A Linear Programming Construction of Fully Optimal Bases in Graphs and Hyperplane Arrangements

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Proceedings EuroComb 2009

Electronic Notes in Discrete Mathematics, pp. 307-311, (2009)

Abstract

The fully optimal basis of a bounded acyclic oriented matroid on a linearly ordered set has been defined and studied by the present authors in a series of papers, dealing with graphs, hyperplane arrangements, and oriented matroids (in order of increasing generality). This notion provides a bijection between bipolar orientations and uniactive internal spanning trees in a graph resp. bounded regions and uniactive internal bases in a hyperplane arrangement or an oriented matroid (in the sense of Tutte activities). This bijection is the basic case of a general activity preserving bijection between reorientations and subsets of an oriented matroid, called the active bijection, providing bijective versions of various classical enumerative results.

Fully optimal bases can be considered as a strenghtening of optimal bases from linear programming, with a simple combinatorial definition. Our first construction used this purely combinatorial characterization, providing directly an algorithm to compute in fact the reverse bijection. A new definition uses a direct construction in terms of a linear programming. The fully optimal basis optimizes a sequence of nested faces with respect to a sequence of objective functions (whereas an optimal basis in the usual sense optimizes one vertex with respect to one objective function). This note presents this construction in terms of graphs and linear algebra.

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1 Introduction

The active bijection in graphs, hyperplane arrangements and oriented matroids on a linearly ordered set is the subject of several papers by the present authors, e.g. [1]...[5]. It defines an activity preserving correspondence between orientations and spanning trees of a graph resp. reorientations and bases of an oriented matroid, with various related bijections and characterizations.

The active bijection can be seen as a far reaching generalization of the well-known bijection between permutations and increasing trees (a particular case obtained from complete graphs [1] or the Coxeter arrangement A_n [2]).

In general, it provides bijective interpretations of the equality of two expressions of the Tutte polynomial, one in terms of base activities [9], the other in terms of orientation activities [7], which contains various enumerative results from the literature, such as counting regions or acyclic orientations, e.g. [8][10][6]. See [1][4] for further references. See [3] for a short survey of its general bijective and enumerative properties.

The basic case of the whole construction is that of a bipolar digraph, or of a bounded region in a hyperplane arrangement or an oriented matroid. It has been proved in [1][4] by combinatorial means that such a structure has a unique so-called fully optimal basis, bijectively associated with the structure. We present here a direct construction of the fully optimal basis, by means of elaborations on linear programming, to be detailed in a forthcoming paper [5].

2 Preliminaries

This note is written in terms of usual linear algebra and graph theory. Still, everything generalizes to oriented matroid theory.

2.1 Hyperplane Arrangements

Here, a *(hyperplane) arrangement* is a finite set $H = h_1, ..., h_n$ of linear forms in a real vector space V.

With any element x of V is associated a sequence of signs $S = (s_1, ..., s_n)$, called the *covector* of x, where $s_i \in \{+, -, 0\}$ is the sign of $h_i(x)$. We may abuse notations and denote also S for the set of $h_i \in H$ with $h_i(x) \neq 0$, and h_i for the set of $x \in V$ with $h_i(x) = 0$. We call this set h_i an *hyperplane* of H. If there exists $x \in V$ whose covector has only $+$ signs, then we say that the arrangement has a *feasible region*, which is the set of x whose covector has no negative sign. A basis of H is a maximal subset of H of linearly independent forms. All bases have same cardinal r.

Let B be a basis of H. For $b \in B$, we denote $C^*(B; b)$ the covector of (an element of the line) $\cap_{b' \in B \setminus b} b'$ where b has a sign $+$. For $e \notin B$, we denote

$C(B;e)$ the sequence of signs $(s_1, ..., s_n)$ defined by $b \in C(B;e)$ if and only if $e \in C^*(B;b)$, and e and b have same sign in one sequence of signs if and only if they have opposite signs in the other.

We will choose a distinguished element $p \in H$ as *hyperplane at infinity.* The feasible region of H is *bounded* if all its elements $x \neq 0$ satisfy $p(x) > 0$.

2.2 Graphs

For geometrical purpose, *directed graphs* can be considered as hyperplane arrangements. Let G be a connected directed graph with set of vertices $\{v_0, ..., v_r\}$. With an edge (v_i, v_j) is associated the linear form $x_j - x_i$ in the real vector space V of dimension $r + 1$, defining the *arrangement* $H(G)$.

It is folklore that: $H(G)$ has a feasible region if and only if G is acyclic, a basis of $H(G)$ is associated with (the set of edges of) a spanning tree of G.

Let B be a spanning tree of G. For $b \in B$, $C^*(B;b)$ is the cocycle joining the two components of $B \setminus \{b\}$. And for $e \notin B$, $C(B;e)$ is the unique cycle contained in $B \cup \{e\}$. Signs are given accordingly with the orientation of G.

Also, it is a reformulation of a result in [1] that $H(G)$ has a bounded feasible region w.r.t. p if and only if G is bipolar w.r.t. p, i.e. G has a unique source and a unique sink which are the extremities of p.

2.3 Linear Programming

A *linear program* P is a finite set of affine inequalities of type $\bar{h}_i(x) \geq t_i$ for x in a vector space $\bar{V}$, together with a linear form f on $\bar{V}$. An *optimum* of P is a vertex of $\bar{V}$ satisfying all inequalities in P and maximizing f. Classically, if the inequalities define a bounded affine region, then there is an optimum.

An arrangement H with hyperplane at infinity p and a chosen $f \in H$ defines a linear program $P = (H; p, f)$ by considering a hyperplane $\bar{V}$ parallel to p, the inequalities $h(x) \geq 0$ for $x \in \bar{V}$ and $h \in H \setminus \{p, f\}$, and the linear form f as the objective function to be maximized. Of course $H \setminus f$ has a feasible bounded region if and only if P defines a bouded affine region.

As a slight formal generalization, we need to define a linear program $(H; p, f)$ for a linear form $f \notin H$ in the same space, to this aim we consider in fact the arrangement obtained by adding f to H.

The *Simplex Criterion* provides a classical combinatorial characterization: a line of H gives an optimum of $(H; p, f)$ if and only if it is $\cap_{b \in B \setminus p} b$ for a basis B of $H \setminus f$ with $p \in B$, $C^*(B;p) \setminus f$ is positive, and $C(B;f) \setminus p$ is positive.

2.4 The Fully Optimal Basis

For H a linearly ordered arrangement, resp. G a graph on a linearly ordered set of edges, with minimal element p, a *fully optimal basis* is a basis B such that:

- for all $b \in B \setminus p$, the signs of b and $min(C^*(B;b))$ are opposite in $C^*(B;b)$
- for all $e \notin B$, the signs of e and $min(C(B;e))$ are opposite in $C(B;e)$.

The main theorem in [4] (see also [1] in the case of graphs) states that if H defines a bounded feasible region, resp. if G is bipolar, w.r.t. the minimal element p, then it has a unique fully optimal basis, denoted $\alpha(H)$, resp. $\alpha(G)$.

Moreover, the mapping α is a bijection between all reorientations of H that define feasible bounded regions (reorienting is reversing some linear forms), resp. all bipolar reorientations of G, w.r.t. p with fixed orientation, and all bases, resp. all spanning trees, with internal activity 1 and external activity 0.

If B is the fully optimal basis of H, a direct consequence [4] of the Simplex Criterion is that $\cap_{b\in B\setminus p} b$ is an optimum of $(H;p,f)$ for the objective function f being the second element in the minimal basis of H. Hence, computing the fully optimal basis is in fact a more general problem than computing an optimal basis in linear programming.

3 Computation of the fully optimal basis

Here, we present an algorithm to compute the fully optimal basis of a bounded region in an ordered hyperplane arrangement. It relies on two elaborations of linear programming, described in the general setting of oriented matroids in [5]. The first one, *multiobjective programming*, consists in optimizing with respect to a sequence objective functions. The second one, *flag programming*, consists in optimizing successive nested faces of successive dimensions.

Let H be a linearly ordered arrangement with a bounded feasible region.

(0) Compute the lexicographically minimal basis $\{p, f_1, \dots, f_{r-1}\}_<$ of H.

(1) Compute the vertex unique optimum of the *multiobjective program* over H w.r.t. the hyperplane at infinity $p_1 = p$ and sequence of objective functions $\{f_1, f_2, \dots, f_{r-1}\}_<$. Into details, we first compute the set of optima F_1 of the linear program $(H;p_1,f_1)$, they belong to a hyperplane parallel to f_1. Then we compute the set of optima F_2 of the linear program $(H;p_1,f_2)$ restricted to F_1. And so on, until we get a unique vertex v_1.

($i \geq 2$) We define a multiobjective program at level i as follows. The hyperplanes of the arrangement H_i are the intersections of p_{i-1} with the hyperplanes of H_{i-1} containing v_{i-1}. The feasible region and the linear forms of H_i are induced by those of H_{i-1}. Note that faces in H_i correspond to faces in H_{i-1} containing v_{i-1}, hence the *flag programming* elaboration. Let p_i be the intersection of p_{i-1} with the smallest hyperplane at level $i-1$ containing v_{i-1}. Then the vertex v_i at level i is the unique optimum of the multiobjective program over H_i w.r.t. the hyperplane at infinity p_i and the sequence of objective functions intersections of p_{i-1} with the objective functions at level $i-1$.

For $i \geq 2$, let b_i be the smallest hyperplane of H containing $v_1, v_2, \ldots, v_{i-1}$ (by construction such a hyperplane exists, note that b_i in H induces p_i in H_i). Then the fully optimal basis of H is $\alpha(H) = \{b_1 = p, b_2, \ldots, b_r\}$.

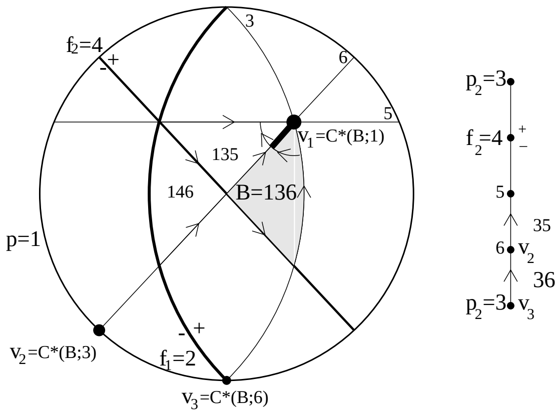
Remarks.

• Levels $i \geq 2$ have important differences with level 1: the objective functions do not necessarily belong to the hyperplane arrangement (and some are useless because of dependencies), and the feasible region is not necessarily bounded.

• Since linear programming can be solved in polynomial time with numerical methods, the fully optimal basis can be computed in polynomial time.

• Some specific properties are available for graphs, for instance $C^*(B; p)$ (corresponding in general to v_1) is the lexicographically smallest directed cocycle.

Example. Arrows indicate increasing directions. Consider the gray region. The vertices $3 \cap 4$ and $3 \cap 5 \cap 6$ are optima w.r.t. $f_1 = 2$, and $v_1 = 3 \cap 5 \cap 6$ is optimum among the two w.r.t. $f_2 = 4$, implying $b_2 = 3$. Note that this vertex is also optimum for region 135. Taking intersections with $p_1 = 1$, we have that $v_2 = 1 \cap 6$ is optimum w.r.t. $f_2 = 4$, implying $b_3 = 6$.



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