Circuit-Cocircuit Reversing Systems in Regular Matroids

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Abstract. We consider that two orientations of a regular matroid are equivalent if one can be obtained from the other by successive reorientations of positive circuits and/or positive cocircuits. We study the inductive deletion-contraction structure of these equivalence classes in the set of orientations, and we enumerate these classes as evaluations of the Tutte polynomial. This generalizes results in digraphs from a previous paper.

Keywords. Oriented matroid, regular matroid, circuit reorientation, Tutte polynomial, duality.

1. Introduction

Three equivalence classes can be defined on reorientations of an oriented matroid. The first allows successive reorientations of positive circuits from a reorientation to another one, the second allows successive reorientations of positive cocircuits, and the third allows successive reorientations of both positive circuits and positive cocircuits. The first and the second one are dual to each other, and the third is self-dual.

These equivalence relations can be thought of as dynamical systems, where the states or configurations are the reorientations of the oriented matroid, and the evolution rule is given by reorientations with respect to allowable subsets. For this reason, and by analogy with [5], we call the three equivalence relations circuit, cocircuit, and circuit-cocircuit reversing systems. We point out that these systems extend the level zero of the chip firing game to oriented matroids. See [5] for further background.

For oriented matroids of directed graphs, we showed in [5] that the number of equivalence classes for these relations are evaluations of the Tutte polynomial. In this paper, we extend these enumerative results to regular matroids. We note that these results do not generalize to any oriented matroid (see Proposition 2).

Since the class of regular matroids is not much larger than the class of graphic and cographic matroids, we might mention that the extension is interesting because of the method used. For example, in [5], we use that acyclic orientations with unique given sink of a graph are in natural bijection with cocircuit reversing acyclic classes, and thus their number is the evaluation of the Tutte polynomial at $(1, 0)$ by a classical result [8]. But this result cannot be used in matroids, where we have to avoid the use of vertices. More specifically, in [5], we use degree sequences, whereas in the present paper we use matroid methods, namely, deletion and contraction, duality decompositions for reorientations and the Tutte polynomial, and the fact that regular matroids are binary and orientable.

2. Preliminaries

The reader is referred to [1] for necessary background on oriented matroids, to [12] on regular matroids, and to [2] on the Tutte polynomial.

Let M be an oriented matroid on E. The oriented matroid obtained by reorientation of M with respect to a subset $A \subseteq E$ is denoted $-_A M$. We may also note $-_A C$ for a signed subset C in which signs are changed for elements of A. Beware that throughout the paper, reorientations are considered at two levels: (1) the set of all possible reorientations of the fixed oriented matroid M, and (2) some reorientations with respect to subsets of E that transform a reorientation of M into another one. Precisely, we call *reorientation of* M any subset $A \subseteq E$, and we consider that a reorientation A of M is associated with $-_A M$. Then, the *reorientation of the reorientation* $A \subseteq E$ *with respect to* $C \subseteq E$ is the reorientation $A \triangle C$. Reorientations of M correspond to signatures of the arrangement associated with M (or to orientations of a graph representing M if it is graphic). This is done to preserve a number of reorientations equal to $2^{|E|}$, similarly with graphs, even if the oriented matroids $-_A M$ and $-_{E \setminus A} M$ are in fact equal. Terminology of oriented matroids is naturally induced on reorientations, for example a reorientation A of M is called acyclic if $-_A M$ is acyclic. As usual, the acyclic reorientations are in canonical bijection with the regions of an arrangement representing the oriented matroid, and thus we may sometimes use geometrical vocabulary.

For an oriented matroid M, the *cyclic part* F of M is the union of positive circuits of M. Then $E \setminus F$ is the union of positive cocircuits, called the *acyclic part* of M, and then M/F is acyclic and $M(F)$ is totally cyclic. This fundamental property for oriented matroid is well-known, in graph theory (Minty lemma [11]) and in linear programming (Farkás lemma [1]). Structurally, it can be seen as a decomposition of the set of reorientations of M into the following disjoint union:

$$2^E = \biguplus_{F \text{ cyclic flat of } M} \Big\{ A' \uplus A'' \mid A' \subseteq F,\ -_{A'} M(F) \text{ totally cyclic},\ A'' \subseteq E \setminus F,\ -_{A''} M/F \text{ acyclic} \Big\}$$

where a *cyclic flat* of M is both a union of circuits and the complementary of a union of cocircuits. See also [6][7] for refinements of such decompositions.

The Tutte polynomial of the matroid M is denoted $t(M;x,y)$. Among its several equivalent definitions, we will use in proofs its famous inductive one [2][14]. A useful numerical extension of the previous duality decomposition is the following formula for the Tutte polynomial. It is implicit in [4] through an explicit bijection, and called "convolution formula" in [9] (see also [3][7] for extensions):

$$t(M;x,y) = \sum_{F \text{ cyclic flat of } M} t(M/F;x,0)\ t(M(F);0,y).$$

It is known that the number of acyclic reorientations of M is $t(M;2,0)$ [8][10][13][15]. Dually, the number of totally cyclic reorientations is $t(M;0,2)$. The reader may check with the previous decomposition and the above formula that the number of reorientations of M is effectively $t(M;2,2) = 2^{|E|}$.

The class of regular matroids contains graphs and is self-dual. One might think of it as the smallest class having both this property and nice constructions. Among several equivalent definitions, regular matroids are binary orientable matroids. Binary matroids (representable over the binary field) are characterized by the fact that $U_{2,4}$ is an excluded minor. In particular, this implies that two cocircuits are modular if and only if their symmetric difference is a cocircuit. Binary matroids are also characterized by the property that a symmetric difference of a set of cocircuits is a disjoint union of cocircuits. We will mainly use these two last properties, together with the oriented matroid structure.

The *cocircuit graph* of an oriented matroid M is the graph of which vertices are positive cocircuits of M and edges are pairs of modular cocircuits. This cocircuit graph is connected, and, moreover, the subgraph of the cocircuit graph of M formed by cocircuits containing an element e is also connected. In the realizable case for instance, the cocircuit graph is the skeleton of a polytope and the set of cocircuits not containing e is the set of vertices of a complementary of a face in this polytope.

Finally, except when it is ambiguous, we use the same notation for a cocircuit of M (signed subset) and the corresponding cocircuit (subset) in the underlying matroid.

3. Main section

We define the *circuit reversing system*, the *cocircuit reversing system*, and the *circuit-cocircuit reversing system* of an oriented matroid M, by analogy with the terminology used in [5], as equivalence relations on the set of reorientations of M. If C is, respectively, a positive circuit, a positive cocircuit, or a positive circuit or cocircuit, of $-_A M$, then the reorientation $C \triangle A$, obtained by reorientation of

A with respect to C, is said to be equivalent with the reorientation A. In each case, the transitive closure of the relation is an equivalence relation on the set of reorientations. When two reorientations A and B are equivalent for the circuit-cocircuit reversing system of M, we denote $A \equiv_M B$.

Note that the circuit reversing system of M is isomorphic to the cocircuit reversing system of M^*, and then the circuit-cocircuit reversing system of M and M^* are isomorphic. Geometrically, the reorientation of a region with respect to a positive cocircuit comes to cross a 0-dimensional face of the region and take the opposite region. Note that two complementary reorientations, corresponding to two opposite regions in the acyclic case, are not necessarily in the same class, even if the associated oriented matroids are the same. Note also that the partitions of the set of reorientations of M into classes are invariant under reorientation of M with respect to $A \subseteq E$, up to symmetric difference with A. At last, observe that reorienting successively some circuits or cocircuits is the same as reorienting at one time their symmetric difference.

Proposition 1. *Let M be an oriented matroid on E, and $A, B \subseteq E$. If $A \equiv_M B$ then $-_AM$ and $-_BM$ have the same cyclic part F.*

Proof. Let $B = A \triangle C$, for C positive circuit of $-_AM$ and let F be the cyclic part of $-_AM$. By definition of F, we have $C \subseteq F$, and there exists a positive maximal vector of $-_AM$ of which support is F. It is generated by conformal composition of some positive circuits $C_1, ..., C_k$. Then the signed subset $(-C) \circ C_1 \circ ... \circ C_k = (-C) \circ -_CC_1 \circ ... \circ -_CC_k$ is a maximal positive vector of $-_{C \triangle A}M$. Hence A and B have same cyclic part. The result is immediately deduced by duality and transitivity. □

From Proposition 1, we get that $A \equiv_M B$ if and only if $A \cap F \equiv_{M(F)} B \cap F$ and $A \setminus F \equiv_{M/F} B \setminus F$ for some F with $-_AM(F)$ totally cyclic and $-_AM/F$ acyclic. In other words, using the duality decomposition of the set of reorientations, the circuit-cocircuit reversing system of M can be completely studied through the circuit reversing systems of totally cyclic reorientations of the minors $M(F)$ and the cocircuit reversing systems of acyclic reorientations of the minors M/F, for all cyclic flats F.

We shall call equivalence classes for acyclic reorientations the equivalence classes for the cocircuit or equally for the circuit-cocircuit reversing system for these reorientations. All results on these reorientations can easily be translated to totally cyclic reorientations by duality. Except in the final theorem, we consider in the sequel only acyclic reorientations

Proposition 2. *A uniform oriented matroid $U_{2,2k}$, resp. $U_{2,2k+1}$, has exactly one, resp. two, equivalence classes of acyclic reorientations, for any $k > 0$.* □

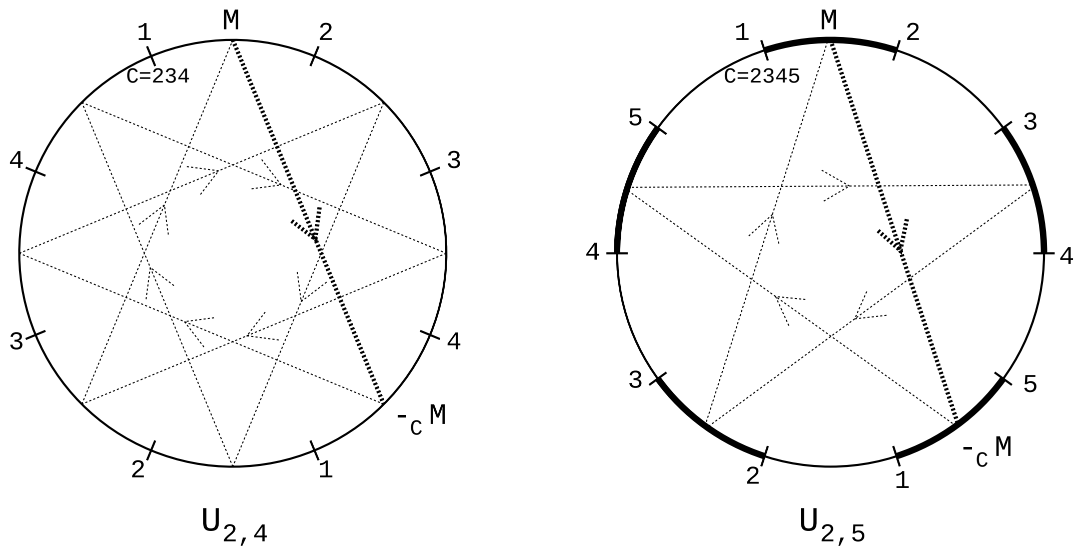

Figure 1. Acyclic classes of $U_{2,2k}$ and $U_{2,2k+1}$

Figure 1 illustrates Proposition 2 for $k = 2$. In each case, a class is built by successive reorientations with respect to cocircuits represented with arrows. The proof of this proposition is left as an easy exercise. It shows that the enumerative results of [5], or of the final theorem, do not extend to general oriented matroids.

We now focus on regular acyclic oriented matroids. We describe essentially the relations between the classes in M, $M \setminus e$, and M/e (notably in order to prove the first assertion in the last enumerative theorem by means of the inductive definition of the Tutte polynomial).

We say that an oriented matroid M on E is *bordered by* $e \in E$ if both M and $-_eM$ are acyclic. Equivalently, M/e is acyclic. Geometrically, e is a frontier separating the regions M and $-_eM$. Observe that if M is bordered by e and C is a positive cocircuit of M with $e \notin M$, then C is a cocircuit of M/e, $-_CM/e$ is acyclic, $\emptyset \equiv_M C$ and $\emptyset \equiv_{M/e} C$.

Let there be a regular oriented matroid M on E, and let $e \in E$ which is not isthmus nor loop.

Lemma 3. *If M is bordered by e then $\emptyset \not\equiv_M e$.*

Proof. If $\emptyset$ and e are in the same acyclic class then e is a symmetric difference of cocircuits. Hence e is a disjoint union of cocircuits, since M is binary. Hence e is a cocircuit, and so e is an isthmus. □

Lemma 4. *If C is a positive cocircuit of M then $\emptyset \equiv_{M\setminus e} C \setminus e$.*

Proof. If $e \in C$ then $C \setminus e$ is a positive cocircuit of $M \setminus e$, and $\emptyset \equiv_{M\setminus e} C \setminus e$ by definition of equivalence classes. Assume now $e \notin C$. If C is a cocircuit of $M \setminus e$, then again $\emptyset \equiv_{M\setminus e} C$. If C is not a cocircuit of $M \setminus e$, then C is obtained in M by modular elimination of e between D and D', with $C = D\triangle D'$ and $D \cap D' = \{e\}$ since M is binary. Hence $D \setminus e$ and $D' \setminus e$ are disjoint positive cocircuits of $M \setminus e$. So $\emptyset \equiv_{M\setminus e} (D \setminus e)\triangle(D' \setminus e) \equiv_{M\setminus e} C$. □

Lemma 5. *If M is bordered by e, C is a positive cocircuit of M with $e \in C$, and $-_CM$ is not bordered by e, then for every positive cocircuit D of $-_CM$ containing e, we have $-_{D\triangle C}M$ bordered by e and $\emptyset \equiv_{M/e} D\triangle C$.*

Proof. In M, D is positive on $D \setminus C$ and negative on $D \cap C$. If C and D are modular, then $C\triangle D$ is a positive cocircuit of M not containing e, hence $\emptyset \equiv_M D\triangle C$, and it is a positive cocircuit of M/e, then $\emptyset \equiv_{M/e} D\triangle C$, and of course $-_{D\triangle C}M/e$ is acyclic. The subgraph of the cocircuit graph of $-_CM$ formed by cocircuits containing e is connected. Hence, by connectivity of this subgraph and by transitivity of the equivalence relation, we get $\emptyset \equiv_{M/e} D\triangle C$ for every positive cocircuit D of $-_CM$ containing e. □

Lemma 6. *If M is bordered by e, C is a positive cocircuit of M with $e \in C$, and $-_CM$ is bordered by e then for every $A \subseteq E \setminus e$ with $\emptyset \equiv_{M/e} A$ and for every positive cocircuit D of $-_AM$ with $e \in D$, we have $-_D -_A M$ bordered by e and $C \setminus e \equiv_{M/e} A\triangle(D \setminus e)$.*

Proof. First we prove the result for $A = \emptyset$. Observe that $-_CM$ is bordered by e implies $-_{C\setminus e}M/e$ is acyclic. Let D be a positive cocircuit of M with $e \in D$ modular with C. Then $C\triangle D$ is a positive cocircuit of both $-_CM$ and $-_DM$ not containing e. Since $-_{C\triangle e}M$ acyclic, we also have $-_{D\triangle e}M$ acyclic. And $C\triangle D$ is a positive cocircuit of both $-_{C\setminus e}M/e$ and $-_{D\setminus e}M/e$. So $C \setminus e \equiv_{M/e} D \setminus e$. The subgraph of the cocircuit graph of M formed by cocircuits containing e is connected. Hence for every positive cocircuit D of M with $e \in C$, we have by transitivity $C \setminus e \equiv_{M/e} D \setminus e$.

Let C' be a positive cocircuit of M with $e \notin C'$. We prove the result for $A = C'$. Observe that C' is also a positive cocircuit of M/e and that $-_{C'}M$ is bordered by e. For the same property of the cocircuit graph as above, there exists a positive cocircuit D' in M modular with C' and such that $e \in D'$. Using the above result we already have $C \setminus e \equiv_{M/e} D' \setminus e$. Moreover $C'\triangle D'$ is a positive cocircuit of $-_{C'}M$ containing e, and $-_{C'\triangle D'} -_{C'} M = -_{D'}M$ is bordered by e. Using the above result this time in $-_{C'}M$, for every D positive cocircuit of $-_{C'}M$ with $e \in D$, we get that $-_D -_{C'} M$ is bordered by e and $(C'\triangle D') \setminus e \equiv_{-_{C'}M/e} (D \setminus e)$. That is $D' \setminus e \equiv_{M/e} C'\triangle(D \setminus e)$.

At last, we get the result for every A such that $\emptyset \equiv_{M/e} A$ by transitivity in the equivalence relation. $\square$

For $A \subseteq E$, we denote $\phi(A) = A \setminus e$, and for $\mathcal{A} \subseteq 2^E$ we denote $\phi(\mathcal{A}) = \{\phi(A) \mid A \in \mathcal{A}\}$. We denote $\mathcal{F}(M)$ the set of acyclic classes, or briefly classes, of M and $\mathcal{R}(M)$ the set of acyclic reorientations of M.

Let $\mathcal{F}_0$ be the set of classes of M such that for any A of which class is in $\mathcal{F}_0$, $-_AM$ is not bordered by e. Let $\mathcal{R}_0$ be the set of acyclic reorientations of which class is in $\mathcal{F}_0$. The definition of $\mathcal{R}_0$ implies that the restriction of ϕ to $\mathcal{R}_0$ is injective. We denote $\mathcal{F}'_0$ the particular classes in $\mathcal{F}(M \setminus e)$ which are images by ϕ of elements of $\mathcal{F}_0$.

Let $\mathcal{F}_+$ be the set of subsets of $\mathcal{R}(M)$ defined by $\mathcal{A} \in \mathcal{F}_+$ when $\phi(\mathcal{A}) \in \mathcal{F}(M/e)$ and $e \notin A$ for every $A \in \mathcal{A}$. Let $\mathcal{R}_+ \subseteq \mathcal{R}(M)$ be the union of elements of $\mathcal{F}_+$. Symmetrically, let $\mathcal{F}_-$ be the set of subsets of $\mathcal{R}(M)$ defined by $\mathcal{A} \in \mathcal{F}_-$ when $\phi(\mathcal{A}) \in \mathcal{F}(M/e)$ and $e \in A$ for every $A \in \mathcal{A}$, and let $\mathcal{R}_- \subseteq \mathcal{R}(M)$ be the union of elements of $\mathcal{F}_-$.

Proposition 7.
(i) The mapping ϕ induces a bijection between F_0 and F'_0.
(ii) The mapping ϕ induces bijections between $\mathcal{F}_+$ and $\mathcal{F}(M/e)$, and between $\mathcal{F}_-$ and $\mathcal{F}(M/e)$.

Proof.
(i) For every positive cocircuit C of a reorientation $A \in \mathcal{R}_0$, we have $C \setminus e$ cocircuit of $M \setminus e$, otherwise there would be a cocircuit D of $-_AM$ with only negative element e and $-_AM$ would be bordered by e. So $C \setminus e$ is a positive cocircuit of the acyclic reorientation $A \setminus e$ of $M \setminus e$. Hence, for every $\mathcal{A} \subseteq \mathcal{R}_0$, by transitivity of the equivalence relations, we have $\mathcal{A} \in \mathcal{F}_0$ if and only if $\phi(\mathcal{A}) \in \mathcal{F}(M \setminus e)$.
(ii) Of course $\phi(\mathcal{R}_+) = \phi(\mathcal{R}_-) = \mathcal{R}(M/e)$. Moreover $A \equiv_{M/e} B$ implies $A \triangle e \equiv_M B \triangle e$, so every element in $\mathcal{F}_+ \uplus \mathcal{F}_-$ has a non-empty intersection with exactly one class of M and is contained in this class. $\square$

Let $\mathcal{G} = (\mathcal{F}_+ \uplus \mathcal{F}_-, \mathcal{E})$ be the bipartite graph of which set of vertices is $\mathcal{F}_+ \uplus \mathcal{F}_-$, and of which set of edges $\mathcal{E}$ is defined by $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}$ if and only if there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $B = A \triangle D$ for a cocircuit D with $e \in D$ and positive in $-_AM$.

Proposition 8. *The connected components of $\mathcal{G}$ are either reduced to an isolated vertex, or reduced to an edge. Moreover, there is a bijection between $\mathcal{F}(M) \setminus \mathcal{F}_0$ and the connected components of $\mathcal{G}$.*

Proof. By Lemma 3, if $A \in \mathcal{R}_+ \uplus \mathcal{R}_-$ then $A \not\equiv_M A \triangle e$. Let $A \in \mathcal{R}_+$. With Lemma 6 if there exists a positive cocircuit C of $-_AM$ containing e such that $-_{C \triangle A}M$ is bordered by e, then for every positive cocircuit C of $-_AM$ containing e we have $-_{C \triangle A}M$ bordered by e. Hence we distinguish two cases depending on the fact that this property is true (second case) or false (first case).

First case: there exists a positive cocircuit C of $-_AM$ with $e \in C$ and $-_{C \triangle A}M$ not bordered by e.

By Lemma 5, if D is a positive cocircuit of $-_{A \triangle C}M$ containing e, then $A \equiv_{M/e} A \triangle C \triangle D$. If D is a positive cocircuit of $-_{A \triangle C}M$ not containing e, then let C' be a positive cocircuit of $-_{A \triangle C}M$ containing e and modular with D (it exists by connectivity of the cocircuit graph of $-_{A \triangle C}M$). Then $C' \triangle D$ is a positive cocircuit of $-_{A \triangle C \triangle C'}M$ containing e. So, a possible sequence of reorientations from A to $A \triangle C \triangle D$ using a sequence of allowable reorientations is made by a sequence from A to $A \triangle C \triangle C'$, which is equivalent to a sequence in M/e by Lemma 5 as above, and at last the reorientation with respect to the cocircuit $C' \triangle D$.

So, finally, a sequence of allowable reorientations beginning from the reorientation A can always be replaced by a sequence of reorientations with respect to cocircuits not containing e followed maybe by one only reorientation with respect to a cocircuit containing e. If the final reorientation is bordered by e, then this last reorientation cannot exist, otherwise, according to the distinction between two cases, the final reorientation would satisfy the property of the second case, and so A would satisfy it too by Lemma 6. The same reasoning holds for $A \in \mathcal{R}_-$.

In conclusion: for $A \in \mathcal{R}_+$, resp. $A \in \mathcal{R}_-$, in this first case, if $A \equiv_M B$ and $B \in \mathcal{R}_+ \uplus \mathcal{R}_-$ then $B \in \mathcal{R}_+$, resp. $B \in \mathcal{R}_-$, and $A \equiv_{M/e} B$, resp. $A \setminus e \equiv_{M/e} B \setminus e$. That is A and B belong to the same element in $\mathcal{F}_+$, resp. $\mathcal{F}_-$. So, in this first case, the class of A in M contains only one class in $\mathcal{F}_+ \cup \mathcal{F}_-$. In other words, the corresponding vertex of the graph G is the only vertex of its connected component.

Second case: there exists a positive cocircuit C of $-_A M$ with $e \in C$ and $-_{C \triangle A} M$ bordered by e.

Then, by Lemma 6, for every $B \equiv_{M/e} A$ and every positive cocircuit D of $-_B M$ containing e, we have $B \triangle (D \setminus e) \equiv_{M/e} A \triangle (C \setminus e)$. Moreover for every $B \equiv_M A$ we have $-_B M$ bordered e and $B \setminus e \equiv_{M/e} A$ or $B \setminus e \equiv_{M/e} A \triangle (C \setminus e)$.

So, if $A \equiv_{M/e} A \triangle C \setminus e$ then the class of A in M contains exactly one class in $\mathcal{F}_+ \cup \mathcal{F}_-$, that is the class of A in $\mathcal{F}_+$. And if $A \not\equiv_{M/e} A \triangle C \setminus e$ then the class of A in M contains exactly two classes in $\mathcal{F}_+ \cup \mathcal{F}_-$, that is the class of A in $\mathcal{F}_+$ and the class of $A \triangle C$ in $\mathcal{F}_-$. In other words, in this case, the connected component in G of the corresponding vertex is reduced to an edge joining two vertices.

Finally, we have built a bijection between $\mathcal{F}(M) \setminus \mathcal{F}_0$ and the connected components of $\mathcal{G}$. $\square$

Let $\mathcal{G}' = (\mathcal{F}_+ \uplus \mathcal{F}_-, \mathcal{E}')$ be the bipartite graph of which set of vertices is $\mathcal{F}_+ \uplus \mathcal{F}_-$, and of which set of edges $\mathcal{E}'$ is defined by $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}$ if and only if there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with either $B = A \triangle e$ or $B = A \triangle D$ for a cocircuit D with $e \in D$ and positive in $-_A M$. This graph is obtained from $\mathcal{G}$ by adding edges between symmetric classes in $\mathcal{F}_+$ and $\mathcal{F}_-$.

Proposition 9. *There is a bijection between $\mathcal{F}(M \setminus e) \setminus F'_0$ and connected components of $\mathcal{G}'$. Moreover, the connected components of $\mathcal{G}'$ are chains.*

Proof. By Lemma 4 and by transitivity in the equivalence relations, for every $A, B \subseteq E$, if $A \setminus e \equiv_{M/e} B \setminus e$ then $A \setminus e \equiv_{M \setminus e} B \setminus e$, and also for every $A, B \subseteq E$, if $A \equiv_M B$ then $A \setminus e \equiv_{M \setminus e} B \setminus e$. Let $A \in \mathcal{R}_+ \uplus \mathcal{R}_-$, let $\mathcal{A}$ be the class of A in M/e, let $\mathcal{A}_+ = \mathcal{A}$ and $\mathcal{A}_-$ be the set of $B \cup e$ for $B \in \mathcal{A}$. The class of A in $M \setminus e$ contains $\mathcal{A}_+$ and $\mathcal{A}_-$. Hence for $A, B \subseteq \mathcal{R}_+ \uplus \mathcal{R}_-$, we have $A \setminus e \equiv_{M \setminus e} B \setminus e$ if and only if the classes of A and B in $\mathcal{F}_+ \uplus \mathcal{F}_-$ can be joined by a sequence of edges in the graph $\mathcal{G}'$. So we get the bijection.

Since a connected component of $\mathcal{G}$ is either reduced to a single vertex or to a single edge by Proposition 8, then a connected component of $\mathcal{G}'$ is either a chain or a cycle. Let $\mathcal{A}_1 \to \mathcal{A}_2 \to \ldots \to \mathcal{A}_{2k+1}$ be a path in a connected component of $\mathcal{G}'$, where no vertex is repeated twice. The edges of this path are alternatively of type $A_{i+1} = A_i \triangle e$ and of type $A_{i+1} = A_i \triangle D$ with D positive in $-_{A_i} M$ containing e, for some $A_i \in \mathcal{A}_i$ and $A_{i+1} \in \mathcal{A}_{i+1}$. Let $C_{ref} = (C^+_{ref}, C^-_{ref})$ be an arbitrary circuit of M with $e \in C_{ref}$. It exists since e is not an isthmus. This circuit will play the part of a reference for our following counting. For $A \subseteq E$, we denote $pos(A)$ the number of positive elements of $-_A C_{ref}$. By the orthogonality property in regular oriented matroids, for $A \subseteq E$, $C = -_A C_{ref}$ and D positive cocircuit in $-_A M$, we have $\mid C^+ \cap D \mid = \mid C^- \cap D \mid$, and so $pos(A \triangle D) = pos(A)$. Hence, for $\mathcal{A} \in \mathcal{F}_+$ or $\mathcal{A} \in \mathcal{F}_-$, we can define $pos(\mathcal{A}) = pos(A)$ for any $A \in \mathcal{A}$. Let $1 \leq i \leq 2k$. If $A_{i+1} = A_i \triangle e$ and e is positive, resp. negative, in $-_{A_i} C_{ref}$ then $pos(\mathcal{A}_{i+1}) = pos(\mathcal{A}_i) - 1$, resp. $pos(\mathcal{A}_{i+1}) = pos(\mathcal{A}_i) + 1$. If $A_{i+1} = A_i \triangle D$ with D positive in $-_{A_i} M$ containing e then $pos(\mathcal{A}_{i+1}) = pos(\mathcal{A}_i)$. Hence, finally, the sequence $pos(\mathcal{A}_{2i+1})$, $0 \leq i \leq k$ is either strictly decreasing or strictly increasing. And so the connected component cannot contain a cycle: it is a chain. $\square$

Note that extremities of chains in G' are exactly the connected components of G reduced to a vertex (first type classes in proof of Proposition 8). We are now able to enumerate classes.

Theorem 10. *Let M be a regular oriented matroid.*
(i) The number of acyclic classes for the cocircuit reversing system of M is $t(M;1,0)$.
(ii) The number of totally cyclic classes for the circuit reversing system of M is $t(M;0,1)$.
(iii) The number of classes for the cocircuit reversing system of M is $t(M;1,2)$.
(iv) The number of classes for the circuit reversing system of M is $t(M;2,1)$;
(v) The number of classes for the circuit-cocircuit reversing system of M is $t(M;1,1)$.

Proof. The main problem is to prove (i), it uses the previous propositions and lemmas. The other assertions follow easily from duality and the convolution formula for the Tutte polynomial.
(i) Let e be a fixed element of E. If e is a loop, then M has no acyclic reorientation. If e is an isthmus, then e is a positive cocircuit in any reorientation of M and the number of classes is 1 if $E = \{e\}$ and

the same as in $M \setminus e$ otherwise. Assume now that e is neither a loop nor an isthmus. It remains to prove that the number of classes equals the number of classes of M/e plus the number of classes of $M \setminus e$. The result is then deduced immediately with the inductive definition of the Tutte polynomial.

The connected components of $\mathcal{G}$ being reduced to a vertex or an edge (Proposition 8), their number equals the number of vertices minus the number of edges $\mid \mathcal{F}_+ \uplus \mathcal{F}_- \mid - \mid \mathcal{E} \mid$. The connected components of $\mathcal{G}'$ being disjoint chains (Proposition 9), they induce disjoint chains on $\mathcal{F}_+$ and their number equals $\mid \mathcal{F}_+ \mid - \mid \mathcal{E} \mid$. Hence the number of connected components in $\mathcal{G}$ equals the number of connected components in $\mathcal{G}'$ plus the number of vertices in $\mathcal{F}_+$. Using the bijections in Propositions 7, 8 and 9, we get

$$\mid \mathcal{F}(M) \setminus F_0 \mid = \mid \mathcal{F}(M \setminus e) \setminus \mathcal{F}'_0 \mid + \mid \mathcal{F}_+ \mid$$

and then

$$\mid \mathcal{F}(M) \mid = \mid \mathcal{F}(M \setminus e) \setminus \mathcal{F}'_0 \mid + \mid \mathcal{F}'_0 \mid + \mid \mathcal{F}_+ \mid = \mid \mathcal{F}(M \setminus e) \mid + \mid \mathcal{F}(M/e) \mid .$$

(ii) By duality with (i).

(iii) The equivalence class of a totally cyclic reorientation is reduced to one element for the cocircuit reversing system, since it has no positive cocircuit. So, thanks to Proposition 1, two reorientations A and B are equivalent for the cocircuit reversing system of M if and only if $-_A M$ and $-_B M$ have same cyclic part F, $A \cap F = B \cap F$, and $A \setminus F$ and $B \setminus F$ are equivalent for the cocircuit reversing system of M/F. So, numerically, the number of classes of M for the cocircuit reversing system is

$$\sum_{\substack{F \text{ cyclic} \\ \text{flat of } M}} \mid \mathcal{F}(M/F) \mid \; \mid \mathcal{R}(M(F)^*) \mid \; = \sum_{\substack{F \text{ cyclic} \\ \text{flat of } M}} t(M/F;1,0)\; t(M(F);0,2) = t(M;1,2)$$

using (i), the known enumeration of totally cyclic reorientations of $M(F)$, and the convolution formula for the Tutte polynomial.

(iv) By duality with (iii).

(v) Thanks to Proposition 1, two reorientations A and B are equivalent for the circuit-cocircuit reversing system of M if and only if $-_A M$ and $-_B M$ have same cyclic part F, $A \cap F$ and $B \cap F$ are equivalent for the circuit reversing system of $M(F)$, and $A \setminus F$ and $B \setminus F$ are equivalent for the cocircuit reversing system of M/F. So, numerically, the number of classes of M for the circuit-cocircuit reversing system is

$$\sum_{\substack{F \text{ cyclic} \\ \text{flat of } M}} \mid \mathcal{F}(M/F) \mid \; \mid \mathcal{F}(M(F)^*) \mid \; = \sum_{\substack{F \text{ cyclic} \\ \text{flat of } M}} t(M/F;1,0)\; t(M(F);0,1) = t(M;1,1)$$

using (i), (ii), and the convolution formula for the Tutte polynomial. □

Since regular matroids generalize digraphs, enumerative results on classes in [5] Section 4 are corollaries of Theorem 10, and enumerative results on indegree sequences in [5] Section 3 are corollaries of Theorem 10 together with the bijections between classes and indegree sequences in [5] Section 4.

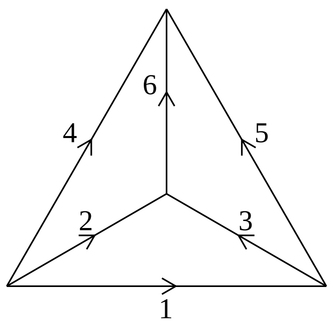

Figure 2.0. The directed graph K_4

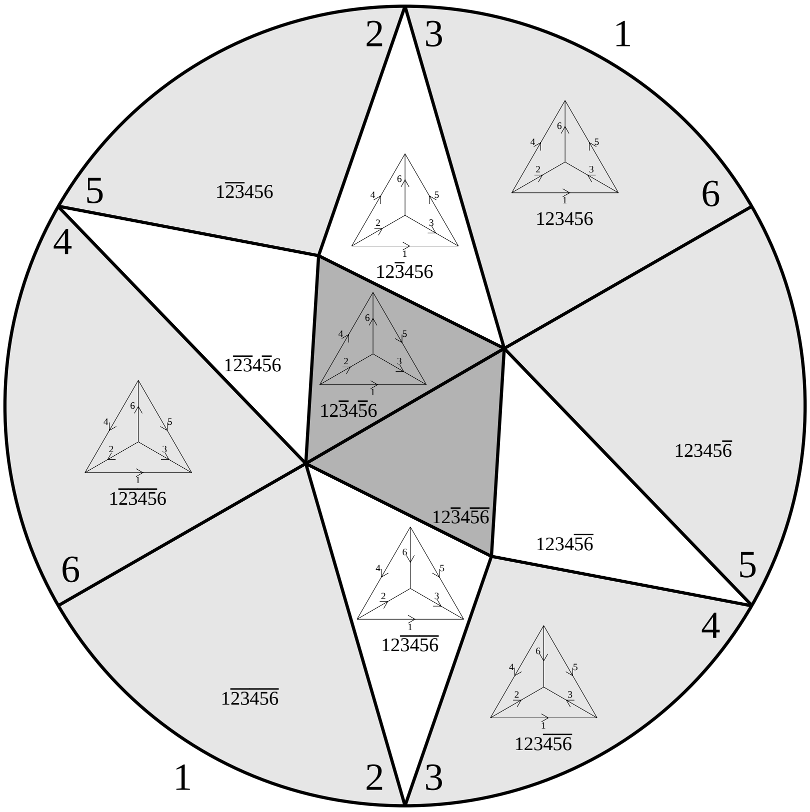

Figure 2.1 The six acyclic classes of K_4, first hemisphere

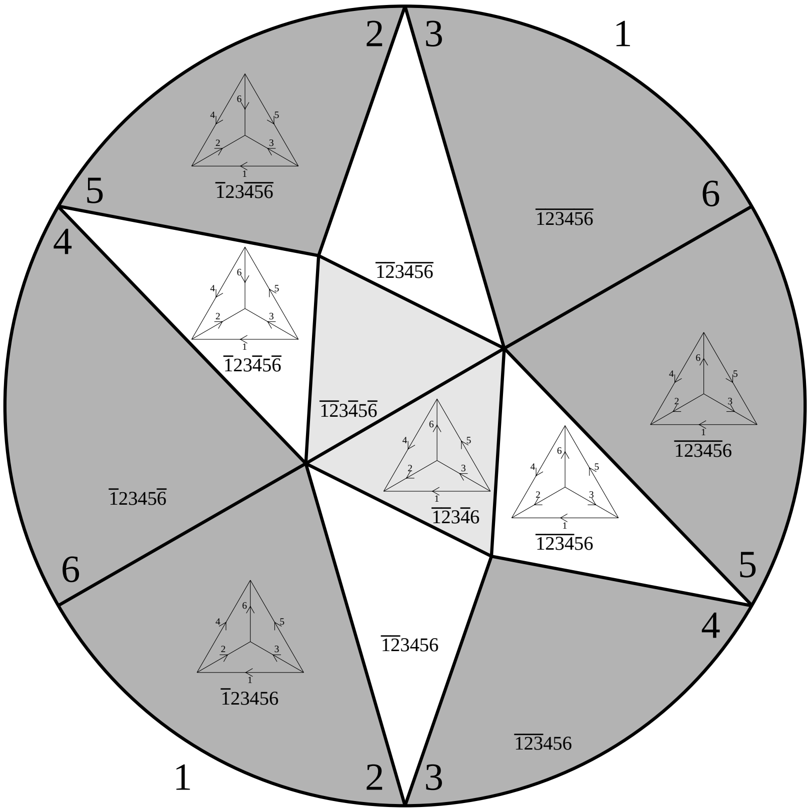

Figure 2.2 The six acyclic classes of K_4, second hemisphere

Example. We consider the regular oriented matroid associated with the acyclic directed graph $M = K_4$ given in Figure 2.0. Acyclic reorientations of K_4 correspond to regions in the two opposite hemispheres of a pseudosphere arrangement represented on Figures 2.1 and 2.2. In each region is written the corresponding maximal covector, with standard notations. Recall that the evolution rule allows crossing a 0-dimensional face and then take the opposite region. In the graph, it allows reversing the edges of a cocycle (minimal cut).

The six acyclic classes are represented on Figures 2.1 and 2.2 the following way: two white classes, two light grey classes and two dark grey classes, and, for each colour, one of the two classes is represented with the corresponding digraphs drawn in the regions. The reference element for the illustration of Propositions 7, 8, 9 is $e = 6$.

The *white* regions are elements of $\mathcal{R}_0$. There are two classes in $\mathcal{F}_0$, namely $\{3, 3456, 1234, 146\}$ (these reorientations are represented with the corresponding digraphs) and $\{235, 56, 12456, 12\}$. These regions and their decomposition into two classes are unchanged when 6 is deleted, illustrating the bijection between $\mathcal{F}_0$ and $\mathcal{F}'_0$.

The *light grey* and *dark grey* regions correspond to the two distinct classes in $\mathcal{F}(M \setminus e) \setminus \mathcal{F}'_0$, and also in $\mathcal{F}(M/e)$. For each colour, the regions of which class is in $\mathcal{F}_+$ (that is: the region is bordered by 6 and 6 is positive) are represented with the digraph drawn inside. The other are opposite with respect to 6. In this particular case, the graph $\mathcal{G}$ is formed by four vertices and no edges. The graph $\mathcal{G}'$ is obtained by joining two classes when they are symmetric with respect to 6, and so is formed by two separate edges. More precisely the light grey classes in M are $\{\emptyset, 2345, 124, 456\}$ and $\{6, 23456, 124, 23\}$, they are joined by an edge in $\mathcal{G}'$, they correspond to the class $\{\emptyset, 2345, 124, 456\}$ in $M/6$ and the same in $M \setminus 6$. Similarly the dark grey classes are opposite to the light grey classes.

Note that the case where $\mathcal{G}$ has a non empty set of edges, and thus $\mathcal{G}'$ has non trivial chains, appears in higher dimensions. Some examples can easily be built using graphs on more than 5 vertices.

References.

[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler, Oriented matroids, Encyclopedia of Mathematics and its Applications 46, 2nd Ed., Cambridge University Press, Cambridge, 1999.

[2] T. Brylawski and J.G. Oxley, The Tutte polynomial and its applications, chapter 6 in: N. White (Ed.), Matroid Applications, Cambridge University Press, Cambridge, 1992.

[3] G. Etienne, On the Möbius algebra geometric lattices, European J. Combin. **19** (8) (1998) 921-933.

[4] G. Etienne and M. Las Vergnas, External and internal elements of a matroid basis, Discrete Math. **179** (1-3) (1998) 111-119.

[5] E. Gioan, Enumerating degree sequences in digraphs and a cycle-cocycle reversing system, European J. Combin., to appear.

[6] E. Gioan and M. Las Vergnas, Activity preserving bijections between spanning trees and orientations in graphs, Discrete Math. **298** (1-3) (2005) 169-188.

[7] E. Gioan and M. Las Vergnas, The active bijection in oriented matroids - II - Decompositions of activities, in preparation. See also chapter 2 in: E. Gioan, Correspondance naturelle entre bases et réorientations des matroïdes orientés, thèse de Doctorat, LaBRI, université Bordeaux 1, 2002.

[8] C. Greene and T. Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs, Trans. Amer. Math. Soc. **280** (1) (1983) 97-126.

[9] W. Kook, V. Reiner and D. Stanton, A convolution formula for the Tutte polynomial, J. Combin. Theory Ser. B **76** (2) (1999) 297-300.

[10] M. Las Vergnas, Acyclic and totally cyclic orientations of combinatorial geometries, Discrete Math. **20** (1) (1977/78) 51-61.

[11] G.J. Minty, On the axiomatic foundations of the theories of directed linear graphs, electrical networks and network-programming, J. Math. Mech. **15** (1966) 485-520.

[12] J.G. Oxley, Matroid theory, Oxford University Press, Oxford, 1992.

[13] R.P. Stanley, Acyclic orientations of graphs, Discrete Math. **5** (1973) 171-178.

[14] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canadian J. Math. **6** (1954) 80-91.

[15] R.O. Winder, Partitions of N-space by hyperplanes. SIAM J. Appl. Math. **14** (1966) 811-818.