Dynamic Distance Hereditary Graphs using Split Decomposition

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Joint work with C. Paul (CNRS - LIRMM)

Dynamic graph representation problem:

Given a representation $R(G)$ of a graph G and a edge or vertex modification of G (insertion or deletion) update the representation $R(G)$.

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Some keys of the problem

Need of a canonical representation (decomposition techniques...) and need of an incremental (dynamic) characterization.

Some known results

	vertex modification	edge modification
proper intervals	$O(d + \log n)$ [HSS02]	$O(1)$ [HSS02]
cographs	$O(d)$ **[CoPeSt85]**	$O(1)$ **[SS04]**
permutations	$O(n)$ [CrPa05]	$O(n)$ [CrPa05]
distance hereditary	$O(d)$ **[GPa07]**	$O(1)$ **[CoT07]**
intervals	$O(n)$ [Cr07]	$O(n)$ [Cr07]

HSS = Hell, Shamir, Sharan
CoPeSt = Corneil, Perl, Stewart
SS = Shamir, Sharan
CrPa= Crespelle, Paul
GPa = Gioan, Paul
CoT = Corneil, Tedder
Cr = Crespelle

1. Revisiting split decomposition

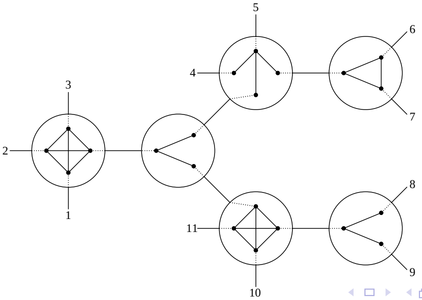
2. Vertex modification of DH graphs

3. Relations with other works

Graph labelled tree

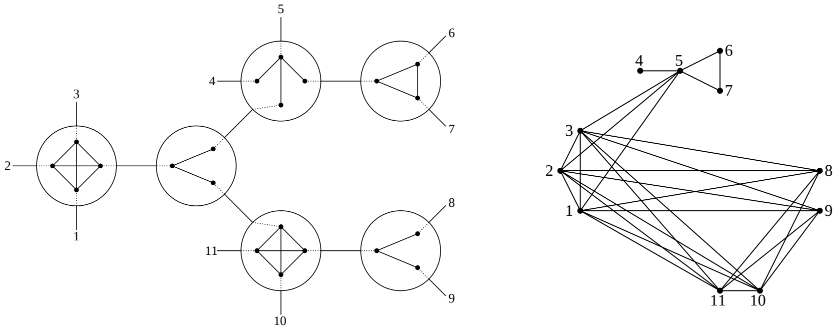
A *graph-labelled tree* is a pair $(T, \mathcal{F})$ with T a tree and $\mathcal{F}$ a set of graphs such that:

- each (internal) node v of degree k of T is labelled by a graph $G_v \in \mathcal{F}$ on k vertices
- there is a bijection ρ_v from the tree-edges incident to v to the vertices of G_v



Given a graph labelled tree $(T, \mathcal{F})$, the *accessibility graph* $G_S(T, \mathcal{F})$ has the leaves of T as vertices and

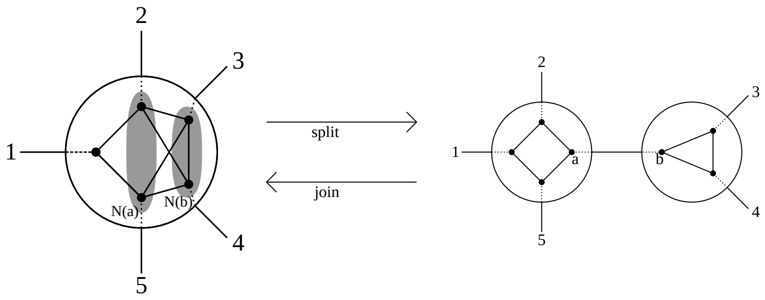
- $xy \in E(G_S(T, \mathcal{F}))$ if and only if $\rho_v(uv)\rho_v(vw) \in E(G_v)$, $\forall$ tree-edges uv, vw on the x, y-path in T



Split

A *split* is a bipartition (A, B) of the vertices of a graph $G = (V, E)$ such that

- $|A| \geqslant 2$, $|B| \geqslant 2$;
- for $x \in A$ and $y \in B$, $xy \in E$ iff $x \in N(B)$ and $y \in N(A)$.



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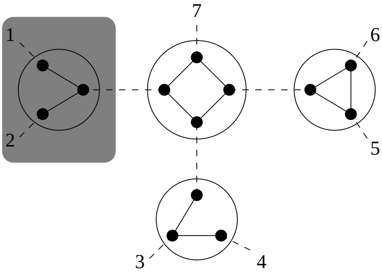
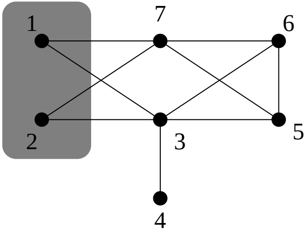
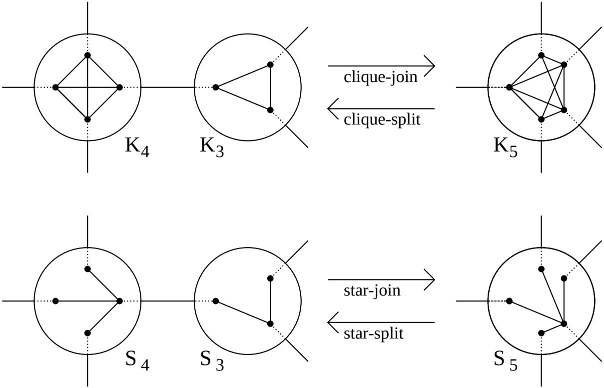




A graph is *prime* if it has no split.
The stars and cliques are called *degenerate*.



Split decomposition [Cunningham'82 reformulated]

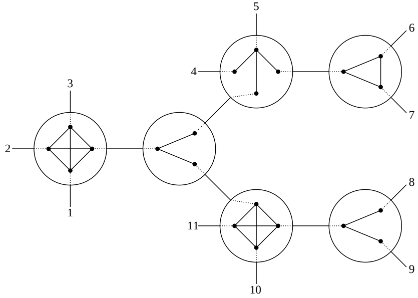
For any connected graph G, there exists a unique graph-labelled tree $(T, \mathcal{F})$ with a minimun number of nodes such that

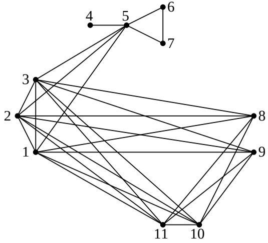
1. $G = G_S(T, \mathcal{F})$,
2. any graph of $\mathcal{F}$ is prime or degenerate for the split decomposition.

$\rightarrow$ We note $(T, \mathcal{F}) = ST(G)$ the *split tree* of G

Distance hereditary graph

A graph is *distance hereditary* if and only if it is totally decomposable for the split decomposition, i.e. its split tree is labelled by cliques and stars.





An intersection model for DH graphs [Gioan and Paul '07]

The *accessibility set* of a leaf a in a clique-star labelled tree is the set of paths (a, b) with b a leaf accessible from a.

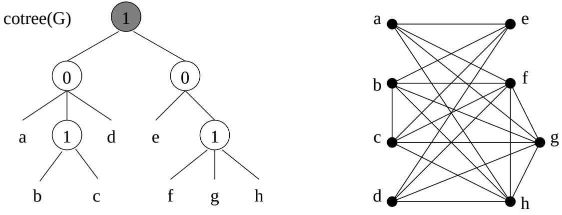
A distance hereditary graph is the intersection graph of a family of accessibility sets of leaves in a set of clique-star labelled trees.

answers a question by Spinrad

Particular case of cographs

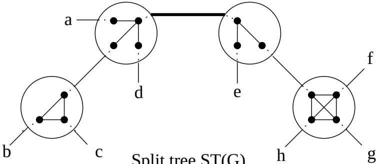
The cographs form the particular case where the centers of all stars are directed towards a **root** of the split tree.

1 = clique
0 = stable

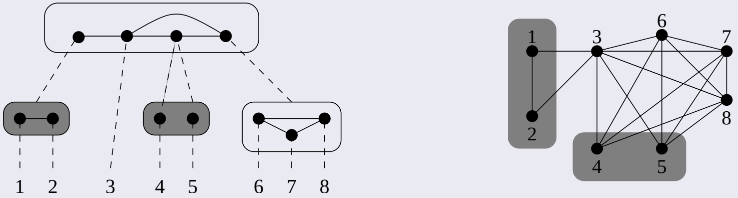


Cograph G

1 = clique (except root)
0 = star (towards root)



Split tree ST(G)



Modules

A subset of vertices M of a graph $G = (V, E)$ is a **module** iff

$$\forall x \in V \setminus M, \text{ either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset$$

Split decomposition

Degenerate graphs

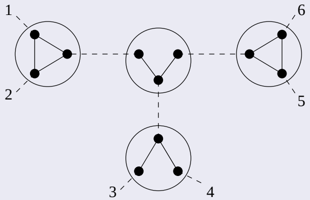
- cliques and stars

Totally decomposable graphs

- Distance hereditary grahs

Unrooted tree decomposition

- [Cunningham 82]



Modular decomposition

Degenerate graphs

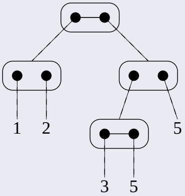
- cliques and stables

Totally decomposable graphs

- Cographs

Rooted tree decomposition

- [Gallai 67]



1. Revisiting split decomposition

2. Vertex modification of DH graphs

3. Relations with other works

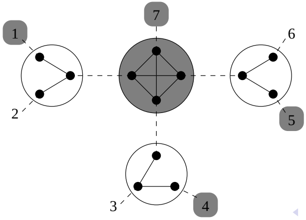
Theorem (Gioan and Paul 07)

Let $G = (V, E)$ be a distance hereditary (DH) graph. It can be tested in

- $O(|S|)$ *whether* $G + (x, S)$, *with* $x \notin E$ *and* $N(x) = S$, *is a DH graph;*
- $O(|S|)$ *whether* $G - x$, *with* $S = N(x)$, *is a DH graph;*

Let $(T, \mathcal{F})$ be a graph-labelled tree, and S be a subset of leaves of T. A node u of $T(S)$ is:

- **fully-accessible** by S if any subtree of $T - u$ contains a leaf of S;



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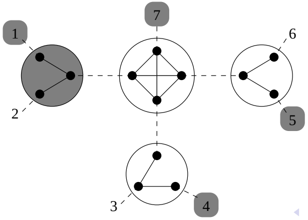
- **fully-accessible** by S if any subtree of $T - u$ contains a leaf of S;
- **singly-accessible** by S if it is a star-node and exactly two subtrees of $T - u$ contain a leaf $l \in S$ among which the subtree containing the neighbor v of u such that $\rho_u(uv)$ is the centre of G_u;



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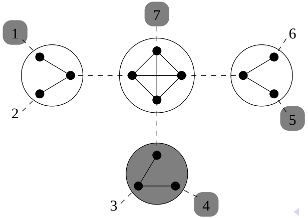
- **partially-accessible** otherwise



Theorem (DH incremental characterization [Gioan, Paul '07])

Let G be a connected DH graph and $ST(G) = (T, \mathcal{F})$ be its split tree. Then $G + (x, S)$ is a DH graph if and only if:

1. *At most one node of $T(S)$ is partially-accessible.*

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3. *If there exists a partially-accessible node u, then any star node $v \neq u$ of $T(S)$ is oriented towards u if and only if it is fully-accessible.*

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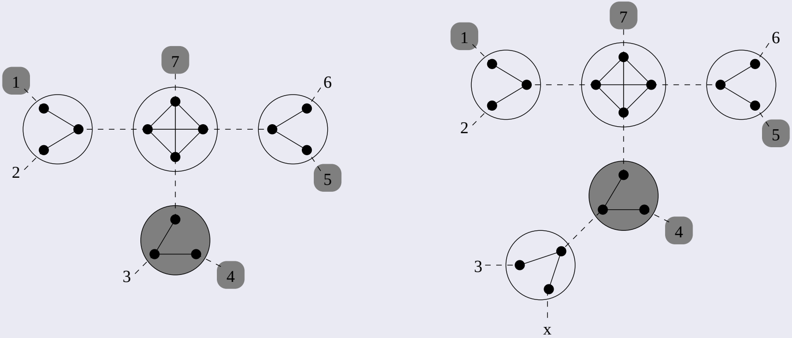
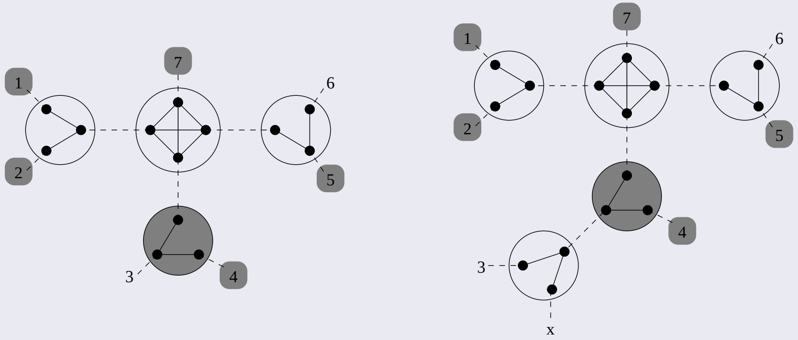
3. *If there exists a partially-accessible node u, then any star node $v \neq u$ of $T(S)$ is oriented towards u if and only if it is fully-accessible.*
4. *Otherwise, there exists a tree-edge e of $T(S)$ towards which any star node of $T(S)$ is oriented if and only if it is fully-accessible.*



The insertion fails: the two singly-accessible nodes are oriented towards the partially-accessible node !



The insertion succeeds: in $G_S(T, \mathcal{F})$, we have $N(x) = S$

Insertion algorithm

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Complexity

1. $O(\mid N(x) \mid)$ dynamic recognition

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Complexity

1. $O(|N(x)|)$ dynamic recognition
2. linear time static recognition

Edge modification of DH graphs

Theorem (Corneil and Tedder 06)

Let $G = (V, E)$ be a distance hereditary (DH) graph. It can be tested in

- $O(1)$ *whether* $G + e$*, with* $e \notin E$*, is a DH graph;*
- $O(1)$ *whether* $G - e$*, with* $e \in E$*, is a DH graph.*

Edge modification of DH graphs

Another approach for this result [GP 07]

A simple algorithm for this result is given by graph-labelled trees: consider the word between the two leaves x and y where $e = xy$ with K a clique, L resp. R a star with center towards x resp. y, and S otherwise.

edge insertion $\longrightarrow$ $\longleftarrow$ edge deletion	
$(R)SS(L)$	$(R)LR(L)$
$(R)SK(L)$	$(R)LK(L)$
$(R)KS(L)$	$(R)KR(L)$
$(R)S(L)$	$(R)K(L)$

Vertex modification of cographs

Theorem (Corneil, Pearl and Stewart '85)

Let $G = (V, E)$ be a cograph. It can be tested in

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Vertex modification of cographs

Theorem (Cograph incremental characterization [CPS'85])

Let G be a cograph and $MD(G) = (T, \mathcal{F})$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph if and only if:

1. *At most one node of $T(S)$ is partially-accessible.*
2. *Any series node of $T(S)$ is either fully or partially-accessible.*
3. *If a partially-accessible node u exists, then a parallel node $v \neq u$ of $T(S)$ is a descendant of u if and only if it is fully-accessible.*
4. *Otherwise, a tree-edge $e = uw$ of $T(S)$ exists such that a parallel node $v \neq u$ of $T(S)$ is a descendant of u if and only if it is fully-accessible.*

Another approach for this result [GP 07]

This result is equivalent to test the insertion/deletion in DH graphs, with the supplementary condition that the split tree is rooted.

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THANKS!