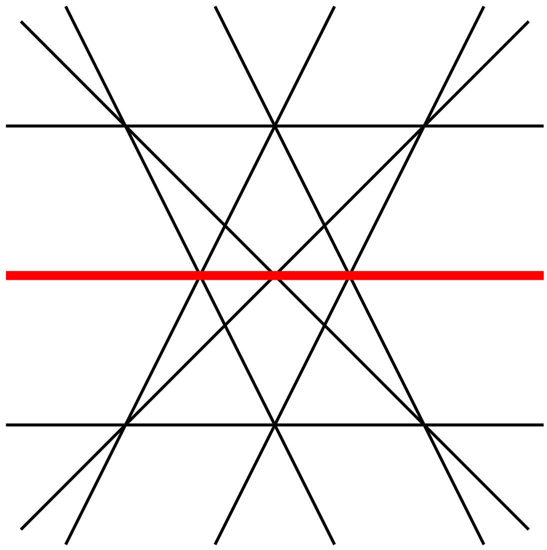
Drawing of lines on a plane: obtained by deleting an hemisphere

(i.e. by choosing the directions at infinity)

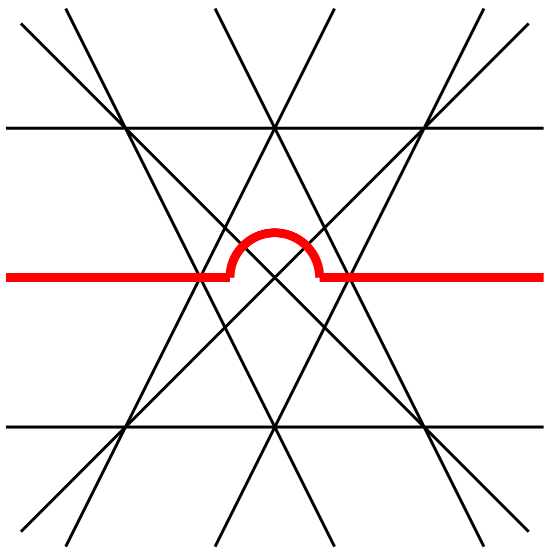
Rk. The 0-dimensional faces represent the cocircuits

From line arrangements to pseudoline arrangements

Pappus' theorem for line arrangements.
The red line must contain the three intersection points if it contains two of them



Non-Pappus pseudoline arrangement:



The relative positions of faces still determine an (oriented) matroid, but it is not realizable

Rk. It is “impossible” to characterize combinatorially realizable oriented matroids
(there is an infinity of classes of excluded minors...)

Axiomatisation of pseudoline arrangements (1)

Topological Representation Theorem. [Folkman, Lawrence 1978]
Pseudosphere arrangements represent the (simple) oriented matroids.
Pseudoline arrangements represent the (simple) rank-3 oriented matroids.

(*simple* means with no loop and no parallel elements)

Pseudoline arrangements: Levi 1926 ...

Oriented matroids: Bland (from linear programming),
Folkman and Lawrence (from polytopes),
Las Vergnas (form directed graphs)
1970' ...

• Axioms for circuits (or cocircuits):

(i) $\emptyset \notin \mathcal{C}$;

(ii) $\mathcal{C} = -\mathcal{C}$; (symmetry)

(iii) for all $X, Y \in \mathcal{C}$ if $\underline{X} \subseteq \underline{Y}$ then $X = Y$ ou $X = -Y$; (non-comparability)

(iv) for all $X, Y \in \mathcal{C}$, $X \neq Y$, and $e \in X^+ \cap Y^-$, there exists $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) - e$ and $Z^- \subseteq (X^- \cup Y^-) - e$ (elimination)

• Many other equivalent axiomatics (bases, covectors...)

Rks.

- Here we test separately the structure and its rank:
rank 3 $\Leftrightarrow$ every subset with 4 elements contains a circuit $\Leftrightarrow$ every basis has cardinal 3

- To draw a rank-3 oriented matroid on a plane we consider the directions at infinity

Axiomatisation of pseudoline arrangements (2)

Theorem. [Cordovil 1982]

Given a (simple) rank-3 matroid M,

M is the (orientable) matroid of a pseudoline arrangement A

if and only if

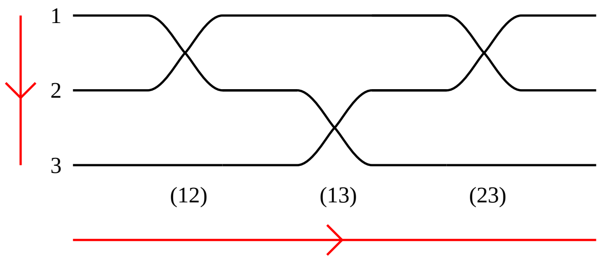
there exist compatible linear orderings of the set of elements of M and the set of cocircuits of M such that if $a < b < c$ for three independent elements then the three corresponding spanned cocircuits have either the order $(ab) < (ac) < (bc)$ or the order $(bc) < (ac) < (ab)$

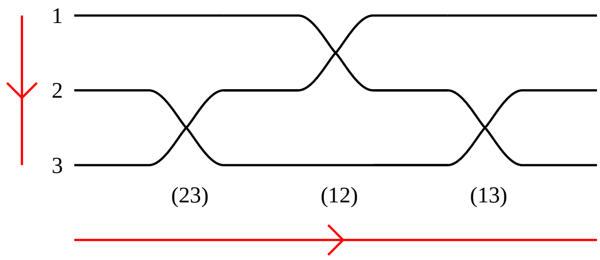
In this case:

- the linear ordering of M is represented by **an** order at infinity and the choice of a direction for each pseudoline

- the cocircuit (ab) is represented by the intersection $a \cap b$ of the pseudolines associated with a and b.

- the linear ordering of cocircuits is represented by the ordering of the intersections w.r.t. the directions of pseudolines

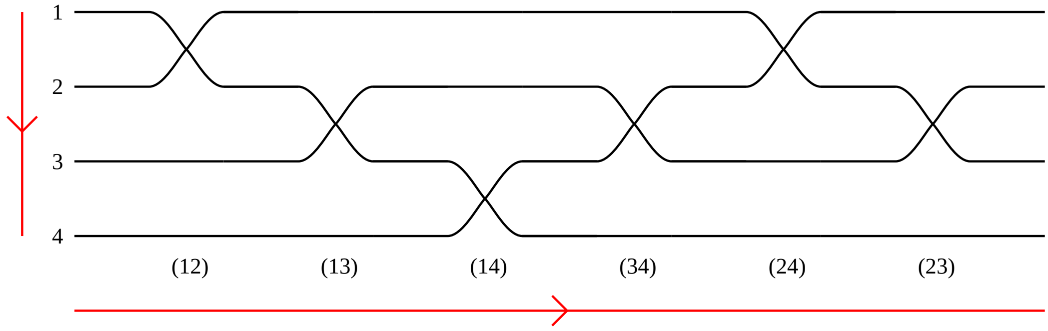




Uniform pseudoline arrangements:
the underlying matroid is uniform (every $k-$subset in n elements is a basis of $U_{n,r}$)

$$\Longleftrightarrow$$

the pseudolines are in general position (no three pseudolines meet at the same point)



Rks.

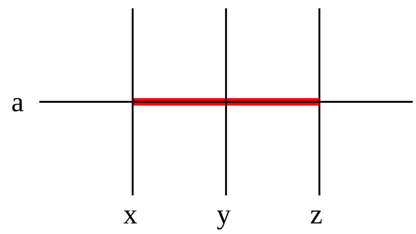
- In the uniform case, the previous hypothesis "M is a matroid" is useless.

- The previous axiomatisation, based on linear orderings, still uses second order logic.

Axiomatisation of pseudoline arrangements (3)

Let A be a pseudoline arrangement.

A relation *between* with arity 4 on A is defined by:
$between(a, x, y, z) \Leftrightarrow a \notin \{x, y, z\}$ and the intersection $a \cap y$ is between $a \cap x$ and $a \cap z$.



Rk. $between(a, x, y, x)$ means that a, x, y meet at the same point.
Hence "x, y, z are in general position" is given by a first order formula using *between*.

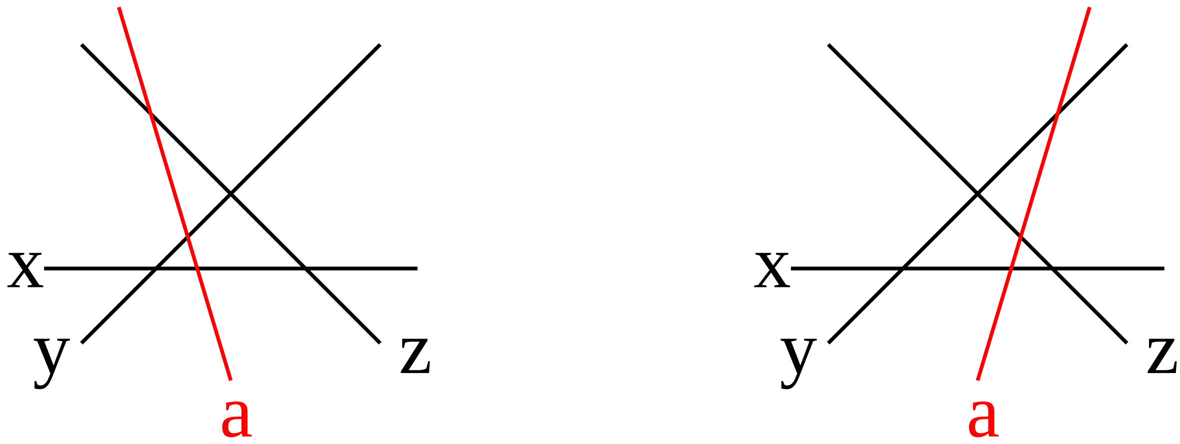
Theorem. [Courcelle, Olive 1999]
An (explicit) set of 9 formulas based on the relation between form a **first order logic** *axiomatisation of pseudoline arrangements.*

Axiom 1: $between(a, x, y, z) \Rightarrow a \notin \{x, y, z\}$

...

Axiom 9: x, y, z in general position $\quad \Rightarrow$

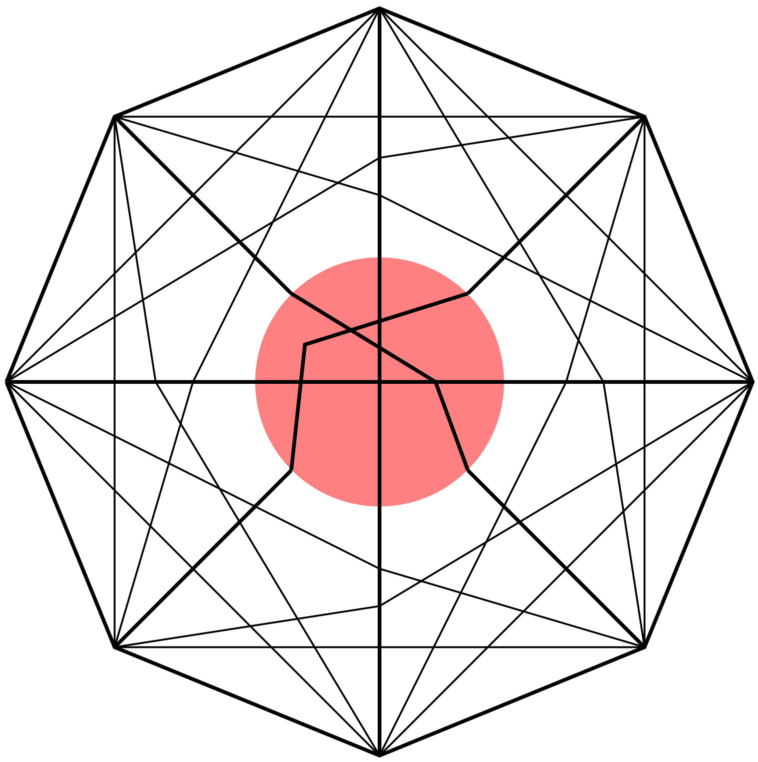
$$\Big(\ between(x, y, a, z) \quad \Leftrightarrow \quad (\ between(y, x, a, z) \quad \oplus \quad between(z, x, a, y)\ )\ \Big)$$



Rk. Axiom 9 "gives" the compatibility property of element and intersection orderings.

FROM PSEUDOLINE ARRANGEMENTS TO GRAPH DRAWINGS WITH EDGE CROSSINGS

An easy example:



Graph drawing

graph: finite, loopless, connected (directed) G(V,E)

drawing D of G: representation of G in the real directed plane

axiomatics of *drawn edges*:

D1 - *topological drawing*:
drawn edge = homeomorphic to a line segment = pseudoline segment

(D1' - *geometrical drawing*: drawn edge = line segment)

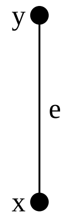
D2 - when two drawn edges meet:
- they do not meet at a vertex
- they meet at most once
- they cross each other

D3 - three drawn edges do not meet at a same point (except at a vertex)

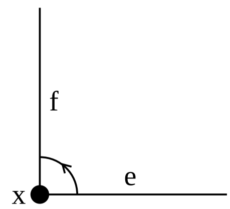
We consider the structure **up to an orientation preserving homeomorphism**.

Map of a drawing

(directed) graph structure:
$(e, x, y) \in Struct_D \subseteq \vec{E}_G \times V_G \times V_G$
iff x and y extremities of e



next edge around a vertex:
$(x, e, f) \in NextAround_D \subseteq V_G \times E_G \times E_G$
iff x extremity of e and f,
and f next drawn edge around x



a corner of $D = (P, \beta, \alpha) \in NextAround_D$
the edges α and β with vertex P touch the unbounded region in a neighbourhood of P

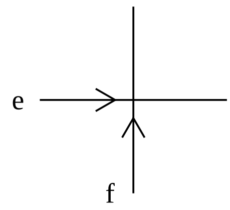
the map of D: $Struct_D$ and $NextAround_D$

Well known result.
The map and a corner determine the drawing of a **planar graph**.

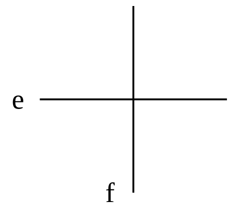
Rk. remove the corner $\sim$ draw on a sphere

Sketch of a drawing [Courcelle 2000]

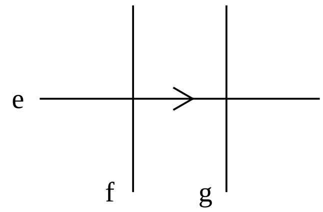
directed crossings:
$(e, f) \in Dcross_D \subseteq \vec{E}_G \times \vec{E}_G$
iff e and f cross in a given direction



crossings:
$(e, f) \in Cross_D \subseteq E_G \times E_G$
iff e and f cross



ordering of crossings:
$(e, f, g) \in OrderCross_D \subseteq \vec{E}_G \times E_G \times E_G$
iff f crosses e before g



the sketch (or predrawing) of D: the map, $Dcross_D$, and $OrderCross_D$

Theorem.
The sketch and a corner determine a drawing of the graph.

drawing $\sim$ *sketch*: up to orientation preserving homeomorphism and with given corner

Axiomatisation of graph drawings [Courcelle 2000]

Let $R =< W, Struct, NextAround, DCross, OrderCross >$
be a finite relational structure with same arities as before and satisfying:

S1 - $< W, Struct >$ is the structure of a directed loop-free graph G

S2 - $< W, Struct, NextAround >$ is a map of G

S3 - $DCross(e, e') \Rightarrow e, e' \in E \wedge e \neq e' \wedge \neg DCross(e', e)$

S4 - for every e, the relation $OrderCross$ induces a strict linear ordering on the set of edges that cross e.

R is called *realizable* if it is the sketch of a drawing

Prop.
(i) That a structure is a sketch can be expressed by an MS-formula
(ii) The realizability of a sketch is expressible by a second-order formula

Question. Is the realizability of a sketch an MS-property ?

Answer for framed sketchs [Courcelle 2000]

A *framed sketch* is the sketch of a graph drawing together with a subgraph such that:
- the subgraph is connected
- the subgraph has the same vertices as the graph
- the subdrawing of this subgraph is planar

the map of the subgraph is the *frame* of the sketch

The 3-Edge Theorem.
A sketch R with frame F is realizable
if and only if
for every subset X of $E - E_F$ of cardinality at most 3, the sketch $R[E_F \cup X]$ is realizable.

Corollary. *The realizability of sketchs having a frame is MS-expressible.*

Back to pseudoline arrangements

pseudoline arrangement described by:

- an ordering at the infinity
- the ordering of crossings along each pseudoline

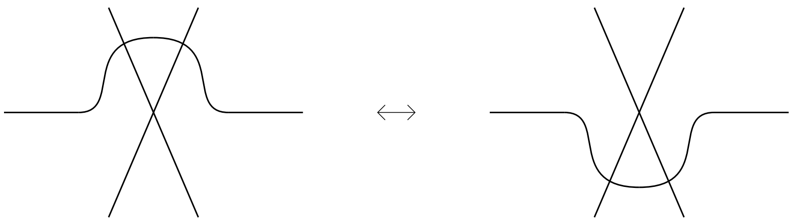
  ∼ *between* relations

  ∼ *OrderCross* relations

  similar to graph drawings but only this relation is not "degenerate"

free triangle:
the topological interior of the region has an empty intersection with the drawing

triangle flip of a free triangle :



Theorem. [Ringel 1957]
Two uniform pseudoline arrangements with same size can be joined by a sequence of triangle flips

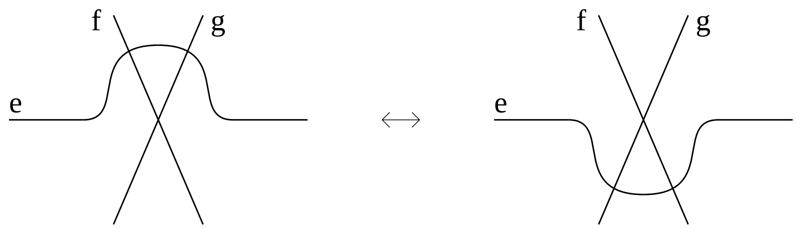
Triangle flips in arrangements or drawings

triangle of a drawing D:
$(e, f, g) \in E_G \times E_G \times E_G$ with e, f and g cross each other

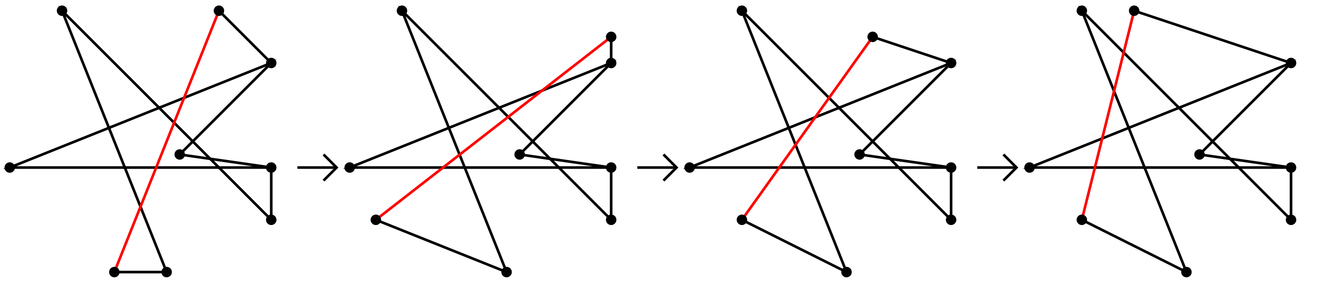
free triangle in D:
the topological interior of the region has an empty intersection with the drawing

permuted triangle (e, f, g) between D and D':
$OrderCross_{D'}(e, f, g) = \neg OrderCross_D(e, f, g)$
and similar properties on f and g.

flip (or switch) D' of a free triangle (e, f, g) in D:
$OrderCross_{D'}(e, f, g) := \neg OrderCross_D(e, f, g)$
and similar properties on f and g.



Graph drawing up to a sequence of flips

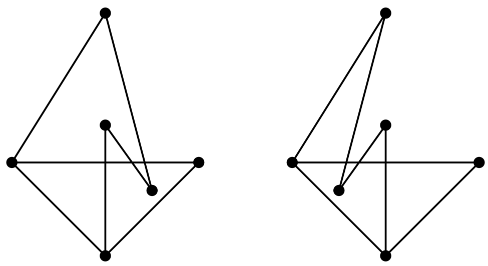


subsketch: the map and the *Cross* relation

(i.e. the sketch without the orderings of crossing)

Observation: a flip does not change the subsketch.

But conversely:
the subsketch *does not determine in general* the drawing up to a sequence of flips.



Complete graph drawing

Theorem. [EG 2005]
A complete graph drawing is determined up to homeomorphism and a sequence of flips only by its size, a corner and the Cross relations (hence by its subsketch and a corner).

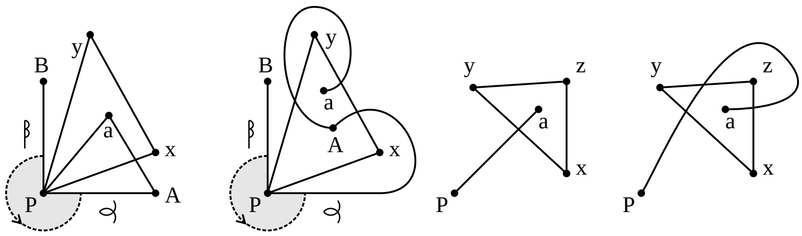
First part:

With **first order logic** formulas, build:
- the map and the *Cross* relations (hence the subsketch)
- the *OrderCross* relations except for triangles,
- relations stating that a vertex belongs to the interior of a triangle.

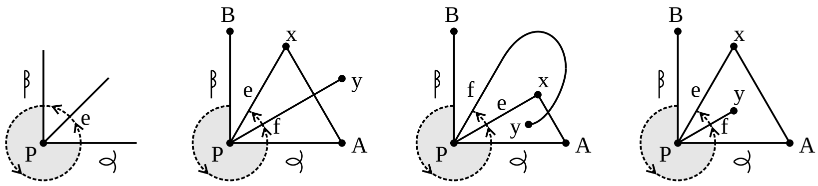
Second part:

With a generalisation of an algorithm for Ringel's theorem, build a sequence of flips from two complete graphs with same subsketch.

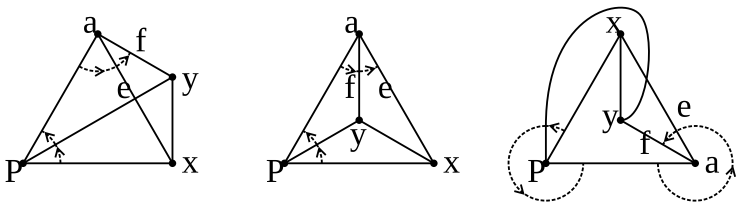
First part: step by step with first order formulas



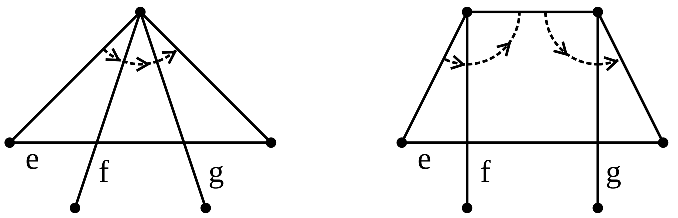
some interior relations



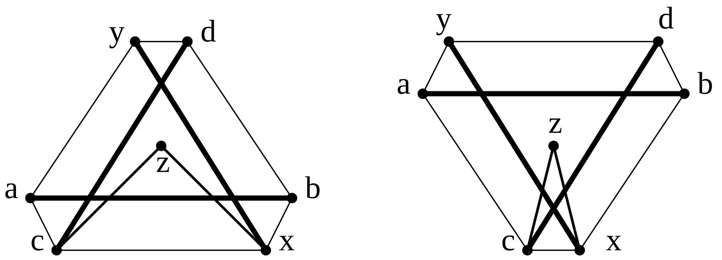
relations around the corner



relations around any vertex



order of crossings except for triangles (geometrical case)



a vertex is in the interior of a triangle

Second part: an algorithm for Ringel's theorem

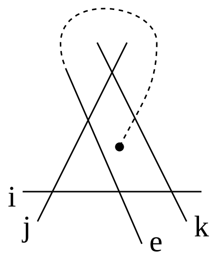
Inductive algorithm:

$A = A_n$ and $A' = A'_n$ arrangements on $E = e_1, ..., e_n$

A_i and A'_i extracted arrangements on $E_i = e_1, ..., e_i$

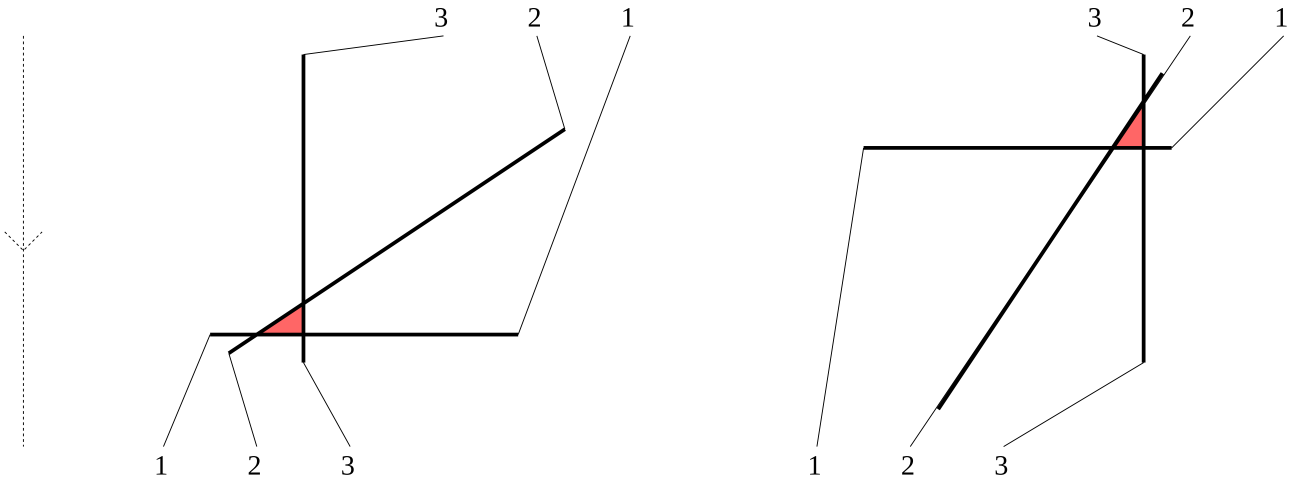
Sequence of flips at level i from A_i to A'_i $:=$

a sequence given by the sequence at level $i - 1$
by "getting e_i out of the muted triangle"



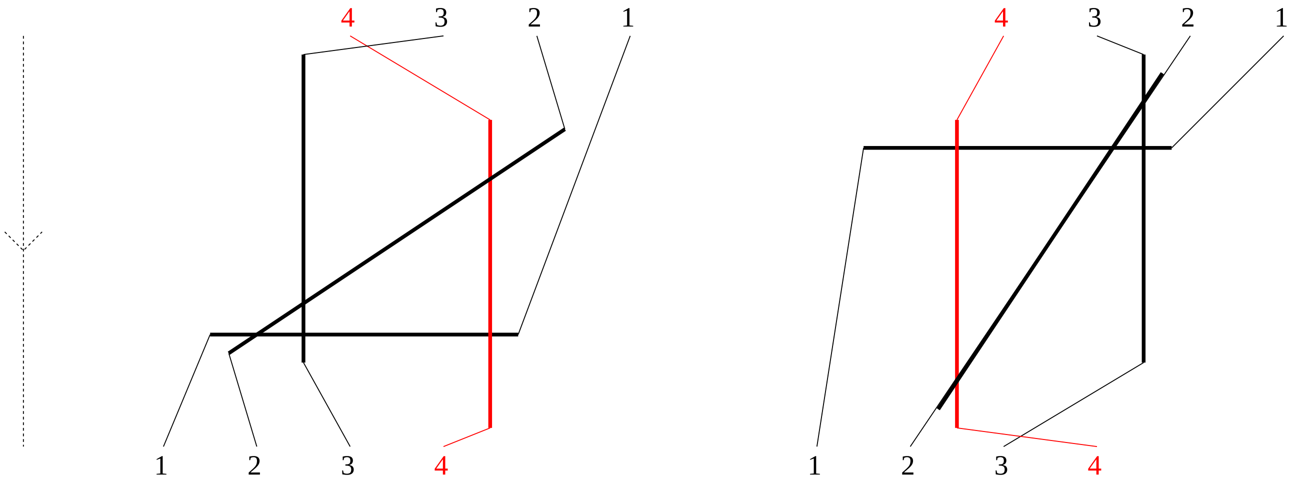
the flip of ijk becomes $eij \to ijk$
(dash is impossible)

+ a sequence "where only e_i moves" (sweeping)



- level 3: $\emptyset$

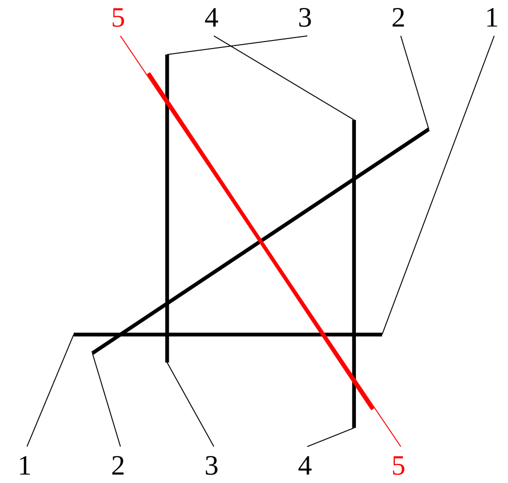
(triangles 123 are the same)

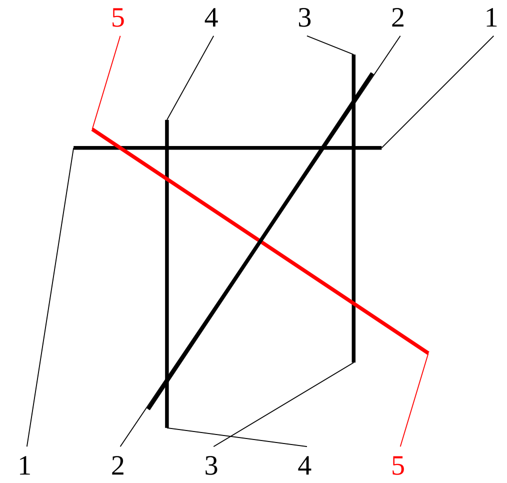


- level 3: $\emptyset$

- level 4: $(\emptyset) \to (234 \to 134 \to 124)$

(only 4 had to be swept)

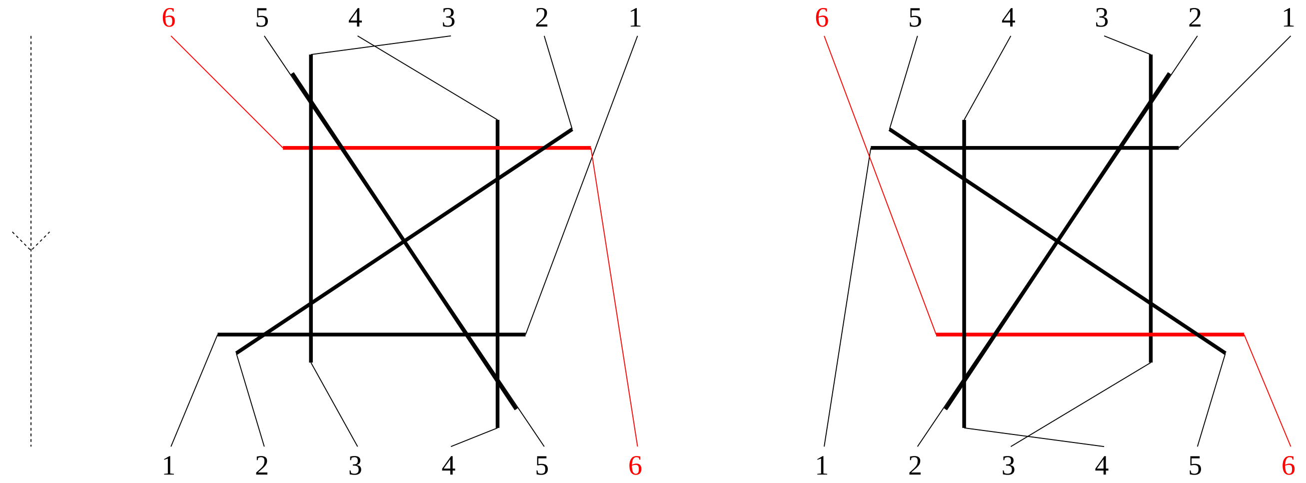




- level 4: $(\emptyset) \rightarrow (234 \rightarrow 134 \rightarrow 124)$

- level 5: $(235 \rightarrow 234 \rightarrow 135 \rightarrow 134 \rightarrow 125 \rightarrow 124) \rightarrow (\emptyset)$

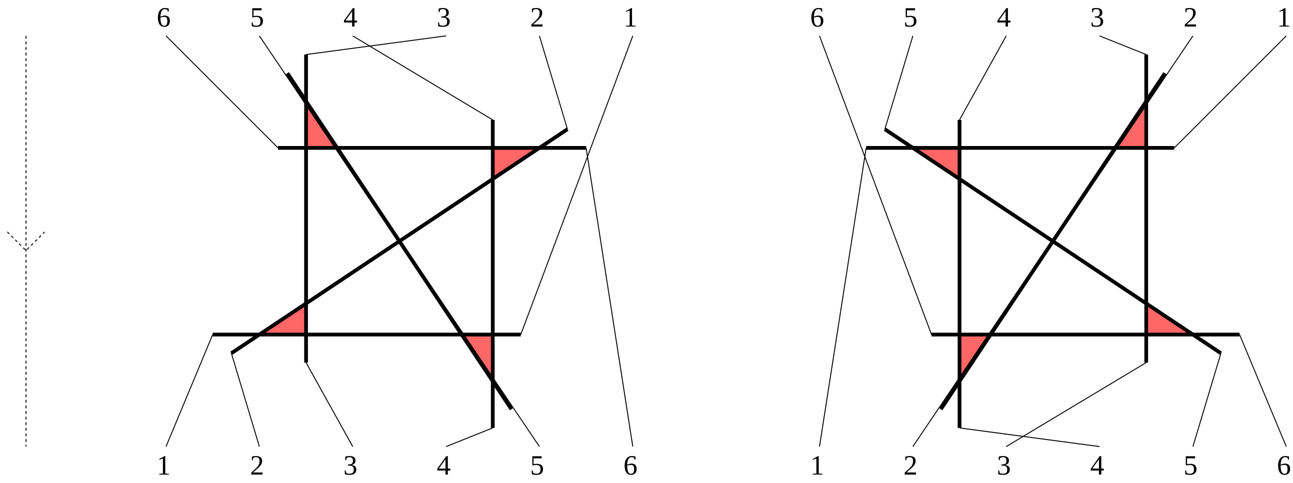
(the sequence given by level 4 is sufficient)



- level 5: $(235 \to 234 \to 135 \to 134 \to 125 \to 124) \to (\emptyset)$

- level 6:

$$(\mathbf{356} \to 235 \to 346 \to 234 \to 135 \to 134 \to 125 \to 124)$$
$$\to (236 \to 126 \to 136 \to 146 \to 156 \to 456 \to 256 \to \mathbf{356})$$



356 $\rightarrow$ 235 $\rightarrow$ 346 $\rightarrow$ 234 $\rightarrow$ 135 $\rightarrow$ 134 $\rightarrow$ 125 $\rightarrow$ 124
$\rightarrow$ 236 $\rightarrow$ 126 $\rightarrow$ 136 $\rightarrow$ 146 $\rightarrow$ 156 $\rightarrow$ 456 $\rightarrow$ 256 $\rightarrow$ **356**

Rk. Here, a triangle has to be flipped twice (356).

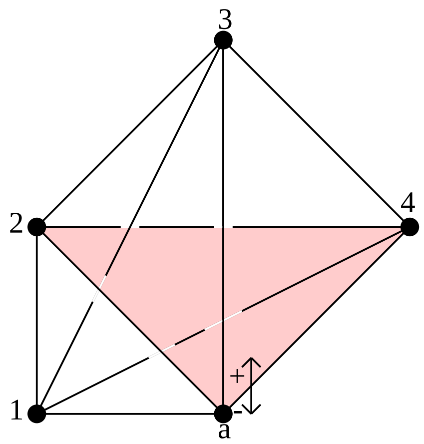
These two arrangements have all free triangles in same positions [Felsner, Weil 2000]

Consequence: the minimal number of flips is unknown in general

This algorithm nevertheless extends to complete graph drawings because:
- any muted triangle is contained in a permuted triangle,
- the subsketch determines the fact that a vertex is in the interior of a triangle.

Spatial graphs and rank-4 oriented matroids

realizable uniform oriented matroid (size n, rank 4) : encodes the relative positions in a set E of n points in general position in the real affine space with 3 dimensions



cocircuit: signed subset $C = (C^+, C^-)$ with
$E \setminus (C^+ \cup C^-) = 3$ points generating a plane $\mathcal{H}$, delimiting the space in two half-spaces
C^+ = points belonging to a half-space
C^- = points belonging to the other *Ex*: plane $[a, 2, 4] \to$ cocircuit $(1, 3)$

faces of the convex hull: positive cocircuit *Ex*: $[a, 1, 2] \to (34, \emptyset)$

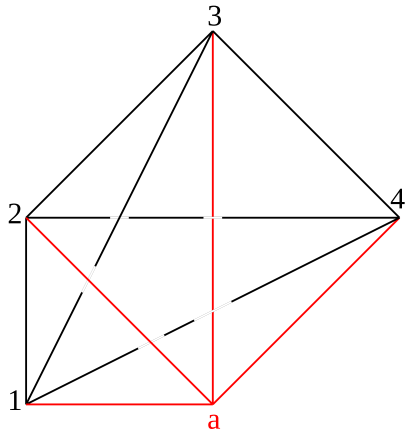
extremal point: only positive element of a cocircuit
Ex: $[1, 3, 4] \to (a, 2)$, hence a extremal

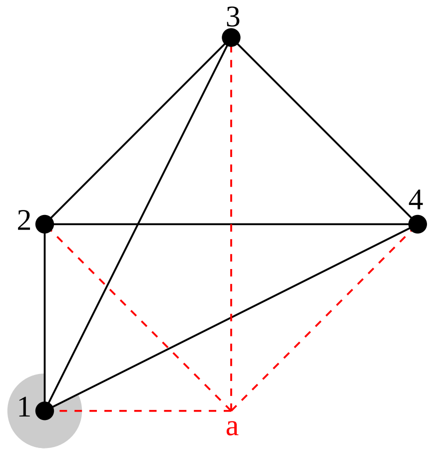
Projection of a spatial graph

$a_1, ..., a_n$ and a in general position

spatial graph: formed by the segments $[a_i, a_j]$

point of view: extremal point a

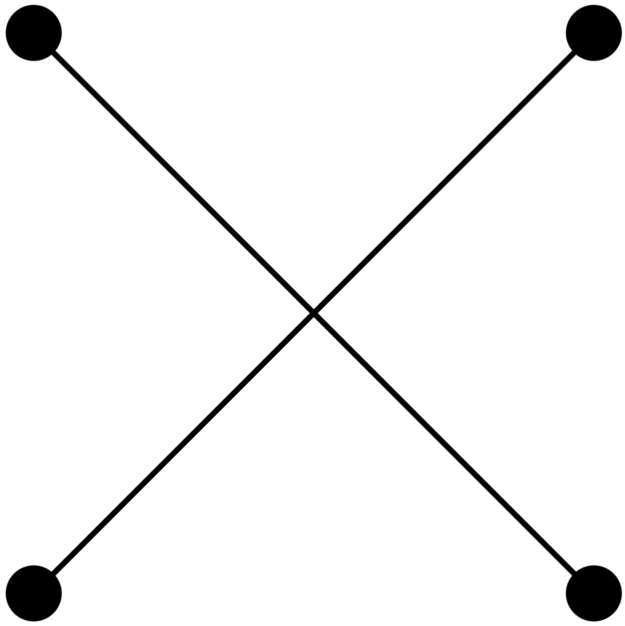




Proposition. *The oriented matroid and the extremal point a determine a corner and the subsketch of the projection drawing from a*

(but not all the drawing)

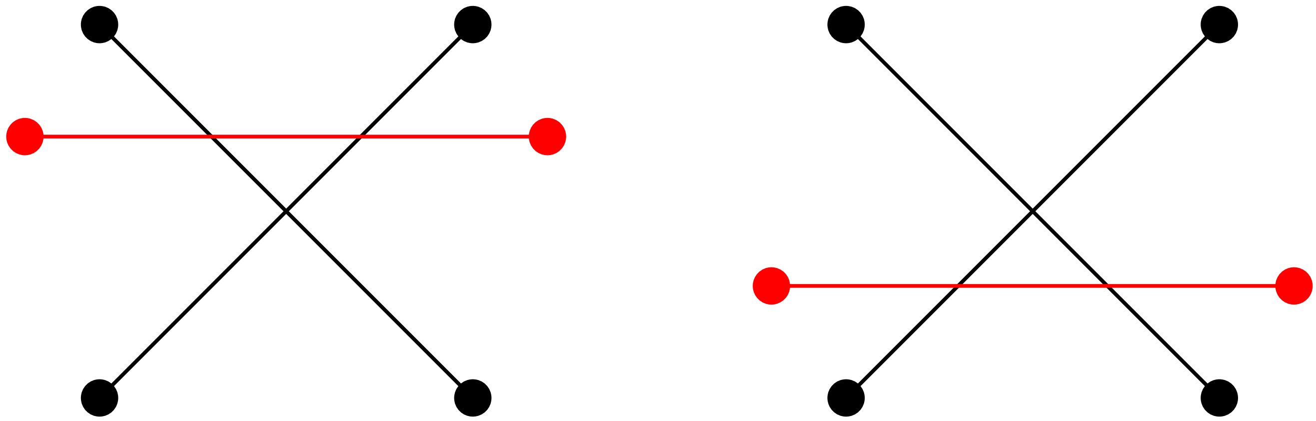
Proof on an example: $[a, 1, 3] \to (2, 4)$ in the o.m. hence $[1, 3]$ cuts $[2, 4]$ in the drawing; $[a, 1, 2]$ and $[a, 1, 4]$ in the convex hull, hence no crossings on $[1, 2]$ and $[1, 4]$, and $(1, [1, 2], [1, 4])$ corner.



Ex. Rank-4 oriented matroid given by 7 points in 3 dimensions:
- the 4 black vertices on the screen,
- my hand, my elbow
- your own head : *extremal point* of the oriented matroid,
  point of view for the projection on the screen

The oriented matroid changes when a point crosses a plane spanned by three other points.

Consider the plane formed by my arm and your head.

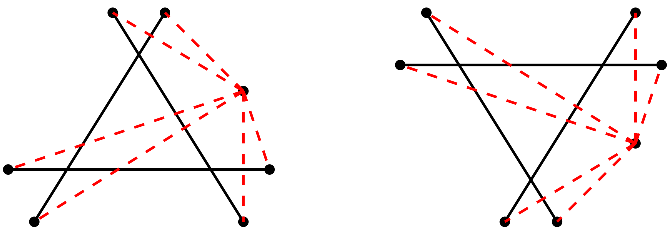


The map and Cross relations of the drawing are determined by the oriented matroid,

but not the triangles.

An encoding with two levels (logical and combinatorial)

Corollary. [EG 2005]
The oriented matroid and the point of view on a spatial graph determine the projection drawing up to a sequence of triangle flips.



Moving the view-point on a spatial graph:

the planes $[a_i, a_j, a_k]$ cut the space into regions,

- when a stays in the same region, the o.m. does not change and the drawing is determined up to flips.

- when the region of a changes, the o.m. structure changes
(simply: if a crosses the plane $[a_i, a_j, a_k]$, the sign of a in the associated cocircuit changes)

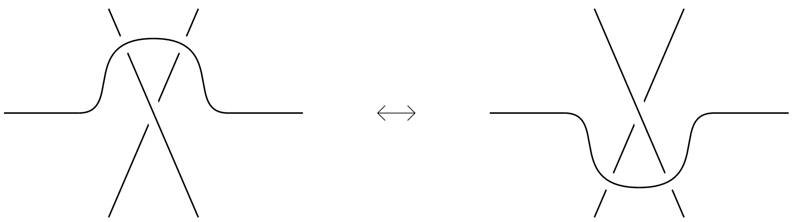
Questions

• *General graph drawings*:
- other intermediate structures between the map and the sketch,
- decompositions, local structures, frames...

• *Logic*:
- characterize sketchs, and subsketchs, by a MS-formula (cf. Courcelle),
- describe sequences of flips, or relaxed structures, with logic formulas

• *Forbidden flips*:
- builiding a minimal sequence of flips (graphs or arrangements)
- projection recording the upper segment when two segments cross
  (topological or geometrical spatial graphs, knots, braids, algebraic geometry...)

triangle flip $\rightarrow$ *third Reidemeister move*



Conjecture. [EG]
A braid of which strands cross each other once is determined by its 3-strands sub-braids.