Orientations of Simplices Determined by Orderings on the Coordinates of their Vertices*

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Abstract

We address the problem of testing when orderings on coordinates of n points in an $(n-1)$-dimensional affine space, one ordering for each coordinate, suffice to determine if these points are the vertices of a simplex (i.e. are affinely independent), and the orientation of this simplex, independently of the real values of the coordinates. In other words, we want to know when the sign (or the non-nullity) of the determinant of a matrix whose columns correspond to affine points is determined by orderings given on the values on each row. We completely solve the problem in dimensions 2 and 3, providing a direct combinatorial characterization, together with a formal calculus method, that can be seen also as a decision algorithm, which relies on testing the existence of a suitable inductive cofactor expansion of the determinant. We conjecture that the method we use generalizes in higher dimensions. The motivation for this work is to be part of a study on how oriented matroids encode shapes of 3-dimensional objects, with applications in particular to the analysis of anatomical data for physical anthropology and clinical research.

Keywords: simplex orientation, determinant sign, chirotope, coordinate ordering, combinatorial algorithm, formal calculus, oriented matroid, 3D model.

1 Introduction

We consider n points in an $(n-1)$-dimensional real affine space. For each of the $n-1$ coordinates, an ordering is given, applied on the n values of the points with respect to this coordinate. We address the problem of testing if these points are the vertices of a simplex (i.e. are affinely independent, i.e. do not belong to a same hyperplane), and of determining the orientation of this simplex, assuming only that their coordinates satisfy the given orderings, independently of their real values.

More formally, we consider the following generic matrix (where each e_i is the label of a point and each b_i is the index of a coordinate)

$$M_{\mathcal{E},\mathcal{B}} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_{e_1,b_1} & x_{e_2,b_1} & \dots & x_{e_n,b_1} \\ x_{e_1,b_2} & x_{e_2,b_2} & \dots & x_{e_n,b_2} \\ \vdots & \vdots & & \vdots \\ x_{e_1,b_{n-1}} & x_{e_2,b_{n-1}} & \dots & x_{e_n,b_{n-1}} \end{pmatrix}$$

together with orderings given on the values on each row, and we want to know when the sign (or the non-nullity) of its determinant is determined by these orderings only.

Equivalently, we consider the above formal matrix and the affine algebraic variety of $\mathbb{R}^{n\times(n-1)}$ whose equation is $det(M_{\mathcal{E},\mathcal{B}}) = 0$. Then we look for which regions of $\mathbb{R}^{n\times(n-1)}$, delimited by the hyperplanes $x_{e_i,b_k} = x_{e_j,b_k}$, for all $1 \leq i,j \leq n$ and all $1 \leq k \leq n-1$, have a non-empty intersection with this variety (obviously, regions delimited by these hyperplanes are in canonical bijection with coordinate linear orderings).

In this paper, we completely solve the problem in dimensions 2 (Section 4) and 3 (Section 5), providing a direct combinatorial characterization, together with a combinatorial formal calculus method, that can be seen also as a decision algorithm, to test if the orientation is determined or not. More precisely, our method relies on testing the existence of a suitable inductive cofactor expansion of the determinant, from which a combinatorial formal calculus is able to determine the sign of the determinant. We conjecture that such a characterization generalizes in higher dimensions (Section 3).

The motivation for this work is to be part of a study on how oriented matroids [1] encode shapes of 3-dimensional objects, with applications in particular to the analysis of anatomical data for physical anthropology and clinical research [3]. In these applications, we usually study a set of models belonging to a given group (e.g. a set of 3D landmark points located on human or primate skulls) and we look for the significant properties encoded by the combinatorial structure. The above results allow us to distinguish chirotopes (i.e. simplex orientations) which are determined by the "generic" form (e.g. in any skull, the mouth is below the eyes) from those which are specific to anatomical variations. As an example, some results on anatomical 3D data are presented in Section 6.

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2 Formalism and terminology of the problem

We warn the reader that we use on purpose a rather abstract formalism throughtout the paper (formal variables instead of real values, indices within arbitrary ordered sets instead of integers). This will allow us to get easier and non-ambiguous constructions and definitions.

Let us fix an (ordered) set $\mathcal{E} = \{e_1, \ldots, e_n\}$, with size n, of *labels*, and an (ordered) canonical basis $\mathcal{B} = \{b_1, \ldots, b_{n-1}\}$, with size $n-1$, of the $(n-1)$-dimensional real space $\mathbb{R}^{n-1}$. We denote $M_{\mathcal{E},\mathcal{B}}$ - or M for short when the context is clear - the formal matrix whose entry at column i and row $j+1$, for $1 \leq i \leq n$ and $1 \leq j \leq n-1$, is the formal variable x_{e_i,b_j}, as represented in Section 1. The determinant $det(M_{\mathcal{E},\mathcal{B}})$ of this formal matrix is a multivariate polynomial on these formal variables, and the main object studied in this paper.

Let $\mathcal{P}$ be a set of n points, labeled by $\mathcal{E}$, in $\mathbb{R}^{n-1}$ considered as an affine space. We denote $M_{\mathcal{E},\mathcal{B}}(\mathcal{P})$ - or $M(\mathcal{P})$ for short - the matrix whose columns give the coordinates of points in $\mathcal{P}$ w.r.t. the basis $\mathcal{B}$, that is specifying real values for the formal variables x_{e_i,b_j} in the matrix $M_{\mathcal{E},\mathcal{B}}$ above. For $e \in \mathcal{E}$ and $b \in \mathcal{B}$, we denote $x_{e,b}(\mathcal{P})$ the real value given to the formal variable $x_{e,b}$ in $\mathcal{P}$. We may sometimes denote $x_{e,b}$ for short instead of $x_{e,b}(\mathcal{P})$ when the context is clear. We call *orientation* of $\mathcal{P}$, or *chirotope* of $\mathcal{P}$ in the oriented matroid terminology, the sign of $det(M(\mathcal{P}))$, belonging to the set $\{+, -, 0\}$. It is the sign of the real evaluation of the polynomial $det(M)$ at the real values given by $\mathcal{P}$. This sign is not equal to zero if and only if $\mathcal{P}$ forms a *simplex* (basis of the affine space).

We call *ordering configuration on* $(\mathcal{E}, \mathcal{B})$ - or *configuration* for short - a set $\mathcal{C}$ of $n-1$ orderings $<_{b_1}, \ldots, <_{b_{n-1}}$ on $\mathcal{E}$, one ordering for each element of $\mathcal{B}$. In general, such an ordering can be any partial ordering. If every ordering $<_b$, $b \in \mathcal{B}$, is linear, then $\mathcal{C}$ is called a *linear ordering configuration.* An element of $\mathcal{E}$ which is the smallest or the greatest in a linear ordering on $\mathcal{E}$ is called *extreme* in this ordering. We call *reversion* of an ordering the ordering obtained by reversing every inequality in this ordering.

Given a configuration $\mathcal{C}$ on $(\mathcal{E}, \mathcal{B})$ and a set of n points $\mathcal{P}$ labeled by $\mathcal{E}$, we say that $\mathcal{P}$ *satisfies* $\mathcal{C}$ if, for all $b \in \mathcal{B}$, the natural order (in the set of real numbers $\mathbb{R}$) of the coordinates b of the points in $\mathcal{P}$ is compatible with the ordering $<_b$ of $\mathcal{C}$, that is precisely :

$$\forall b \in \mathcal{B}, \quad \forall e, f \in \mathcal{E}, \quad e <_b f \ \Rightarrow \ x_{e,b}(\mathcal{P}) < x_{f,b}(\mathcal{P}).$$

One may observe that the set of all $\mathcal{P}$ satisfying $\mathcal{C}$ forms a convex polyhedron, more precisely: a (full dimensional) region of the space $\mathbb{R}^{n \times (n-1)}$, delimited by some hyperplanes of equations of type $x_{e,b} = x_{f,b}$ for $b \in \mathcal{B}$ and $e, f \in \mathcal{E}$.

We say that a configuration $\mathcal{C}$ is *fixed* if all the sets of points $\mathcal{P}$ satisfying $\mathcal{C}$ form a simplex and have the same orientation. In this case, the sign of $det(M(\mathcal{P}))$ is the same for all $\mathcal{P}$ satisfying $\mathcal{C}$. Then we call *sign of* $det(M)$ this sign, belonging to $\{\boxed{+}, \boxed{-}\}$ accordingly, and we denote it $\sigma_{\mathcal{C}}(det(M))$. If $\mathcal{C}$ is *non-fixed*, then its *sign* is $\sigma_{\mathcal{C}}(det(M)) = \boxed{\pm}$.

The following lemma is easy to prove.

Lemma 1 *The following propositions are equivalent:*

(a) The configuration $\mathcal{C}$ is non-fixed, that is $\sigma_{\mathcal{C}}(det(M)) = \boxed{\pm}$.

(b) There exist two sets of points $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfying $\mathcal{C}$ and forming simplices that do not have the same orientation, that is $det(M(\mathcal{P}_1)) > 0$ and $det(M(\mathcal{P}_2)) < 0$;

(c) There exists a set of points $\mathcal{P}$ satisfying $\mathcal{C}$ and such that the points of $\mathcal{P}$ belong to one hyperplane, that is $det(M(\mathcal{P})) = 0$.

Two configurations on $(\mathcal{E}, \mathcal{B})$ are called *equivalent* if they are equal up to a permutation of $\mathcal{B}$, a permutation of $\mathcal{E}$ (relabelling), and some reversions of orderings (symmetries from the geometrical viewpoint). Note that changing a configuration into an equivalent one comes, in a matricial setting, to change the orderings of rows, of columns, and to multiply some rows by -1. Obviously those operations do not change the non-nullity of the determinant, hence two equivalent configurations are fixed or non-fixed simultaneously.

Now, given an ordering configuration $\mathcal{C}$, the aim of the paper is to determine if $\mathcal{C}$ is fixed or non-fixed.

3 Computable fixity criteria and conjectures

3.1 From partial orderings to linear orderings

We recall that a linear extension of an ordering on a set $\mathcal{E}$ is a linear ordering on $\mathcal{E}$ compatible with this ordering. A *linear extension* of an ordering configuration $\mathcal{C}$ on $(\mathcal{E}, \mathcal{B})$ is a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$ obtained by replacing each ordering on $\mathcal{E}$ in C by one of its linear extensions.

Lemma 2 *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$. If there exists a set $\mathcal{P}$ of n points satisfying $\mathcal{C}$ and contained in an hyperplane, then there exists a set of n points $\mathcal{P}'$ contained in an hyperplane and a linear extension $\mathcal{C}'$ of $\mathcal{C}$ satisfied by $\mathcal{P}'$.*

From the previous (easy) lemma, we get (directly) the above proposition.

Proposition 1 *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$. The configuration $\mathcal{C}$ is non-fixed if and only if there exists a non-fixed linear extension of $\mathcal{C}$. The configuration $\mathcal{C}$ is fixed if and only if every linear extension of $\mathcal{C}$ is fixed.*

The above result allows to test only the fixity of linear ordering configurations to deduce the fixity of any configuration. In what follows, we will concentrate on linear ordering configurations.

3.2 Formal fixity

Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$. We consider formal expression of type $x_{e,b} - x_{f,b}$ for $e, f \in \mathcal{E}$, $e \neq f$, and $b \in \mathcal{B}$, which we may sometimes denote $x_{e-f,b}$ for short. Such a formal expression gets a *formal sign w.r.t.* $\mathcal{C}$ denoted $\sigma_{\mathcal{C}}(x_{e,b} - x_{f,b})$ and belonging to $\{\boxed{+}, \boxed{-}\}$, the following way:

$$\begin{aligned} \sigma_{\mathcal{C}}(x_{e,b} - x_{f,b}) &= \boxed{+} \quad \text{if} \quad f <_b e; \\ \sigma_{\mathcal{C}}(x_{e,b} - x_{f,b}) &= \boxed{-} \quad \text{if} \quad e <_b f. \end{aligned}$$

Recall that the polynomial $det(M_{\mathcal{E},\mathcal{B}})$ is a multivariate polynomial on variables $x_{e,b}$ for $b \in \mathcal{B}$ and $e \in \mathcal{E}$. Assume a particular formal expression of $det(M_{\mathcal{E},\mathcal{B}})$ is a sum of multivariate monomials where each variable is replaced by some $x_{e,b} - x_{f,b}$, for $b \in \mathcal{B}$ and $e, f \in \mathcal{E}$. Various expressions of this type can be obtained by suitable transformations and determinant cofactor expansions from the matrix M, as we will do more precisely below. This particular expression of $det(M_{\mathcal{E},\mathcal{B}})$ gets a *formal sign w.r.t.* $\mathcal{C}$ belonging to $\{\boxed{+}, \boxed{-}, \boxed{?}\}$, by replacing each expression of type $x_{e,b} - x_{f,b}$ with its formal sign $\sigma_{\mathcal{C}}(x_{e,b} - x_{f,b})$ and applying the following formal calculus rules:

$$\begin{array}{ccccc} \boxed{+} \cdot \boxed{+} & = & \boxed{-} \cdot \boxed{-} & = & \boxed{+}, \\ \boxed{+} \cdot \boxed{-} & = & \boxed{-} \cdot \boxed{+} & = & \boxed{-}, \\ \boxed{+} + \boxed{+} & = & \boxed{+} - \boxed{-} & = & \boxed{+}, \\ \boxed{-} + \boxed{-} & = & \boxed{-} - \boxed{+} & = & \boxed{-}, \\ \boxed{+} + \boxed{-} & = & \boxed{-} + \boxed{+} & = & \boxed{?}, \end{array}$$

and the result of any operation involving a $\boxed{?}$ term or factor is also $\boxed{?}$.

We say that $\mathcal{C}$ is *formally fixed* if $det(M_{\mathcal{E},\mathcal{B}})$ has such a formal expression whose formal sign is not $\boxed{?}$.

Example. Consider the following matrix $M = M_{\mathcal{E},\mathcal{B}}$ for $\mathcal{E} = \{a, b, c\}$ and $\mathcal{B} = \{1, 2\}$:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ x_{a,1} & x_{b,1} & x_{c,1} \\ x_{a,2} & x_{b,2} & x_{c,2} \end{pmatrix}$$

and consider the configuration $\mathcal{C}$ defined by:

$$\begin{array}{ccccc} a & <_1 & b & <_1 & c \\ b & <_2 & c & <_2 & a \end{array}$$

A formal expression of $det(M)$ is:

$$det(M) = x_{b-a,1} \cdot x_{c-a,2} - x_{b-a,2} \cdot x_{c-a,1}$$

whose formal sign w.r.t. $\mathcal{C}$ is

$$\boxed{+} \cdot \boxed{-} - \boxed{-} \cdot \boxed{+} \quad = \quad \boxed{?}.$$

Another formal expression of $det(M)$ is:

$$det(M) = x_{b-a,1} \cdot x_{c-b,2} - x_{b-a,2} \cdot x_{c-b,1}$$

whose formal sign w.r.t. $\mathcal{C}$ is

$$\boxed{+} \cdot \boxed{+} - \boxed{-} \cdot \boxed{+} \quad = \quad \boxed{+}.$$

This second expression shows that $\mathcal{C}$ is formally fixed.

Observation 1 *If $\mathcal{C}$ is formally fixed, then $\mathcal{C}$ is fixed.*

More precisely, given an expression as above whose formal sign w.r.t. $\mathcal{C}$ is $\boxed{+}$ or $\boxed{-}$, the evaluation of this determinant for any set of real values $\mathcal{P}$ satisfying $\mathcal{C}$ necessarily provides a real number whose sign is consistent with the formal sign of this expression. In this case, this resulting sign does not depend on the chosen expression as soon as it is not $\boxed{?}$, and $\sigma_C(det(M))$ equals this sign.

Conversely, one may wonder if for every fixed configuration there would exist a suitable expression of the determinant showing formally that $\mathcal{C}$ is fixed by this way. That is, equivalently: if every formal expression of $det(M_{\mathcal{E},\mathcal{B}})$ has formal sign $\boxed{?}$, then $\sigma_C(det(M)) = \boxed{\pm}$. We strongly believe in this result, which we state as a conjecture, and which we will prove for $n \leq 4$.

Conjecture 1 *Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$. Then $\mathcal{C}$ is fixed if and only if $\mathcal{C}$ is formally fixed.*

3.3 Formal fixity by expansion

Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$, and $\mathcal{E}' = \mathcal{E} \setminus \{e\}$, $\mathcal{B}' = \mathcal{B} \setminus \{b\}$ for some $e \in \mathcal{E}$, $b \in \mathcal{B}$. We call *configuration induced by $\mathcal{C}$ on $(\mathcal{E}', \mathcal{B}')$* the configuration on $(\mathcal{E}', \mathcal{B}')$ obtained by restricting every ordering $<_{b'}$, $b' \in \mathcal{B}'$, of $\mathcal{C}$ to $\mathcal{E}'$. Moreover, we say that *all the configurations induced by $\mathcal{C}$ on $\mathcal{E}'$ are fixed* if, for every $b \in \mathcal{B}$, the configuration induced by $\mathcal{C}$ on $(\mathcal{E}', \mathcal{B} \setminus \{b\})$ is a fixed configuration.

Let $M = M_{\mathcal{E},\mathcal{B}}$ as previously, with $\mathcal{E} = \{e_1, ..., e_n\}_<$ and $\mathcal{B} = \{b_1, ..., b_{n-1}\}_<$. Let $e_i, e_j \in \mathcal{E}$, with $e_i \neq e_j$. Consider the matrix obtained from M by substracting the j-th column (corresponding to e_j), from the i-th column (corresponding to e_i), that is:

$$\begin{pmatrix} 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ x_{e_1,b_1} & \dots & x_{e_{i-1},b_1} & x_{e_i,b_1} - x_{e_j,b_1} & x_{e_{i+1},b_1} & \dots & x_{e_n,b_1} \\ x_{e_1,b_2} & \dots & x_{e_{i-1},b_2} & x_{e_i,b_2} - x_{e_j,b_2} & x_{e_{i+1},b_2} & \dots & x_{e_n,b_2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{e_1,b_{n-1}} & \dots & x_{e_{i-1},b_{n-1}} & x_{e_i,b_{n-1}} - x_{e_j,b_{n-1}} & x_{e_{i+1},b_{n-1}} & \dots & x_{e_n,b_{n-1}} \end{pmatrix}$$

The determinant of this matrix equals $det(M)$. The cofactor expansion formula for the determinant of this matrix with respect to its i-th column yields:

$$det(M_{\mathcal{E},\mathcal{B}}) = \sum_{k=1}^{n-1} (-1)^{i+k+1} \cdot (x_{e_i,b_k} - x_{e_j,b_k}) \cdot det\big(M_{\mathcal{E}\setminus\{e_i\},\mathcal{B}\setminus\{b_k\}} \big)$$

which we call *expression of $det(M)$ by expansion with respect to (e_i, e_j)*.

Then the above particular expression of $det(M)$ gets a *formal sign* w.r.t. $\mathcal{C}$ the following way. First, replace each expression of type $x_{e,b} - x_{f,b}$ with its formal sign w.r.t. $\mathcal{C}$ in $\{\boxed{+}, \boxed{-}\}$, and replace each $det(M_{\mathcal{E}\setminus\{e_i\},\mathcal{B}\setminus\{b_k\}})$, $1 \leq k \leq n-1$, with its sign $\sigma_{\mathcal{C}_k}(det(M_{\mathcal{E}\setminus\{e_i\},\mathcal{B}\setminus\{b_k\}})) \in \{\boxed{+}, \boxed{-}, \boxed{\pm}\}$, where $\mathcal{C}_k$ is the configuration induced by $\mathcal{C}$ on $(\mathcal{E} \setminus \{e_i\}, \mathcal{B} \setminus \{b_k\})$. This leads to the formal expression:

$$\sum_{k=1}^{n-1} (-1)^{i+k+1} \cdot \sigma_{\mathcal{C}}(x_{e_i,b_k} - x_{e_j,b_k}) \cdot \sigma_{C_k}\Big(det\big(M_{\mathcal{E}\setminus\{e_i\},\mathcal{B}\setminus\{b_k\}} \big)\Big),$$

Then, provide the formal sign of this expression by using the same formal calculus rules as previously, completed with the following one:

$$\boxed{+} \cdot \boxed{\pm} = \boxed{-} \cdot \boxed{\pm} = \boxed{?}.$$

If there exists such an expression of $det(M)$ by expansion whose formal sign is $\boxed{+}$ or $\boxed{-}$, then $\mathcal{C}$ is called *formally fixed by expansion.*

Observation 2 *If $\mathcal{C}$ is formally fixed by expansion, then $\mathcal{C}$ is fixed.*

The above observation is similar to Observation 1: if $\mathcal{C}$ is formally fixed by expansion then $\sigma_{\mathcal{C}}(det(M))$ is given as the formal sign of any expression certifying that $\mathcal{C}$ is formally fixed by expansion. Notice that if $\mathcal{C}$ is formally fixed by expansion then all those configurations $\mathcal{C}_k$ induced by $\mathcal{C}$ are fixed, since one must have $\sigma_{\mathcal{C}_k}(det(M_{\mathcal{E}\setminus\{e_i\},\mathcal{B}\setminus\{b_k\}})) \in \{\boxed{+}, \boxed{-}\}$.

Conjecture 2 *Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$. Then $\mathcal{C}$ is fixed if and only if $\mathcal{C}$ is formally fixed by expansion.*

We point out that if Conjecture 1 is true in dimension $n-1$, then Conjecture 2 in dimension n implies Conjecture 1 in dimension n. Indeed, in this case, the fixity of the $(n-1)$-dimensional configurations corresponding to cofactors can be determined using formal expressions.

Finally, the point of this paper is to deal with the property of being formally fixed by expansion as an inductive criterion for fixity. In what follows, we will prove Conjecture 2 for $n = 4$, together with more precise and direct characterizations in this case.

3.4 A non-fixity criterion

The following Lemma 3 will be our main tool to prove that a configuration is non-fixed. We point out that, when $n = 4$, the sufficient condition for being non-fixed provided by Lemma 3 turns out to be a necessary and sufficient condition (see Theorem 4). However, the authors feel that this equivalence result is too hazardous to be stated as a general conjecture in dimension n.

Lemma 3 *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$. If the configuration $\mathcal{C}'$ induced by $\mathcal{C}$ on $(\mathcal{E} \setminus \{e\}, \mathcal{B} \setminus \{b\})$ for some $e \in \mathcal{E}$ and $b \in \mathcal{B}$ satisfies the following properties: $\mathcal{C}'$ is non-fixed and e is extreme in the ordering $<_b$ of $\mathcal{C}$, then $\mathcal{C}$ is non-fixed.*

4 Results in dimension 2

In this section we fix $n = 3$ and $\mathcal{E} = \{A, B, C\}$. In order to lighten notations of variables $x_{e,b}$ for $e \in \mathcal{E}$ and $b \in \mathcal{B}$, we rather denote:

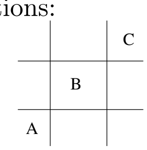
$$M = \begin{pmatrix} 1 & 1 & 1 \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix}$$

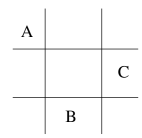
We will denote also $\mathcal{B} = \{x, y\}$ and $<_x, <_y$ the orderings in a configuration.

The following theorems are easy to prove. First it is easy to check that, up to equivalence of configurations, there exist exactly two linear ordering configurations:

$A <_x B <_x C$	$A <_x B <_x C$
$A <_y B <_y C$	$B <_y C <_y A$

They correspond respectively to the following grid representations:





Theorem 1 *Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 3$, $\mathcal{E} = \{A, B, C\}$ and $\mathcal{B} = \{x, y\}$. The following properties are equivalent:*

a) $\mathcal{C}$ is non-fixed;

b) the two orderings on $\mathcal{E}$ in $\mathcal{C}$ are either equal or equal to reversions of each other;

c) up to equivalence, $\mathcal{C}$ is equal to $\begin{matrix} A <_x B <_x C \\ A <_y B <_y C \end{matrix}$

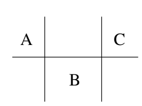
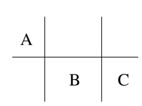
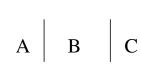
Theorem 2 *Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 3$, $\mathcal{E} = \{A, B, C\}$ and $\mathcal{B} = \{x, y\}$. The folowing properties are equivalent:*

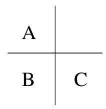
a) $\mathcal{C}$ is fixed;

b) $\mathcal{C}$ is formally fixed;

c) up to equivalence, $\mathcal{C}$ is equal to $\begin{matrix} A <_x B <_x C \\ B <_y C <_y A \end{matrix}$

Now that we have listed fixed and non-fixed linear ordering configurations, we are able to determine all fixed and non-fixed configurations using Proposition 1 (up to equivalence, and omitting those obviously non-fixed for which two elements of $\mathcal{E}$ are comparable in no ordering in the configuration):

$A <_x B <_x C$ $B <_y A$ $B <_y C$		fixed
$A <_x B <_x C$ $B <_y A$ $C <_y A$		non-fixed
$A <_x B <_x C$		non-fixed

$A <_x C$ $B <_x C$ $B <_y A$ $C <_y A$		non-fixed

5 Results in dimension 3

In this section we fix $n = 4$ and $\mathcal{E} = \{A, B, C, D\}$. In order to lighten notations of variables $x_{e,b}$ for $e \in \mathcal{E}$ and $b \in \mathcal{B}$, we rather denote:

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_A & x_B & x_C & x_D \\ y_A & y_B & y_C & y_D \\ z_A & z_B & z_C & z_D \end{pmatrix}$$

We will denote also $\mathcal{B} = \{x, y, z\}$ and $<_x, <_y, <_z$ the orderings in a configuration.

As noticed in Section 3, in order to prove that a configuration $\mathcal{C}$ is formally fixed by expansion, we need to find an element $e \in E$ such that all the configurations induced by $\mathcal{C}$ on $\mathcal{E} \setminus \{e\}$ are fixed. The proposition below characterizes such induced configurations.

Proposition 2 *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 4$, $\mathcal{E} = \{A, B, C, D\}$ and $\mathcal{B} = \{x, y, z\}$. All the configurations induced by $\mathcal{C}$ on $\{A, B, C\}$ are fixed if and only if $\mathcal{C}$ is equivalent to a configuration whose orderings satisfy:*

$$\begin{array}{ccccc} B & <_x & C & <_x & A \\ C & <_y & A & <_y & B \\ A & <_z & B & <_z & C \end{array}$$

We will now state Theorem 3 which is the main result of the paper. Its detailed proof is about five pages long. Briefly, it consists in separating configurations having a triplet such that all induced configurations w.r.t. this triplet are fixed, and the other ones. In the first group, characterized by Proposition 2, we prove a sufficient condition for fixity. Then we prove that every configuration in the first group not satisfying this condition, and every configuration in the second group, is non-fixed, by analysing several cases, and always using Lemma 3 together with Theorem 1. So, it turns out that Lemma 3 completely characterizes non-fixed configurations, which proves also Theorem 4.

Theorem 3 *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 4$, $\mathcal{E} = \{A, B, C, D\}$ and $\mathcal{B} = \{x, y, z\}$. The following propositions are equivalent:*

a) $\mathcal{C}$ is fixed;
b) $\mathcal{C}$ is formally fixed;
c) $\mathcal{C}$ is formally fixed by expansion;
d) up to equivalence, $\mathcal{C}$ satsifies:

$$\begin{array}{ccccc} B & <_x & C & <_x & A \\ C & <_y & A & <_y & B \\ A & <_z & B & <_z & C \end{array}$$

and there exists $X \in \{A, B, C\}$ such that either $X <_b D$ for every $b \in \mathcal{B}$, or $D <_b X$ for every $b \in \mathcal{B}$.

Theorem 4 *Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 4$. Then $\mathcal{C}$ is non-fixed if and only if conditions of Lemma 3 are satisfied, that is: there exist $e \in \mathcal{E}$ and $b \in \mathcal{B}$ such that the configuration $\mathcal{C}'$ induced by $\mathcal{C}$ on $(\mathcal{E} \setminus \{e\}, \mathcal{B} \setminus \{b\})$ is non-fixed and e is extreme in the ordering $<_b$ of $\mathcal{C}$.*

We computed the result provided by Theorem 3 to list the fixed linear ordering configurations when $n = 4$. Up to equivalences, there are exactly 4 such configurations, within 21 linear ordering configurations:

$$\begin{array}{ll} B <_x C <_x A <_x D & B <_x C <_x D <_x A \\ C <_y A <_y B <_y D & C <_y A <_y B <_y D \\ A <_z B <_z C <_z D & A <_z B <_z C <_z D \end{array}$$

$$\begin{array}{ll} B <_x D <_x C <_x A & B <_x C <_x D <_x A \\ C <_y A <_y B <_y D & C <_y D <_y A <_y B \\ A <_z B <_z C <_z D & A <_z B <_z C <_z D \end{array}$$

The interest of the results of this section is to provide a combinatorial characterization as well as an (easily computable) algorithm deciding if a configuration is fixed or not. Also, we point out that our result statements deal with being fixed or not, but not with the exact value $\boxed{+}$ or $\boxed{-}$ of the considered fixed configuration. This sign can be derived easily from the construction stating the fixity. As well, this sign can be obtained by choosing any set of points $\mathcal{P}$ satisfying the configuration and evaluating the sign of the real number $det(M(\mathcal{P}))$. Finally, from the list of fixed linear ordering configurations given above, one may compute the list of all fixed (partial) ordering configurations using Proposition 1, but we do not give this list here.

6 An example from applications to anatomical data

Let us consider ten anatomical landmark points in $\mathbb{R}^3$ chosen by experts on the 3D model of a skull from [2], as shown on Figure 1. We choose a canonical basis $(O, \vec{x}, \vec{y}, \vec{z})$ such that the axis $\vec{x}$ goes from the right of the skull to its left, the axis $\vec{y}$ goes from the bottom of the skull to its top, and the axis $\vec{z}$ goes from the front of the skull to its back. The specificity of this 3D model as being a skull implies that some coordinate ordering relations are satisfied by those points: for instance the point 9 (right internal ear) will always be on the right, above and behind w.r.t. point 5 (right part of the chin). Figures 2 and 3 show respectively those points from the front and from the right of the model, together with a grid representing those coordinate ordering relations. By this way, the ordering configurations are represented on Figures 2 and 3, with $\mathcal{E}$ any set of four points, and $\mathcal{B}$ corresponding to the three axis $\{x, y, z\}$.

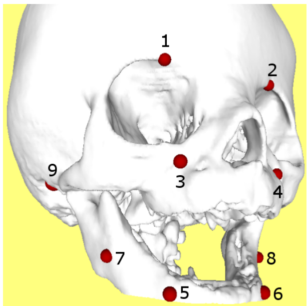


Figure 1: Ten anatomic points on a skull model [2]

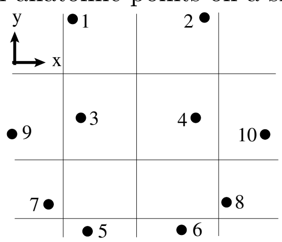


Figure 2: View from the front

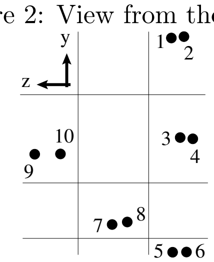


Figure 3: View from the right

We are usually given a set of such models, coming from various individuals (with possibly some pathologies) and species (e.g. primates and humans), by experts of those fields who are interested in characterizing and classifying mathematically those models. In this paper, our aim is to detect which configurations are fixed, independently of the real values of the landmarks, meaning that the relative positions of points satisfying these configurations do not depend on some anatomical variabilities (e.g. being a primate or a human skull), but just on the generic shape of the model (i.e. being a skull).

Example 1. Fixed linear ordering configurations providing a fixed partial ordering configuration: the configuration on $\mathcal{E} = \{2, 5, 8, 9\}$ is fixed.

This configuration is given by the orderings:

$$9 <_x 5 <_x 2 <_x 8$$
$$5 <_y 8 <_y 9 <_y 2$$
$$2 <_z 8 <_z 9 \quad \text{and} \quad 5 <_z 8 <_z 9$$

Its two linear extensions, respectively $\mathcal{C}_1$ and $\mathcal{C}_2$, are the following:

$$\begin{array}{ll} 9 <_x 5 <_x 2 <_x 8 & 9 <_x 5 <_x 2 <_x 8 \\ 5 <_y 8 <_y 9 <_y 2 & 5 <_y 8 <_y 9 <_y 2 \\ 2 <_z 5 <_z 8 <_z 9 & 5 <_z 2 <_z 8 <_z 9 \end{array}$$

Let us write these orderings in another way:

$$\begin{array}{ll} 2 <_z 5 <_z 8 <_z 9 & 5 <_z 2 <_z 8 <_z 9 \\ 5 <_y 8 <_y 9 <_y 2 & 5 <_y 8 <_y 9 <_y 2 \\ 9 <_x 5 <_x 2 <_x 8 & 9 <_x 5 <_x 2 <_x 8 \end{array}$$

By this way, we see that, up to a permutation of $\mathcal{B}$, that is for $\{i, j, k\} = \{x, y, z\}$, and if we choose $A = 9$, $B = 2$, $C = 8$ and $D = 5$, then the orderings in those configurations both satisfy:

$$B <_i C <_i A$$
$$C <_j A <_j B$$
$$A <_k B <_k C$$

as required by Theorem 3. Moreover, for each of these orderings we have D smaller than C (i.e. $5 <_x 8$, $5 <_y 8$, $5 <_z 8$). So, by Theorem 3, those two configurations are fixed. Then, $\mathcal{C}$ is fixed by Proposition 1.

Example 2. A non-fixed ordering configuration implied by a non-fixed linear ordering configuration: the configuration on $\mathcal{E} = \{1, 3, 7, 10\}$ is non-fixed.

It is given by the orderings:

$$\begin{array}{lcl} 7 <_x 3 <_x 10 & \text{and} & 7 <_x 1 <_x 10 \\ 7 <_y 3 <_y 1 & \text{and} & 7 <_y 10 <_y 1 \\ 1 <_z 7 <_z 10 & \text{and} & 3 <_z 7 <_z 10 \end{array}$$

One of its linear extensions is $\mathcal{C}'$:

$$7 <_x 3 <_x 1 <_x 10$$
$$7 <_y 10 <_y 3 <_y 1$$
$$3 <_z 1 <_z 7 <_z 10$$

whose configuration induced on $(\{7, 3, 1\}, \{x, y\})$ is

$$7 <_x 3 <_x 1$$
$$7 <_y 3 <_y 1$$

which is non-fixed by Theorem 1. Then 10 is extreme in the ordering $<_z$ of configuration $\mathcal{C}'$, hence $\mathcal{C}'$ is non-fixed by Lemma 3, and so is $\mathcal{C}$ by Proposition 1.

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