On the Evaluation at (j, j^2) of the Tutte Polynomial of a Ternary Matroid

Emeric GIOAN

Université Bordeaux I
LaBRI
351 cours de la Libération, 33405 Talence (France)

gioan@labri.u-bordeaux.fr

Michel LAS VERGNAS*

Université Pierre et Marie Curie
case 189 - Combinatoire
4 place Jussieu, 75005 Paris (France)

mlv@ccr.jussieu.fr

Abstract

F. Jaeger has shown that up to a $\pm$ sign the evaluation at (j, j^2) of the Tutte polynomial of a ternary matroid can be expressed in terms of the dimension of the bicycle space of a representation over $GF(3)$. We give a short algebraic proof of this result, which moreover yields the exact value of $\pm$, a problem left open in Jaeger's paper. It follows that the computation of $t(j, j^2)$ is of polynomial complexity for a ternary matroid.

AMS classification. Primary: 05B35. Secondary: 68Q17 57M27

Keywords: matroid, ternary matroid, Tutte polynomial, graph, knot theory, Jones polynomial, computational complexity

* C.N.R.S., Paris

In the seminal paper [4] on the complexity of Tutte polynomials, it is shown that the point (j, j^2) and its conjugate (j^2, j) are two out of eight 'easy' special points, where 'easy' is intended from a computational point of view. Each of these eight points have remarkable combinatorial interpretations. A result of F. Jaeger [3] relates $t(j, j^2)$ and $t(j^2, j)$ to ternary matroids. Specifically, let E be a finite set, V be a subspace of $GF(3)^E$, and $M(V)$ be the matroid of V. Then $t(M(V); j, j^2) = \pm j^{|E|+\dim V}(i\sqrt{3})^{\dim(V\cap V^\perp)}$. Graphs, via graphic matroids, are a special case of ternary matroids. We refer the reader to the introduction of [3] (see also [4] Section 6) for the relevance of these properties to the Jones polynomial in Knot Theory. We also mention the related paper [5], where the problem of the complexity of the computation of $t(M; x, y)$, for x, y algebraic numbers and M vectorial over a given finite field, is addressed in full generality.

The main step of the proof in [3] is to establish that $\sum_{u\in V} j^{|s(u)|} = \pm(i\sqrt{3})^{\dim V+\dim(V\cap V^\perp)}$, where $s(u)$ denotes the support of u. The proof of this last property in Jaeger's paper uses deletion/contraction of elements of E, and is about four page long. Our purpose in the present note is to provide a short algebraic proof. Moreover, we obtain the exact value of $\pm$, a question left open in Jaeger's paper. As a consequence $t(M; j, j^2)$ is of polynomial complexity for a ternary matroid M.

We will use two classical results about orthogonal bases. We include proofs for completeness.

Lemma 1.1

Let V be a finite dimensional vector space over a field of characteristic $\neq 2$ endowed with a bilinear form. Then V has an orthogonal basis.

More specifically an orthogonal basis of V can be constructed from any given basis in polynomial time.

Proof

Let $(u_k)_{1\leq k\leq d}$ be a basis of V. If there is an index $1 \leq \ell \leq d$ such that $< u_\ell, u_\ell > \neq 0$, then reindex such that $\ell = 1$ and set $u'_1 = u_1$. Otherwise, if there is an index $2 \leq \ell \leq d$ such that $< u_1 + u_\ell, u_1 + u_\ell > \neq 0$, then set $u'_1 = u_1 + u_\ell$. In both cases, update u_k as $u_k - < u'_1, u_k >< u'_1, u'_1 >^{-1} u'_1$ for $2 \leq k \leq d$. We have $< u'_1, u_k >= 0$ for $2 \leq k \leq d$.

Otherwise we have $< u_k, u_k >= 0$ for $1 \leq k \leq d$, and $< u_1 + u_k, u_1 + u_k >= 0$ for $2 \leq k \leq d$. From $< u_1 + u_k, u_1 + u_k >= 0$, we get $< u_1, u_1 > +2 < u_1, u_k > + < u_k, u_k >= 2 < u_1, u_k >= 0$, hence $< u_1, u_k >= 0$ in characteristic $\neq 2$. We set $u'_1 = u_1$.

In all three cases, $\{u'_1, u_2, \ldots, u_d\}$ is a basis of V such that u'_1 is orthogonal to the space generated by u_k for $2 \leq k \leq d$. Lemma 1.1 follows by induction. $\square$

Let K be a field, E be a finite set. The *canonical bilinear form* on the space K^E is defined by $< u, v >= \sum_{e \in E} u(e)v(e)$ for $u, v \in K^E$. In K^E, the subspace *orthogonal* to a subspace V is defined by $V^\perp = \{u' \in K^E \mid < u, u' >= 0 \text{ for all } u \in V\}$.

Lemma 1.2

The isotropic vectors of any orthogonal basis of V constitute a basis of $V \cap V^\perp$.

Proof

Let $(u_k)_{1 \le k \le d}$ be an orthogonal basis of V, and $u = \sum_{1 \le k \le d} a_k u_k \in V \cap V^\perp$. For $1 \le \ell \le d$, we have $0 =< u, u_\ell >= \sum_{1 \le k \le d} a_k < u_k, u_\ell >= a_\ell < u_\ell, u_\ell >$. Hence if $< u_\ell, u_\ell > \ne 0$, we have $a_\ell = 0$. It follows that u is generated by the isotropic vectors of the basis. These vectors being independant, they constitute a basis of $V \cap V^\perp$. □

Our basic result is the following strengthening of Jaeger's proposition.

Proposition 1

Let E be a finite set, and V be a subspace of $GF(3)^E$. We have

$$\sum_{u \in V} j^{|s(u)|} = (-1)^{d+d_1} (i\sqrt{3})^{d+d_0}$$

where $d = \dim V$, $d_0 = \dim V \cap V^\perp$ and d_1 is the number of vectors with support of size congruent to 1 modulo 3 in any orthogonal basis of V with respect to the canonical bilinear form.

Proof

We have $GF(3) \approx Z/3Z$, in other words the elements of $GF(3)$ can be assimilated to integer residues modulo 3. We observe that for $u \in GF(3)^E$ we have $|s(u)|$ modulo $3 =< u, u >$, where $< u, v >= \sum_{e \in E} u(e)v(e)$ is the canonical bilinear form. It follows that $j^{|s(u)|} = j^{< u,u >}$.

By Lemma 1.1, there is an orthogonal basis $(u_k)_{1 \le k \le d}$ of V. We have

$$
\begin{aligned}
\sum_{u\in V} j^{|s(u)|} &= \sum_{u\in V} j^{<u,u>} \\
&= \sum_{(a_1,a_2,\ldots,a_d)\in GF(3)^d} j^{<\sum_{1\le k\le d} a_k u_k, \sum_{1\le k\le d} a_k u_k>} \\
&= \sum_{(a_1,a_2,\ldots,a_d)\in GF(3)^d} j^{\sum_{1\le k\le d} a_k^2 <u_k,u_k>} \\
&= \sum_{(a_1,a_2,\ldots,a_d)\in GF(3)^d} \prod_{1\le k\le d} j^{a_k^2 <u_k,u_k>} \\
&= \prod_{1\le k\le d} \sum_{a_k\in GF(3)} j^{a_k^2 <u_k,u_k>} \\
&= \prod_{1\le k\le d} (1+2j^{<u_k,u_k>}) \\
&= 3^{d_0}(1+2j)^{d_1}(1+2j^2)^{d_2}
\end{aligned}
$$

where d_0 resp. d_1, d_2 is the number of vectors u_k $1 \le k \le d$ such that $< u_k, u_k >= 0$ resp. $= 1 = 2$. We have $1+2j = i\sqrt{3}$, $1+2j^2 = -i\sqrt{3}$, $d = d_0 + d_1 + d_2$, and $d_0 = \dim V \cap V^{\perp}$ by Lemma 1.2. Proposition 1 follows. □

It follows from Proposition 1 and Lemma 1.1 that

Corollary 2

Let E be a finite set, and V be a subspace of $GF(3)^E$.

The parity of the number of vectors with support of cardinality congruent to 1 resp. 2 modulo 3 in an orthogonal basis of V does not depend of the particular orthogonal basis. □

By Corollary 2 the residue modulo 2 of the number of vectors with support of cardinality congruent to 1 resp. 2 modulo 3 in an orthogonal basis of a subspace V of $GF(3)^E$ is a 0-1 invariant of V. We will denote it by $\overline{d}_1(V)$ resp. $\overline{d}_2(V)$. It follows from Lemma 1.1 that $\overline{d}_1(V)$ can be computed in polynomial time from any given basis of V.

We recall that by a theorem of Greene [2], given a subspace V of $GF(q)^E$, q a prime power, we have $\sum_{u\in V} z^{|s(u)|} = z^{|E|-d}(1-z)^d t(M; 1/z, 1 + qz/(1-z))$, where $d = \dim V$.

Theorem 3

Let M be a ternary matroid on a finite set E. We have

$t(M; j, j^2) = (-1)^{d_2} j^{|E|+d} (i\sqrt{3})^{d_0}$

where $d = \dim V$, $d_0 = \dim V \cap V^\perp$, and d_2 is the number of vectors with support of cardinality congruent to 2 modulo 3 in any orthogonal basis of a subspace V of $GF(3)^E$ such that $M = M(V)$.

Proof

As in Jaeger's paper, we derive Theorem 3 from Proposition 1 by means of Greene's theorem. Specializing this formula to $z = j$ and $q = 3$, and applying Proposition 1, we get

$$t(M; j^2, j) = (-1)^{d+d_1} j^{-|E|-d} (i\sqrt{3})^{d_0}$$

Since $t(M; j, j^2)$ is the complex conjugate of $t(M; j^2, j)$, Theorem 2 follows. □

A short proof of Greene's theorem is given in [3] Proposition 7 (see also [1] for another short proof).

Theorem 3 provides the exact value of $\pm$ in Jaeger's formula for $t(M; j, j^2)$ when M is a ternary matroid. This answers the question in [3] p.25 asking for an interpretation of the parameter $\epsilon(M)$, defined by $t(M; j, j^2) = \epsilon(M) j^{|E|+d} (i\sqrt{3})^{d_0}$. By Corollary 2, $\overline{d}_1 = \overline{d}_1(V) = \overline{d}_1(M)$ is a 0-1 invariant of polynomial complexity of a ternary matroid M. By Theorem 3, we have

$$\epsilon(M) = (-1)^{d+\overline{d}_1} = (-1)^{d_0+\overline{d}_2}$$

As well-known $d_0 = \dim V \cap V^\perp$ is of polynomial complexity (also a corollary of Lemmas 1.1 and 1.2). Hence

Corollary 4

The evaluation $t(M; j, j^2)$ of the Tutte polynomial of a ternary matroid M is of polynomial complexity. □

Corollary 4 strengthens the previously known polynomial complexity of the modulus $|t(M; j, j^2)|$, used in [4][5].

As noted by Jaeger (see [3] Proposition 9) $\epsilon(M)$ and $\epsilon(M^*)$ are related.

Corollary 5

Let M be a ternary matroid on a set E. We have $\overline{d}_1(M^*) \equiv \overline{d}_1(M) + d_0(M) + |E|$ modulo 2, where M^* denotes the dual matroid of M.

Corollary 5 follows from the relation $\epsilon(M) = (-1)^{d+\overline{d}_1}$, combined with [3] Proposition 9.(i). It can also be easily derived directly from Theorem 2.

Finally, we mention that the initial motivation of the present note was the computation of $\sum_{u\in V} j^{|s(w+u)|}$, where w is any vector of $GF(3)^E$.

Corollary 6

Let $w \in GF(3)^E$.

- *If $w \in V + V^\perp$, say $w = w' + w''$ with $w' \in V$ and $w'' \in V^\perp$, then, with notation of Proposition 1, we have*

$$\sum_{u\in V} j^{|s(w+u)|} = (-1)^{d+d_1}(i\sqrt{3})^{d+d_0} j^{|s(w'')|}$$

- *If $w \notin V + V^\perp$, we have*

$$\sum_{u\in V} j^{|s(w+u)|} = 0$$

Proof

If $w = w' + w''$ with $w' \in V$ and $w'' \in V^\perp$, we have

$$\begin{aligned}\sum_{u\in V} j^{|s(w+u)|} &= \sum_{u\in V} j^{|s(w'+w''+u)|} = \sum_{u\in V} j^{|s(w''+u)|}\\ &= \sum_{u\in V} j^{<w''+u,w''+u>} = \sum_{u\in V} j^{<w'',w''>+<u,u>}\\ &= j^{<w'',w''>}\sum_{u\in V} j^{<u,u>}\end{aligned}$$

Then we obtain Corollary 4 by applying Proposition 1.

If $w \notin V + V^\perp$, then there is $v \in V \cap V^\perp = (V + V^\perp)^\perp$ such that $< w, v > \neq 0$. Let V' be a supplement of $< v >$ in V. We have

$$\begin{aligned}\sum_{u\in V} j^{|s(w+u)|} &= \sum_{u\in V'}\sum_{a\in GF(3)} j^{|s(w+u+av)|}\\ &= \sum_{u\in V'}\sum_{a\in GF(3)} j^{<w+u+av,w+u+av>}\\ &= \sum_{u\in V'}\sum_{a\in GF(3)} j^{<w+u,w+u>+a<w,v>}\\ &= \sum_{u\in V'} j^{<w+u,w+u>}\Big(\sum_{a\in GF(3)} j^{a<w,v>}\Big)\\ &= \sum_{u\in V'} j^{<w+u,w+u>}(1+j+j^2) = 0\end{aligned}$$

□

References

[1] G. Etienne, M. Las Vergnas, The Tutte polynomial of a morphism of matroids III. Vectorial matroids, Advances in Applied Mathematics, to appear

[2] C. Greene, Weight enumeration and the geometry of linear codes, Studies in Applied Math. 55 (1976), 119-128

[3] F. Jaeger, Tutte polynomials and bicycle dimension of ternary matroids, Proc. Amer. Math. Soc. 107 (1989), 17-25

[4] F. Jaeger, D.L. Vertigan, D.J.A. Welsh, On the computational complexity of the Jones and Tutte polynomials, Math. Proc. Camb. Phil. Soc. 108 (1990), 35-53

[5] D. Vertigan, Bicycle dimension and special points of the Tutte Polynomial, J. Combinatorial Theory B 74 (1998), 378-396