Enumerating Degree Sequences in Digraphs and a Cycle-Cocycle Reversing System

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Abstract. We give some new enumerations of indegree sequences of orientations of a graph using the Tutte polynomial. Then we introduce some discrete dynamical systems in digraphs consisting in reversing cycles, cocycles, or both, which extend the edge firing game (reversing sinks) by considering all orientations (reversing cocycles) and by introducing duality (reversing cycles). We show that indegree sequences can represent the configurations of these systems, and we enumerate equivalence classes of these systems. In particular, concerning the cycle-cocyle reversing system, we show that its configurations are in bijection with indegree sequences of orientations having a given vertex (quasi-sink of the system) reachable from any other. We also briefly discuss its generalization to oriented matroids, and relate structural and enumerative properties of its configurations to those of the sandpile model or chip firing game.

Keywords. Directed graph, degree sequence, cycle reversing, Tutte polynomial, duality, discrete dynamical system, edge firing game, chip firing game, sandpile model, matroid.

1. Introduction

The indegree sequence of a digraph is the sequence of integers formed by the numbers of incoming edges at each vertex. The number of indegree sequences of orientations, resp. acyclic orientations, resp. acyclic orientations with a given unique sink, of a given graph can be calculated as an evaluation of its Tutte polynomial [5][15][22][23]. The Tutte polynomial of a graph is a generalization to two dual variables of the chromatic polynomial [24]. There are several equivalent definitions [5][18][24]. This polynomial encodes much enumerative and structural properties, and appears in various contexts, notably in some physical models [5]. Its coefficients are important invariants in graph theory, and it can be also generalized for instance to hyperplane arrangments, matroids and oriented matroids.

In Section 3, we use this polynomial again to enumerate indegree sequences of strongly connected orientations of a graph, and then, using a duality decomposition, to enumerate indegree sequences of orientations for which a given vertex is reachable from any other with a directed path (we say that these orientations have a unique quasi-sink at this vertex, since this vertex induces the unique sink when cycles are contracted).

Discrete dynamical systems on graphs have been studied for about twenty years, originally in connection with physical models, and because of their intriguing structural, enumerative and algebraic properties. The general motivation is the modelization with cellular automata of self-organized criticality in complex phenomena [9]. Similarities have been noticed between the behaviour of the mathematical models and of some physical events as sandpile formation, avalanche release, or earthquake propagation [1]. Briefly, the sandpile model consists in topple of grains of sand from a vertex to its neighbours when the degree of the vertex is reached. This model is sometimes also called chip firing game [4]. It has intriguing algebraic properties like a not well understood group structure [2][7]. The edge firing game on a digraph consists in reorienting sinks when the graph is directed. Its main interest concerns acyclic orientations (see for instance [17]). For these orientations, it is equivalent to the sandpile model at level 0, and its equivalence classes are represented by acyclic orientations with given unique sink (a consequence of [4], see also Proposition 4.3 for a direct proof).

In Section 4, we introduce first a cocycle reversing system which allow to reorient directed cuts, not only sinks, extending properties of the edge firing game to all orientations. Then we introduce by duality a cycle reversing system which consists in reorienting cycles, and at last a cycle-cocycle reversing system allowing both rules. We prove that the equivalence classes of this system are in bijection with indegree sequences with unique quasi-sink at a given vertex. We obtain enumerations of equivalence classes of these systems as evaluations of the Tutte polynomial.

We mention that these systems can be defined more generally, and exactly the same way, in oriented matroids, in terms of circuit and cocircuit reversing. These combinatorial structures [3] extend cycle and cocycle spaces of graphs, by considering them in some sense as hyperplane arrangements, and the notion of duality is fundamental. Many results on the Tutte polynomial generalize to oriented matroids. The previous enumerative results extend to regular matroids (but not to general oriented matroids), see [12]. However, the proofs of these results are quite different, since there is no notion of vertex, neither of sink, nor of indegree sequence, in matroids.

Stable recurrent configurations at level 0 of the sandpile model are in bijection with equivalence classes of configurations of the edge or cycle-cocyle reversing systems on acyclic orientations. Using the Tutte polynomial, we got an enumerative equality between the number of equivalence classes of configurations for the cycle-cocyle reversing system and the number of stable recurent configurations of the sandpile model [19], all levels being counted. These properties lead to the question of finding an appropriate bijection, see Section 5. Finally, this paper may be a first step towards a study of the sandpile model of a graph using orientations of this graph, towards a systematic study of duality in this model, or towards its generalization in the more geometrical context of regular matroids or oriented matroids.

2. Preliminaries

An *orientation* $\vec{G}$ of a graph $G = (V, E)$ is the digraph defined by the choice of a direction for every edge of G. For $A \subseteq E$, we denote $\vec{G}/A$ the minor of $\vec{G}$ obtained by contraction of A, and $\vec{G} \setminus A = G(E \setminus A)$ the minor of $\vec{G}$ obtained by deletion of A or equivalently by restriction to $E \setminus A$. The *reorientation* of $\vec{G}$ with respect to $A \subseteq E$, denoted $-_A\vec{G}$ is the digraph obtained by changing the directions of the edges in A. A *directed cycle* of a digraph is a set of (undirected) edges forming a cycle of the graph such that they are all directed accordingly with a direction for the cycle. A vertex is said to be *(contained) in* a cycle if it is the extremity of an edge in the cycle. A *directed cocycle* is a set of (undirected) edges forming a cut of the graph such that all edges are directed from the same part of the set of vertices defined by the cut towards the other. A *sink* of a digraph is a vertex with no outgoing edge. Note that the set of edges with a same sink as extremity is a directed cocycle. A digraph $\vec{G}$ is *acyclic* if it has no directed cycle, and *totally cyclic* if every edge belongs to a directed cycle. A connected graph is totally cyclic if and only if it is *strongly connected*, that is if for every pair of vertices there is a directed cycle containing the two vertices.

Duality wil be introduced thanks to a fundamental property of orientations of graphs: every edge belongs to a directed cycle or a directed cocycle but not both [29]. Hence a directed graph is acyclic, resp. totally cyclic, if and only if every edge, resp. no edge, belongs to a directed cocycle. For a directed graph $\vec{G}$, the *cyclic part* F of $\vec{G}$ is the union of (edges of) all directed cycles of $\vec{G}$, and $E \setminus F$ is called the *acyclic part* of $\vec{G}$ (it is the union of all directed cocycles of $\vec{G}$). Then $\vec{G}/F$ is acyclic and $\vec{G}(F)$ is totally cyclic. More structurally this result can be seen as a decomposition of the set of (re)orientations of a (di)graph:

$$2^E = \biguplus_{\substack{F \text{ cyclic} \\ \text{flat of } G}} \left\{ \ A' \uplus A'' \mid A' \subseteq F,\ -_{A'}\vec{G}(F) \text{ totally cyclic},\ A'' \subseteq E \setminus F,\ -_{A''}\vec{G}/F \text{ acyclic } \ \right\}$$

where a *cyclic flat* F of G is a union of (edges of) cycles of G such that its complementary $E \setminus F$ is a union of (edges of) cocycles, or equivalently, such that if $C \subseteq E$ is a cycle with $C \setminus e \subseteq F$ and $e \in C$, then $e \in F$ (see [13][14] for refinements of such decompositions).

Given a vertex s of G, we say that an orientation $\vec{G}$ has *a unique quasi-sink at s* if the vertex induced by s in $\vec{G}/F$ is the unique sink of $\vec{G}/F$, where F is the cyclic part of $\vec{G}$. Equivalently, there is a directed path from v to s in $\vec{G}$ for every vertex v of G.

More precisely, we call *quasi-sink* of a connected digraph $\vec{G} = (V, E)$ a set $S \subseteq V$ of vertices such that, with E_S the set of edges with extremities in S, E_S is contained in the cyclic part F of $\vec{G}$, $G(E_S)$ is a connected component of $G(F)$, and the vertex induced by S in $\vec{G}/F$ is a sink of $\vec{G}/F$. Hence an orientation has a unique quasi-sink at s if and only if it has a unique quasi-sink S and this quasi-sink contains s.

Note that the definitions of sink and quasi-sink coincide for acyclic digraphs, and that a strongly connected digraph has a unique quasi-sink $S = V$.

We call V-vector a mapping which associates a positive integer with every vertex of a graph or a digraph on set of vertices V. Assuming $V = \{v_1, ..., v_m\}$ is labeled with increasing indices, a V-vector will be denoted $(a_1, ..., a_m)$ where a_i is associated with v_i, $1 \leq i \leq m$, and a_i is called the *value* of v_i.

The *indegree sequence*, or *in-sequence* for short, of a digraph $\vec{G}$ is the V-vector associating with $v \in V$ the number of incoming edges in v. Obviously the in-sequence $(a_1, ..., a_m)$ of $\vec{G}$ satisfies

$$\sum_{1 \leq i \leq m} a_i = \#E.$$

Note that a vertex is a sink of $\vec{G}$ if and only if its value in the in-sequence of $\vec{G}$ equals its degree in G.

Lemma 2.1. *The set of in-sequences of acyclic orientations of a graph G is in bijection with the set of acyclic orientations of G.*

Proof. An algorithm calculating the unique acyclic orientation with a given in-sequence of acyclic orientation is the following: since the orientation is acyclic, there exists a vertex of which value equals its degree in G, it is a sink of $\vec{G}$. Then, when this vertex is removed, the in-sequence of the subgraph is also obtained by deleting the vertex, the subgraph is also acyclic, and so on... □

The Tutte polynomial of the graph G is denoted $t(G; x, y) = \sum_{i,j} b_{i,j} x^i y^j$. Among its several equivalent definitions [5][18][24], we will use in proofs its famous following inductive one. When a graph is not connected, its Tutte polynomial equals the product of the Tutte polynomials of its connected parts.

$t(\emptyset; x, y) = 1$ and for all $e \in E$
if e is not a loop nor an isthmus of G then

$$t(G; x, y) = t(G \setminus e; x, y) + t(G/e; x, y)$$

if e is an isthmus of G then $t(G; x, y) = xt(G/e; x, y) = xt(G \setminus e; x, y)$
if e is a loop of G then $t(G; x, y) = yt(G/e; x, y) = yt(G \setminus e; x, y)$

It is known [15] that the number of (in-sequences of) acyclic orientations of a connected graph with unique given sink is $t(G; 1, 0) = \sum_i b_{i,0}$. The number of (in-sequences of) acyclic orientations of a graph G is $t(G; 2, 0)$ [22]. The number of in-sequences of orientation of a graph G is $t(G; 2, 1)$ [5][23]. The number of totally cyclic orientations is $t(G; 0, 2)$ (by duality in [15] extending [22])

A useful numerical extension of the previous duality decomposition is this formula for the Tutte polynomial, implicit in [11] since proved bijectively, and called "convolution formula" in [16] (see also [10][14] for extensions)

$$t(G; x, y) = \sum_{F \text{ cyclic flat of } G} t(G/F; x, 0)\ t(G(F); 0, y).$$

The *edge firing game* on a graph $G = (V, E)$ is defined by its set of configurations which are the orientations of the graph and by an evolution rule: if a vertex $v \in V$ is a sink of $\vec{G}$ with A as set of incoming edges, then $-_A\vec{G}$ is a *successor* of $\vec{G}$. These two orientations are said to be *equivalent*, and the transitive closure of this relation is an equivalence relation on the orientations of G. Obviously, if an orientation is acyclic, then every equivalent orientation is acyclic. On the contrary, if a vertex belongs to a directed cycle of an orientation then this vertex will never be a sink in an equivalent orientation. Thus the main interest of the edge firing game is its restriction to acyclic orientations.

Note that usually this system is defined on a digraph, and one studies the configurations reachable from this digraph [17]. Here we consider all edge firing games on the same underlying undirected graph, and call this structure the edge firing game of the undirected graph. If the graph is connected, and if we choose a vertex $s \in V$ to be the *sink* of the system, there exists one and only one acyclic orientation of G with unique sink s in each equivalence class of acyclic orientations. Since the edge firing game is somehow equivalent to the sandpile model at level 0 (see last Section), this result may be seen for instance as a consequence of a classical property of sandpile model configurations [4]. However, a direct proof is given in the present paper for completeness in Proposition 4.3. These particular orientations are called *s-configurations* of the edge firing game on acyclic orientations of G. Hence the number of these configurations is $t(G;1,0)$.

We introduce in the sequel, first, a cocycle reversing system with larger equivalent classes, equivalent to the edge firing game in the acyclic part. We introduce, secondly, a cycle reversing system, allowing to extend the dynamic by duality to all orientations, and we introduce, finally, a cycle-cocycle reversing system which is a combination of the two.

In every dynamical system that we will consider in a digraph : edge firing game, cycle, cocycle, cycle-cocycle reversing systems, sandpile model, our aim is to represent every equivalence class of configurations with a particular object associated with this class, and compare these representative objects and their enumeration between these systems. Such a representative of a class will be called *s-configuration* of the class, it may be an orientation (edge firing game, cocycle reversing system) or a V-vector (cycle reversing system, cycle-cocycle reversing system).

The *sandpile model* of a graph G and its s-configurations are also described in terms of V-vectors, this is done in Section 5 since this system is not directly used in the other sections.

3. Enumerations of indegree sequences

Given a graph G, it is known that the number of in-sequences of orientations is $t(G;2,1)$ (by [5][23]), of acyclic orientations is $t(G;2,0)$ (by [22] and bijection of Lemma 2.1), of acyclic orientations with unique given sink when G is connected is $t(G;1,0)$ (by [15] and bijection of Lemma 2.1). An example is given at the end of Section 4, illustrating also the enumerations of this Section.

Theorem 3.1. *Let G be a connected graph. The number of in-sequences of strongly connected orientations of G is $t(G;0,1)$.*

Proof. Recall that a digraph is strongly connected if and only if it is connected and every edge belongs to a directed cycle.

First, we prove that if there exists a in-sequence $(a_1, a_2, a_3..., a_m)$ of strongly connected orientation, then the set of in-sequences of strongly connected orientations, when $a_3, ..., a_m$ are fixed, is of the form $(a - j, a' + j, a_3..., a_m)$ for positive integers a, a', k and $0 \leq j \leq k$. Indeed on the one hand two orientations with same values for $v_3, ..., v_m$ must differ on a union of paths from v_1 to v_2, and on the other hand the reorientation of a path from v_1 to v_2 changes by one the values of v_1 and v_2, in positive and negative respectively, or the contrary, and does not affect the strong connectivity if we keep a path from v_1 to v_2 and a path from v_2 to v_1. Hence we get the property.

Secondly, let e be an edge of G. Up to relabeling we assume that v_1 and v_2 are the extremities of e. If e is an isthmus of G, then no orientation is strongly connected. If e is a loop of G, then it can be deleted and the number of in-sequences remains the same. Assume now that e is not a loop nor an isthmus.

If $G \setminus e$ has an isthmus f, then its number of strongly connected orientations equals zero, and there is a bijection between strongly connected orientations of G/e and G: the direction of e is opposite to the direction of f in the cut $\{e, f\}$.

Hence there exists a strongly connected orientation of $\vec{G}$ with in-sequence $(a_1, ..., a_m)$ if and only if there exists a strongly connected orientation of $\vec{G}/e$ with in-sequence $(a_1 + a_2, a_3, ..., a_m)$. Since $a_1 + a_2 = \#E - (a_3 + ... + a_m)$, the number of in-sequences of strongly connected orientations of G when $a_3, ..., a_m$ are fixed is the number of in-sequences of strongly connected orientations of G/e when $a_3, ..., a_m$ are fixed, it equals one or zero.

Assume now that $G \setminus e$ has no isthmus, and $(a_1, a_2, ...a_m)$ is a in-sequence of strongly connected orientation of $\vec{G} \setminus e$. On the one hand, the set of in-sequences of strongly connected orientations of $G \setminus e$ when $a_3, ..., a_m$ are fixed is of the form $(a-j, a'+j, a_3..., a_m)$. If $\vec{G} \setminus e$ is a strongly connected orientation of $G \setminus e$, then e can be directed in any way in G, and the orientation remains strongly connected. So the set of in-sequences of strongly connected orientations of G when $a_3, ..., a_m$ are fixed is of the form $(a + 1 - j, a' + j, a_3..., a_m)$. Then the number of in-sequences of strongly connected orientations of G when $a_3, ..., a_m$ are fixed is one plus the number of in-sequences of strongly connected orientations of $G \setminus e$ when $a_3, ..., a_m$ are fixed. A strongly connected orientation $\vec{G}$ of G with in-sequence of the above form induces a strongly connected orientation $\vec{G}/e$ of G/e with unique possible in-sequence $(a_1 + a_2, a_3, ..., a_m)$.

Thus, finally, in both cases, the numerical relations satisfie the inductive definition of the Tutte polynomial, hence the result. □

Corollary 3.2. *The number of in-sequences of totally cyclic orientations of a graph G is $t(G; 0, 1)$.*

Proof. A digraph is totally cyclic if and only if the connected components of the underlying graph are strongly connected. Its Tutte polynomial equals the product of the Tutte polynomial of its connected components. □

Lemma 3.3. *An orientation with in-sequence $(a_1, ..., a_m)$ has a directed cocycle C from A towards B where $V = A \uplus B$ is the partition induced by C if and only if the sum of values a_i of vertices in A equals the number of edges in the subgraph induced on A.* □

With the previous lemma, the fact that an orientation has a unique quasi-sink, and the fact that this quasi-sink contains a given vertex s, can be read on its in-sequence : we can calculate the acyclic part with the in-sequence, and the directions of edges in this acyclic part (lemma 2.1), then we know if the contraction of the cyclic part has a unique sink. Hence we define a *quasi-sink* of an indegree sequence as a *quasi-sink* of an associated orientation.

Theorem 3.4. *Let G be a connected graph. The number of in-sequences of orientations of G having a unique quasi-sink at a given vertex is $t(G; 1, 1)$.*

Proof. By definition, the orientations $\vec{G}$ of G having a unique quasi-sink at s are the orientations such that, F being the cyclic part of $\vec{G}$, $\vec{G}/F$ has a unique sink equal to the vertex of G/F induced by s. As noted above, with Lemma 3.3, the in-sequence of $\vec{G}$ determines its cyclic part. As mentioned at the beginning of the section, it is known that the number of in-sequences of acyclic orientations of G/F with unique given sink is $t(G/F; 1, 0)$. Besides, by Corollary 3.2, the number of in-sequences of $G(F)$ is $t(G(F); 0, 1)$. Hence, using the duality decomposition of the set of orientations, the number of in-sequences of orientations of G having a unique quasi-sink at a given vertex equals $\sum_{F \text{ cyclic flat of } G} t(G/F; 1, 0)\, t(G(F); 0, 1)$. Hence it equals $t(G; 1, 1)$ using the convolution formula for the Tutte polynomial. □

Remark. The enumeration of in-sequences for all orientations with $t(G; 2, 1)$ (from [23]) could be deduced from the enumeration for acyclic orientations with $t(G; 2, 0)$ ([22] and Lemma 2.1) and the enumeration for totally cyclic orientations with $t(G; 0, 1)$ (corollary 3.2), exactly the same way as in Theorem 3.4. All these enumerations may also be seen as corollaries of enumerations of [12] in regular matroids together with bijections of Section 4.

4. A cycle-cocycle reversing system

In this Section, $G = (V, E)$ is always a connected graph. We introduce successively a cocycle reversing system, a cycle reversing system and a cycle-cocycle reversing system on G. We enumerate their configurations (these enumerations may be seen also as corollaries of [12] in regular matroids), and relate them to indegree sequences. We end with a remark on the planar case, a remark on the generalization oriented matroids, a detailed example and a sum up of the bijections and enumerations involved in the paper.

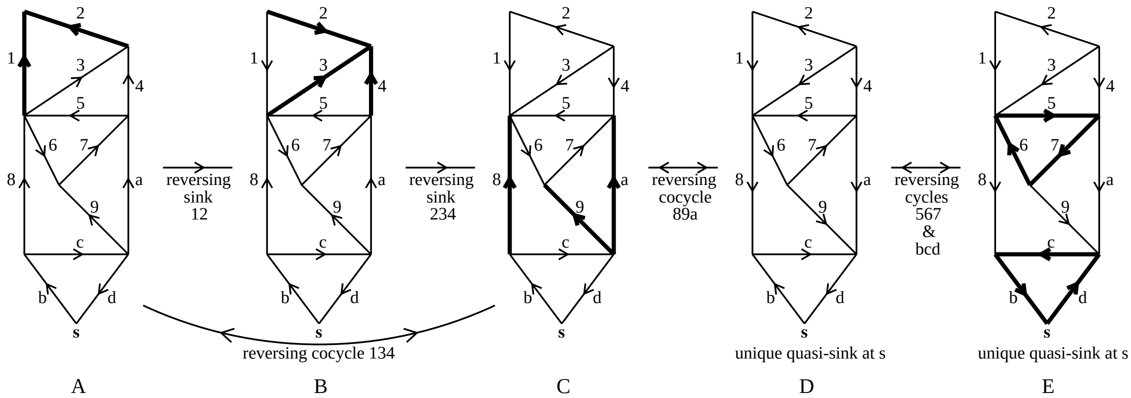

Figure 1. Reversing cycles and cocycles

The *cocycle reversing system* on G is a dynamical system on G with set of configurations the orientations of G, and if C is a directed cocycle of $\vec{G}$, then $-_C\vec{G}$ is equivalent with $\vec{G}$. The transitive closure of this relation is an equivalence relation, defining equivalence classes for the cocycle reversing system. For example, in Figure 1, the directed graphs A, B, C and D are equivalent.

Lemma 4.1. *Given an acyclic digraph $\vec{G} = (V, E)$, there exists a sequence of vertices $v_{i_1}, ..., v_{i_m}$ with $\{v_{i_1}, ..., v_{i_m}\} = V$ and a sequence of digraphs $G = G_0, G_1, ..., G_{m-1}, G_m = G$ such that v_{i_k} is a sink of G_{k-1} and G_k is obtained from G_{k-1} by reorienting edges incoming to v_{i_k}.*

Proof. Easy by induction: the first sink is reoriented and deleted, then there exists a sequence in the subgraph following the first reorientation. Of course, this procedure comes to reorient every edge twice, and so the final configuration equals the first one. □

Proposition 4.2. *The equivalence classes of acyclic orientations for the edge firing game equal the equivalence classes of acyclic orientations for the cocycle reversing system.*

Proof. If C is a directed cocycle of $\vec{G}$, and $\vec{G}$ is acyclic, then let $V = A \uplus B$ be the partition defined by the cut $C \subseteq E$, with the edges in C directed from A towards B. The subgraph of G induced on B is acyclic, then it has a sink. By Lemma 4.1, there exists a sequence of reorientations of sinks in this subgraph. Since C is directed towards B, sinks in the subgraph with set of vertices B remain sinks in G. Every edge in this subgraph has been reoriented twice. Hence the sequence of reorientations in G comes to the reorientation of C. □

For example, in Figure 1, the digraphs A, B and C, restricted to edges 12345, are acyclic and equivalent for the edge firing game and the cocycle reversing system. Reversing 12 and 234 is the same as reversing cocycle 134.

Lemma 4.3.1 *Let $s, v \in V$. Let k be the minimal number of edges in a path (undirected) from s to v in G. Let $\vec{G}_1 \to ... \to \vec{G}_j$ be a sequence of orientations of G with sink s and such that, for $1 \leq i \leq j-1$, $\vec{G}_{i+1}$ is obtained by reorienting a sink v_i of $\vec{G}_i$. The number of digraphs G_i in the sequence for which $v = v_i$ is at most $k-1$.*

Proof. We prove this lemma by induction on k. If $k = 1$ the result is obvious since s and v cannot be sinks at the same time. Assume now $k > 1$. Let $w \in V$ be a neigbourg of v at distance $k-1$ from s. Assume i and i', $1 \leq i < i' \leq j$, are such that $v = v_i$ and $v = v_{i'}$. The edge (v, w) is directed from v to w in $\vec{G}_{i+1}$, and from w to v in $\vec{G}_{i'}$. So there exists i'', with $i < i'' < i'$, such that $w = v_{i''}$. By induction, there exist at most $k-2$ possible values for i'', hence there exist at most $k-1$ possible values for i and i'. □

Lemma 4.3.2 *If two orientations $\vec{G}$ and $\vec{G}'$ are equivalent for the cocycle reversing system, then either $\vec{G} = \vec{G}'$, or there exists a directed cocycle of $\vec{G}$ which is reversed between $\vec{G}$ and $\vec{G}'$.*

Proof. With Proposition 4.2, there exists a sequence of orientations $\vec{G} = \vec{G}_1 \to \ldots \to \vec{G}_j = \vec{G}'$ such that, for $1 \leq i \leq j-1$, $\vec{G}_{i+1}$ is obtained by reorienting a sink v_i of $\vec{G}_i$. We prove that for every i, $1 \leq i \leq j$, we have either $\vec{G}_i = \vec{G}$ or there exists a directed cocycle C of $\vec{G}$ which is reversed between $\vec{G}_i$ and $\vec{G}$. This is true for $i = 1$. Assume it is true for some i, if $\vec{G} = \vec{G}_i$ then the set of edges incident to v_i is a directed cocycle of $\vec{G}$ reversed between $\vec{G}$ and $\vec{G}_{i+1}$. Assume now $\vec{G} \neq \vec{G}_i$ then let C be a directed cocycle of $\vec{G}$ reversed between $\vec{G}_i$ and $\vec{G}$. Let D be the set of edges in C incident to v_i, and D' the set of edges not in C incident to v_i. If $D = \emptyset$, then C remains reversed between $\vec{G}_{i+1}$ and $\vec{G}$. Otherwise, $C \setminus D \cup D'$ is a cocycle of G, reversed between $\vec{G}_{i+1}$ and $\vec{G}$. Hence the property is proved for all i, in particular $i = j$. □

Proposition 4.3. *There exists one and only one acyclic orientation with unique given sink in every equivalence class of acyclic orientations for the cocycle reversing system.*

Proof. Let $s \in V$, chosen to be the unique sink. First, we prove the existence. Consider an equivalence class of acyclic orientations for the cocycle reversing system. With Lemma 4.1, we immediately get that there exists an orientation $\vec{G}$ in the class such that s is a sink of $\vec{G}$. Let S be the set of orientations of G belonging to the equivalence class of $\vec{G}$ for which s is a sink. If every orientation in S had a sink different from s, then this sink should not be a neighbourg of the sink s, and it would be possible to build a sequence of orientations in S with arbitrary length, with the property that every orientation is obtained by reorientation of a sink different from s in the previous one. But this would be a contradiction with Lemma 4.3.1. Hence S contains an orientation with unique sink s. Secondly, we prove the unicity. Assume two orientations $\vec{G}$ and $\vec{G}'$ with unique sink s are equivalent for the cocycle reversing system. With Lemma 4.3.2, either $\vec{G} = \vec{G}'$ or there exists a directed cocycle C of $\vec{G}$ reversed between $\vec{G}$ and $\vec{G}'$. If $\vec{G} \neq \vec{G}'$, let $V = A \uplus B$ be the partition associated with C. Since $\vec{G}$ and $\vec{G}'$ have a sink s, and C is a directed cocycle reversed between $\vec{G}$ and $\vec{G}'$, the digraph $\vec{G}'$ must have a sink in both sets of vertices A or B, which is a contradiction with $\vec{G}'$ has a unique sink. Hence $\vec{G} = \vec{G}'$. □

Remark. The result of Proposition 4.3 is not really new, but the author did not find such a direct proof in previous works. It may be seen for instance as a consequence of a classical property of sandpile model configurations at level 0, see [4].

Corollary 4.4. *The number of equivalence classes of acyclic orientations for the cocycle reversing system of a graph G is $t(G; 1, 0)$.* □

Note that for non-acyclic orientations, equivalence classes of the cocycle reversing system are larger than equivalence classes of the edge firing game. For instance, if a directed cocycle C defines a cut $V = A \uplus B$, and if the subgraphs induced on A and B are strongly connected, then the directed cocycle C can be reoriented in the cocyle reversing system, but not in the edge firing game. In fact, in the cocycle reversing system, it is like if one can reorient at the same time several vertices. More precisely, we have the following proposition.

Lemma 4.5. *If C is a directed cycle or a directed cocycle of $\vec{G}$, then $\vec{G}$ and $-_C\vec{G}$ have same cyclic part and same acyclic part.*

Proof. Let C be a directed cycle and C' be a directed cocyle of $\vec{G}$. Then $C \cap C' = \emptyset$. Hence C is a cycle in a connected component of $G \setminus C'$. Thus C' is a directed cocycle of $-_C\vec{G}$, and C is a directed cycle of $-_{C'}\vec{G}$. Hence the acyclic part of $\vec{G}$ is included in the acyclic part of $-_C\vec{G}$, and the cyclic part of $-_{C'}\vec{G}$ is included in the cyclic part of $\vec{G}$. By symmetry, the three graphs $\vec{G}$, $-_C\vec{G}$ and $-_{C'}\vec{G}$ have same cyclic and acyclic parts. □

Proposition 4.6. *Let $\vec{G}$ be an orientation of G with cyclic part F. All configurations for the cocycle reversing system equivalent to $\vec{G}$ have same cyclic part F. The equivalence classes of $\vec{G}$ and $\vec{G}/F$ for the cocycle reversing system, and the equivalence classes of $\vec{G}/F$ for the edge firing game are isomorphic.*

Proof. First, edges in a directed cycle cannot be reoriented in the cocycle reversing system, thus the cyclic part is invariant (cf. Lemma 4.5). Moreover if C is a directed cycle of $\vec{G}$, and A the set of vertices of C, then the cut $V = A \uplus (V \setminus A)$ correspond to a vertex of $\vec{G}/F$, and so the rule is the same applied in $\vec{G}$ or in $\vec{G}/F$. At last, $\vec{G}/F$ is acyclic, so for this graph edge firing game and cocycle reversing system give the same structure to the configurations by Proposition 4.2. □

Proposition 4.7. *Let s be a vertex of G. In each equivalence class of the cocycle reversing system, there exists a unique orientation with unique quasi-sink at s.*

Proof. The directions of the edges belonging to F are fixed, then using Proposition 4.6 the unique orientation of G/F with unique sink s gives a unique orientation of G with unique cocycle-sink s. □

For example, in Figure 1, for the equivalent digraphs A, B, C and D, we have $F = \{5, 6, 7, b, c, d\}$. The digraph D is the unique in its class with unique quasi-sink at s. And when F is contracted, D is the unique in its class with a unique sink s.

Corollary 4.8. *The number of equivalence classes of the cocycle reversing system of G is $t(G; 1, 2)$.*

Proof. On the one hand, the rule of the system does not affect the cyclic part of the digraph (cf. Lemma 4.5). On the other hand, an edge belonging to the cyclic part is never reoriented when the rule is applied. Hence two configurations with same cyclic part F such that the restriction to F are not the same do not belong to the same equivalent class. The number of totally cyclic orientations of G equals $t(G; 0, 2)$ (cf. Preliminaries), and the number of equivalence classes of G/F equals $t(G; 1, 0)$ Hence the number of equivalence classes equals $\sum_{F \text{ cyclic flat of } G} t(G/F; 1, 0)\ t(G(F); 0, 2) = t(G; 1, 2)$ (convolution formula). □

Finally, when a vertex s is chosen to be the *sink* of the system, the orientations of G with unique quasi-sink at s are in bijection with equivalence classes of configurations. They are called *s-configurations* of the cocycle reversing system on G.

Dually, we introduce the *cycle reversing system* on G. Configurations are still the orientations of G, but the evolution rule is dual and defined by: if C is a directed cycle of $\vec{G}$, then $-_C\vec{G}$ is equivalent with $\vec{G}$. The transitive closure of this relation is an equivalence relation, defining equivalence classes for the cycle reversing system. For example, in Figure 1, the digraphs D and E are equivalent.

Note that the definition of the cocycle reversing system and the cycle reversing system can be readily extended to oriented matroids, since there is no choice of a particular sink in these definitions. In this context, the cocyle reversing system for an oriented matroid M is exactly the cycle reversing system in the dual oriented matroid M^*, see last remarks or [12].

Lemma 4.9. *Reorienting a cycle in a digraph does not change its indegree sequence.* □

Proposition 4.10. *Two orientations of a graph are equivalent for the cycle reversing system if and only if they have same indegree sequence.*

Proof. If two orientations are equivalent then they can be joined, by definition, by a sequence of reorientatons of cycles, hence they have same in-sequence with Lemma 4.9. Conversely, let $\vec{G}$ and

$\vec{G}'$ be two orientations with same in-sequence. Suppose there exists an edge e directed in opposite directions in these two digraphs. Assume e is directed from v_1 to v_2 in $\vec{G}$. Since the in-sequence is the same, there exist an edge (v_2, v_3) directed from v_2 towards v_3 in $\vec{G}$ which has the opposite direction in $\vec{G}'$. Then inductively we get a sequence of vertices containing a cycle C of which edges have opposite directions in the two digraphs. Then the digraph $-_C\vec{G}$ is equivalent to $\vec{G}$ and has a strictly smaller number of edges directed differently from $\vec{G}'$. Thus we get the result by induction. □

Corollary 4.11. *The number of equivalence classes of strongly connected orientations of a connected graph G for the cycle reversing system is $t(G;0,1)$. The number of equivalence classes of orientations of a connected graph G for the cycle reversing system is $t(G;2,1)$.*

Proof. This is a direct consequence of Proposition 4.10 and Theorems 3.1 and 3.4. □

Finally, we have a bijection between equivalence classes of configurations for the cycle reversing system of G and indegree sequences of orientations of G. These sequences are called *s-configurations* of the cycle reversing system on G.

Note that on the contrary with edge firing games or cocycle reversing systems, these s-configurations, that represent equivalence classes, are not orientations of the graph but V-vectors. However in the planar case the cycle reversing system can be described as the cocycle reversing system of the dual, and then these s-configurations can be described as particular orientations of the dual graph. A statement generalizing this description is not clear in general: see remark on planar graphs at the end of this Section, and questions on sandpile model in next Section.

At last, we define a *cycle-cocycle reversing system* on G, of which configurations are the orientations of G with the folowing rule: if C is a directed cycle or a directed cocycle of $\vec{G}$, then $-_C\vec{G}$ is equivalent with $\vec{G}$. The transitive closure of this relation is an equivalence relation, defining equivalence classes for the cycle-cocycle reversing system. For example, in Figure 1, all digraphs are equivalent.

Proposition 4.12. *Let s be a vertex of G. Two orientations with unique quasi-sink at s are equivalent for the cycle-cocycle reversing system if and only if they have same indegree sequence.*

Proof. The evolution rule does not change the cyclic part of the digraph, hence the directions of the edges in the contraction G/F are determined by the in-sequence and the fact that this acyclic digraph has a unique sink induced by s. Then the directions in the strongly connected graph $G(F)$ are determined by the in-sequence up to cycle reorientations thanks to Lemma 4.10. □

Corollary 4.13. *The number of equivalence classes for the cycle-cocycle reversing system of G is $t(G;1,1)$.*

Proof. This is a direct consequence of the bijection below and Theorem 3.4. □

Thus, finally, we have a bijection between equivalence classes of the cycle-cocycle reversing system and indegree sequences of G with a unique quasi-sink at a given vertex s. These in-sequences are called *s-configurations* for the cycle-cocycle reversing system with quasi-sink at s.

Planar case. When the graph is planar, it has a dual G^*. As a particularization of the well known duality property of the Tutte polynomial in matroids [5], we get that $t(G;x,y) = t(G^*;y,x)$. Strongly connected orientations of G, resp. G^*, are acyclic orientations of G^*, resp. G. Then the cycle reversing systems and cocycle reversing systems on G and G^* are dual, and the cycle-cocycle reversing system on G is isomorphic up to duality to the cycle-cocycle reversing system on G^*. Thus, moreover, for any given vertex of G and any given vertex of G^* (particular cycle of G), we get a bijection between in-sequences of G and in-sequences of G^* with unique quasi-sinks at these respective vertices.

Generalization to oriented matroids. The reader is refered to [3] for necessary background on oriented matroids, and to [12] for the detailed constructions and results described below. In an oriented matroid M on E (or in particular an hyperplane arrangement), the oriented matroid obtained by *reorientation of M with respect to a subset $A \subseteq E$* is denoted $-_A M$. By extension, we call *reorientation* a subset $A \subseteq E$, of which associated oriented matroid is $-_A M$. These reorientations correspond to signatures of the arrangement associated with M. This is done to preserve a number of reorientations equal to $2^{|E|}$, similarly with graphs. But of course, structurally, the oriented matroids $-_A M$ and $-_{E\setminus A} M$ are in fact equal. We define a *circuit reversing*, resp. a *cocircuit reversing*, resp. a *circuit-cocircuit reversing system*, of which configurations are the reorientations of M, generalizing the previous systems in digraphs. Indeed a (positive) circuit, resp. cocircuit, of a matroid associated with a graph is an elementary (directed) cycle, resp. cocycle. The following rule is used: if C is a positive circuit, resp. a positive cocircuit, resp. a positive circuit or a positive cocircuit, of $-_A M$, then the reorientation $C \triangle A$ (associated with $-_C(-_A M)$) is equivalent with the reorientation A (associated with $-_A M$). The transitive closure of this relation is an equivalence relation, defining equivalence classes for the system. Note that the circuit reversing system for M is isomorphic to the cocircuit reversing system on M^*, and then the circuit-cocircuit reversing system of M and M^* are isomorphic. However in oriented matroids there is not a notion of vertex as in graphs, and so there is not a notion of in-sequence. It is easy to check that the enumerative results on equivalence classes do not generalize to any oriented matroid. For instance a uniform oriented matroid $U_{2,2k}$, resp. $U_{2,2k+1}$, has exactly one, resp. two, equivalence classes of acyclic reorientations, for any $k > 0$. Nevertheless, enumerative results of Section 4 all extend to regular matroids, which are binary orientable matroids, and form the smaller usual class of oriented matroids containing graphs and self-dual.

Example. Figures 2, 3, 4 and 5 show all orientations of a same small graph and their in-sequences. We chose that *the directions of edges are represented with grains put at the outgoing extremities*, in order to bring out visually in-sequences and by analogy with level 0 configurations of the sandpile model. The in-sequences are written with respect to the ordering (s, a, b, c). Configurations equivalent for the cycle-cocycle reversing system are on a same line. The various Figures correspond to various cyclic parts F. Here, for each F, we have the same useful quasi-sink S for the different configurations associated with F (it is not the case in general). The Tutte polynomial of this graph is

$$t(G; x, y) = x^3 + 2x^2 + x + 2xy + y + y^2.$$

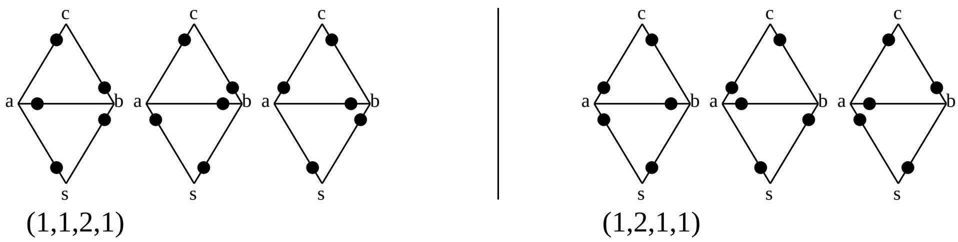

Figure 2. $F = E$, $S = V$ **(strongly connected orientations)**

The number of orientations equals $t(G; 2, 2) = 2^5 = 32$. The number of in-sequences, that is the number of equivalence classes for the cycle reversing system, equals $t(G; 2, 1) = 24$ (there are 2 in Figure 2, 2 in Figure 3 and Figure 4, and 18 in Figure 5).

Figure 2 shows the equivalence classes of strongly connected orientations. For these orientations, the cyclic part is the set of edges E, and the unique quasi-sink is V. The number of strongly connected orientations is $t(G; 0, 2) = 6$. Two configurations on a same line have same in-sequence: they can be deduced by a sequence of reorientations of cycles. According to Theorem 3.4, the number of equivalence classes, or s-configurations, or in-sequences, of strongly connected orientations, is $t(G; 0, 1) = 2$.

Figure 3 and Figure 4 show respectively the orientations with cyclic part $F = \{ab, ac, bc\}$ and $F = \{sa, sb, ab\}$. Two configurations on a same line have same in-sequence if and only if they can be deduced by a sequence of reorientations of cycles. And the in-sequences are different if and only if a cocycle has been reoriented. The first orientation in Figure 3 has a unique quasi-sink $\{s\}$ which is in fact a (unique) sink (s does not belong to the cycle $\{ab, ac, bc\}$). The first orientation in Figure 4 has a unique quasi-sink $\{s, a, b\}$, but has no sink (s belongs to the cycle $\{sa, sb, ab\}$). In both cases, the unique quasi-sink contains s. Hence the first in-sequence of each line is the s-configuration of the equivalence class.

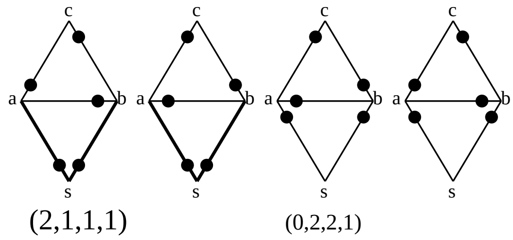

Figure 3. $F = \{ab, ac, bc\}$, $S = \{s\}$

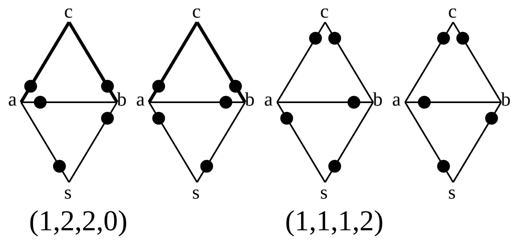

Figure 4. $F = \{sa, sb, ab\}$, $S = \{s, a, b\}$

Figure 5 illustrates the (trivial) bijection between acyclic orientations and their in-sequences, and the first elements of each line give the bijection between acyclic orientations with unique sink and their in-sequences. For these orientations, the cyclic part F is empty. The number of in-sequences of acyclic orientations equals the number of acyclic orientations $t(G; 2, 0) = 18$. The number of equivalence classes for acyclic orientations, or s-configurations, or in-sequences of acyclic orientations with unique given sink, is $t(G; 1, 0) = 4$. These configurations are exactly the same as for the edge firing game of acyclic orientations. Their in-sequences are sandpile configurations at level 0 (see Section 5).

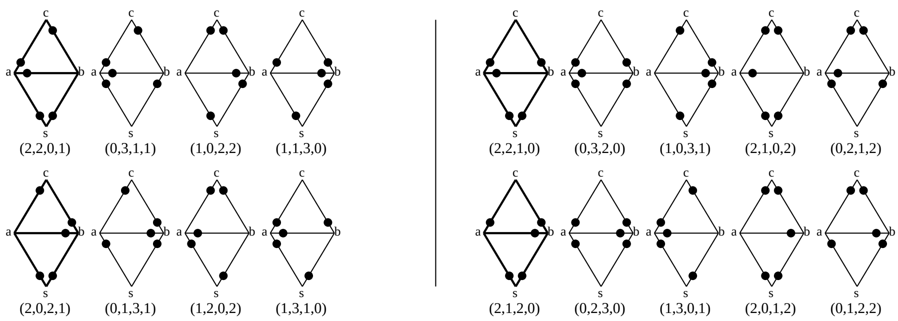

Figure 5. $F = \emptyset$, $S = \{s\}$ **(acyclic orientations)**

The number of equivalence classes for the cocycle reversing system is $t(G; 1, 2) = 14$. There are 6 in Figure 2 (all strongly connected orientations), 2 in Figure 3 and in Figure 4 (the two first of each line), and 4 in Figure 5 (the first of each line).

Finally, the number of equivalence classes, or s-configurations, or in-sequences of orientations with unique quasi-sink at s, is $t(G; 1, 1) = 8$. It is the number of lines in the four Figures. The first in-sequences, one for each line, is the set of s-configurations. Their number is the same as s-configurations for the sandpile model.

Sum up. The bijections, and the related evaluations of the Tutte polynomial, involved in the paper, are the following. Let s be a fixed vertex of G.

$t(G; 1, 0)$ = # classes of the (cycle-)cocycle reversing system of acyclic orientations
= # acyclic orientations with unique sink at s
= # in-sequences of acyclic orientations with unique sink at s

$t(G; 1, 2)$ = # classes of the cocycle reversing system
= # orientations with unique quasi-sink at s

$$
\begin{aligned}
t(G;0,1) &= \#\text{ classes of the cycle(-cocycle) reversing system of strongly connected orientations}\\
&= \#\text{ in-sequences of strongly connected orientations}\\
t(G;2,1) &= \#\text{ classes of the cycle reversing system}\\
&= \#\text{ in-sequences of all orientations}\\
t(G;1,1) &= \#\text{ classes of the cycle-cocycle reversing system}\\
&= \#\text{ in-sequences of orientations with unique quasi-sink at } s
\end{aligned}
$$

5. Open link with the sandpile model

Let $G = (V, E)$ be a connected graph with $V = (v_1, ..., v_m)$, and let d_i be the degree of v_i, $1 \leq i \leq m$. The sandpile model on a graph [2][4][9] can be defined as a dynamical system, of which configurations $(a_1, ..., a_m)$ are V-vectors on G, defined by the following rule: if a vertex v_i has a value a_i superior to its degree d_i, then its value is decreased by d_i and the value of each neighbourg is increased by the number of edges joining the two vertices (one if we forbid multiple edges). A vertex s is chosen to be the *sink* of the system, and a configuration is said to be *stable* if s is the only vertex of which value is superior to its degree. A configuration is a *stable recurent configuration* if when the rule is applied to the sink we necessarily get the same configuration after a sequence of successors. The set of configurations reachable from any stable recurent configuration is an *equivalence class of configurations*, and there exist one and only one stable recurent configuration in every equivalence class. We call *s-configurations* the stable recurent configurations for which the sink has value exactly its degree (this restriction is due to enumerative reasons, but in fact the value of the sink, superior to its degree has no importance). These configurations are the main object in the study of the sandpile model, they have an intriguing group structure [9] which has already remarkable properties, see for instance [8], or concerning duality, since we noticed that the cycle-cocycle reversing system of a planar graph is isomorphic to the one of its dual, we mention here that the sandpile group associated to a planar graph is also isomorphic to the one of its dual [7].

The *level* of a configuration $(a_1, ..., a_m)$ is $(\sum_{1 \leq i \leq m} a_i) - \#E$. The V-vectors which are s-configurationsy are characterized by a lattice structure, graded by the level. The s-configurations at level 0 are exactly in-sequences of acyclic orientations with unique sink s (this result is implicit for example in [6]). There exists a unique s-configuration at maximal level $(\sum_{1 \leq i \leq m} d_i - 1) - \#E + 1$, it is the V-vector having all values equal to the degree less one, except for the sink which has value equal to its degree (as for all s-configurations). An s-configuration is a V-vector superior to a configuration at level 0 and inferior to the maximal s-configuration with respect to the Gale ordering $((a_1, ..., a_m) \leq (b_1, ..., b_m)$ if $a_i \leq b_i$ for all $1 \leq i \leq m)$. The number of s-configurations of the sandpile model on G at level k is $\sum_i b_{i,k}$ and hence the number of s-configurations of the sandpile model is $t(G; 1, 1)$ [19].

So, on the one hand, s-configurations of acyclic orientations for the edge firing game or the cycle-cocycle reversing system on G are in canonical bijection, via the in-sequence, with s-configurations at level 0 of the sandpile model on G. Note moreover that the evolution rule is compatible with this bijection: configurations of the sandpile model at level 0 are in bijection with acyclic orientations and their in-sequences, and their equivalence classes are in bijection with equivalence classes for the edge firing game or the cycle-cocycle reversing system.

On the other hand, s-configurations of the sandpile model and of the cycle-cocycle reversing system are particular V-vectors, in same number. Hence the following problems raise.

Problems. *Find a bijection between in-sequences of orientations with unique quasi-sink at a given vertex and s-configurations of the sandpile model with given sink. If possible, find an interpretation of the sandpile model levels $k > 0$ in terms of in-sequences or in terms of orientations. If possible, find an interpretation of the algebraic law between configurations of the sandpile model in terms of in-sequences or orientations.*

Given a linear order on E, the number $\sum_i b_{i,k}$ is also the number of spanning trees with external activity k [24]. This equality leads to a bijection [6] between s-configurations of the sandpile model and spanning trees, transforming level into external activity. On the other hand, the same coefficients have an interpretation in terms of activities of orientations [18]. An activity preserving bijection between these spanning trees and activity classes of orientations is done in [13], providing with the previous result a bijection between s-configurations of the sandpile model and some classes of orientations tranforming the level into the orientation-activity. However this bijection is not - a priori - in question here, since these activity classes of orientations depend on a linear ordering, whereas the dynamical equivalence classes of orientations studied here do not.

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