Introduction to treewidth

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Rencontres virtuelles en théorie des graphes
JCRAALMA – 29 mars 2021

Outline of the talk

1. Definition and simple properties
2. Dynamic programming on tree decompositions
   - Two simple algorithms
   - Courcelle's theorem
   - Introduction to parameterized complexity
3. Brambles and duality
4. Computing treewidth

Next section is...

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The multiples origins of treewidth

- 1972: Bertelè and Brioschi (dimension).
- 1976: Halin (S-functions of graphs).
- 1984: Arnborg and Proskurowski (partial k-trees).
- 1984: Robertson and Seymour (treewidth).

A measure of the similarity with a tree

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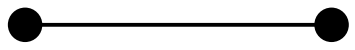

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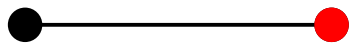

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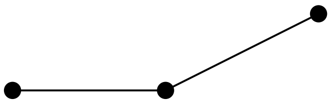

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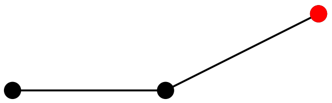

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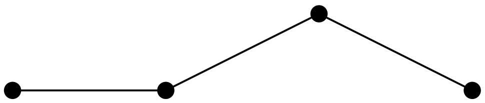

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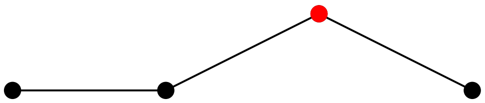

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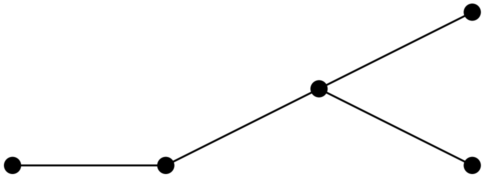

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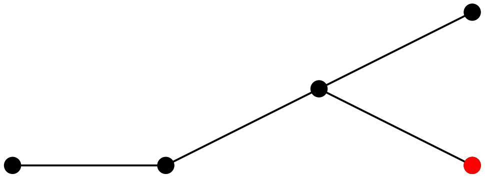

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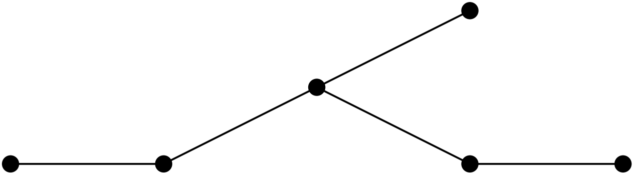

Treewidth via k-trees

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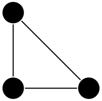

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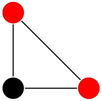

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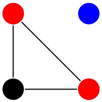

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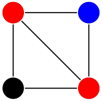

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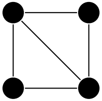

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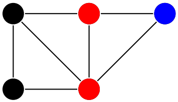

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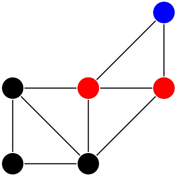

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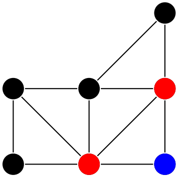

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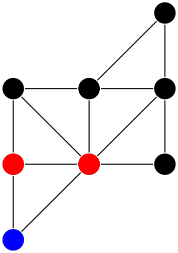

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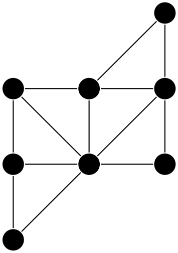

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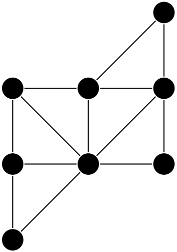

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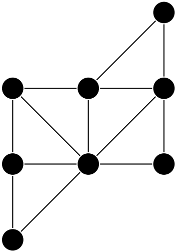

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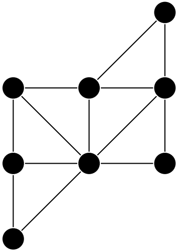

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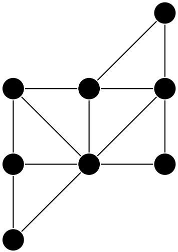

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Construction suggests the notion of tree decomposition: small separators.

An equivalent (and more common) definition of treewidth

- Tree decomposition of a graph G:

  pair $(T, \{X_t \mid t \in V(T)\})$, where
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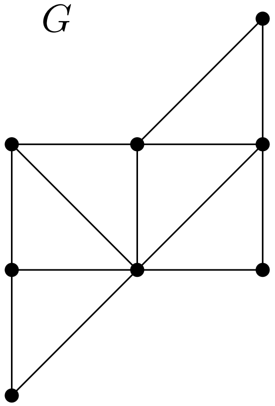

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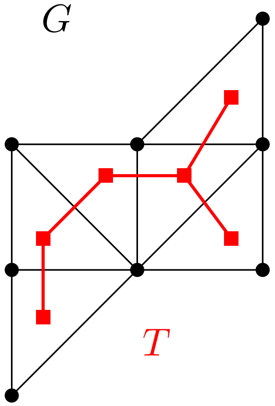

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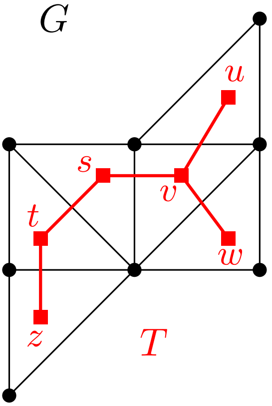

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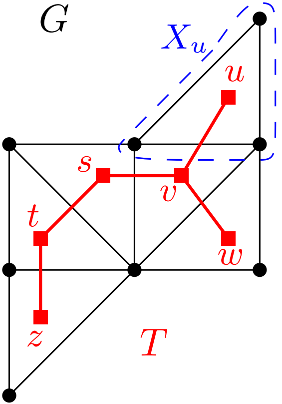

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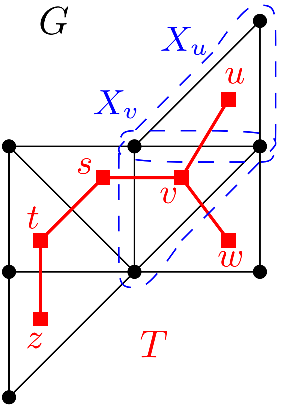

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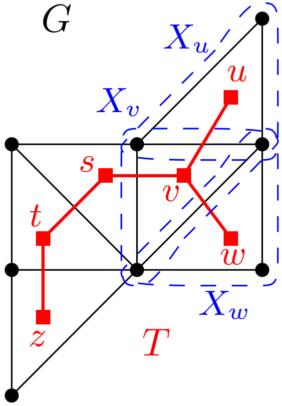

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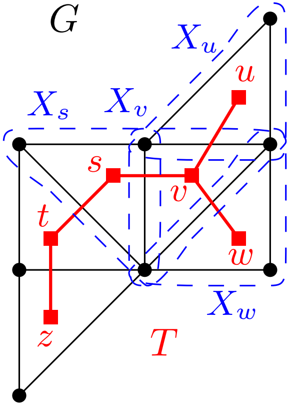

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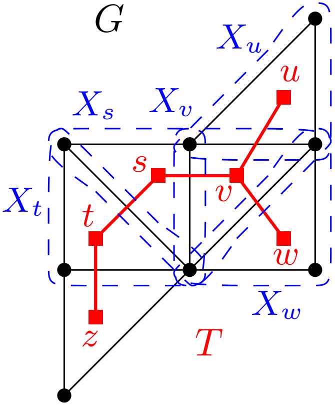
- Treewidth of a graph G, $\mathrm{tw}(G)$: minimum width of a tree decomposition of G.



An equivalent (and more common) definition of treewidth

- Tree decomposition of a graph G:

  pair $(T, \{X_t \mid t \in V(T)\})$, where
  T is a tree, and
  $X_t \subseteq V(G) \;\; \forall t \in V(T)$ (bags),

  satisfying the following:

  - $\bigcup_{t \in V(T)} X_t = V(G)$,
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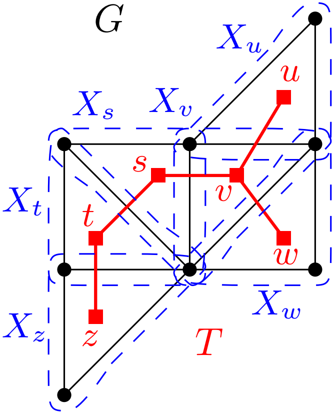

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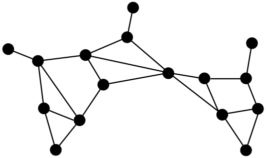

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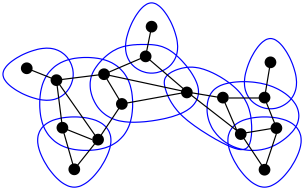

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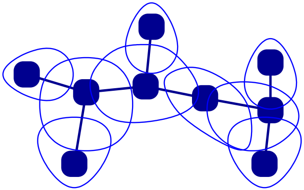

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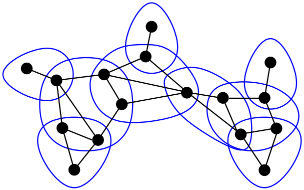

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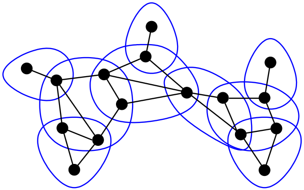

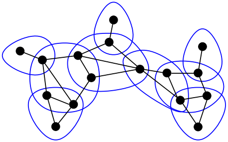

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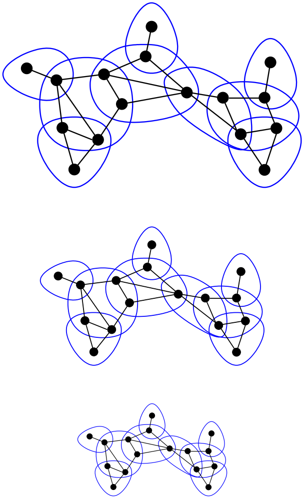

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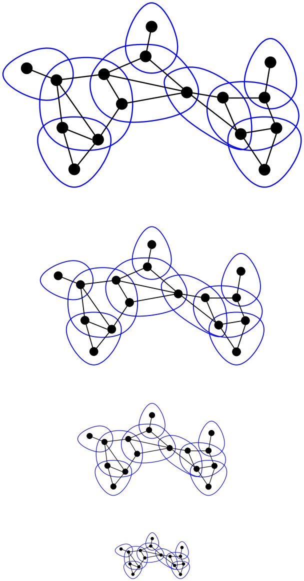

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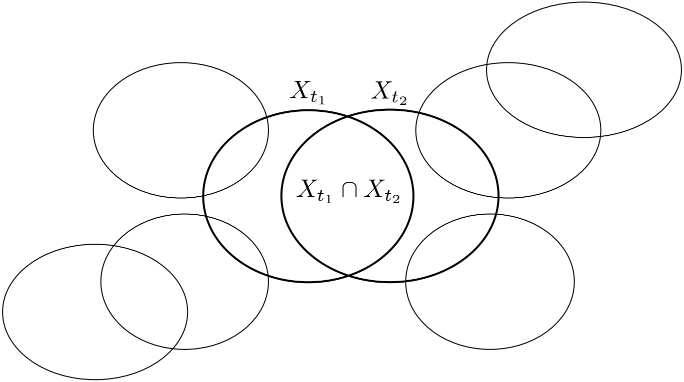

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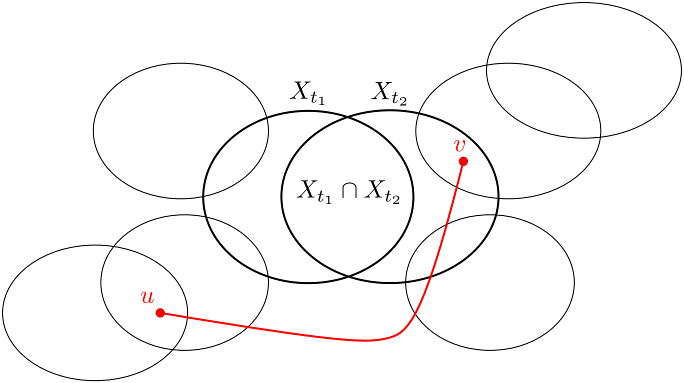

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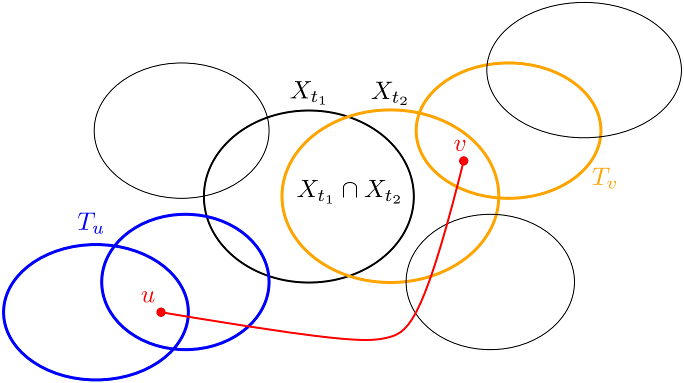

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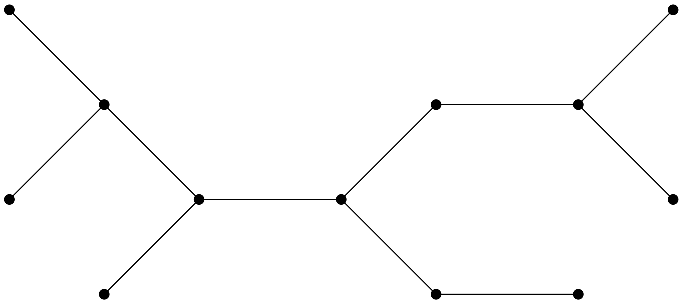

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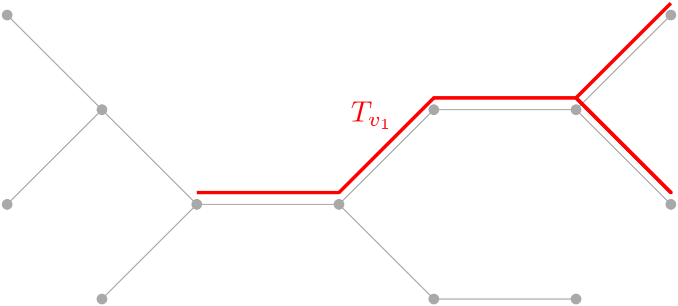

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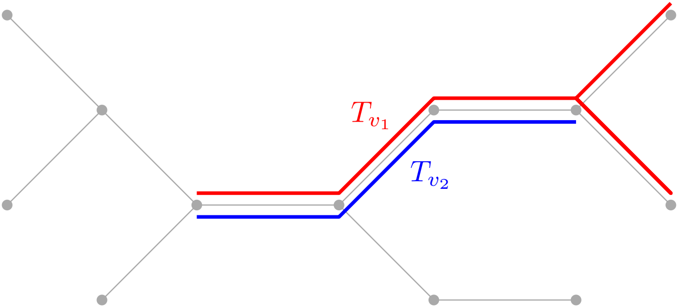

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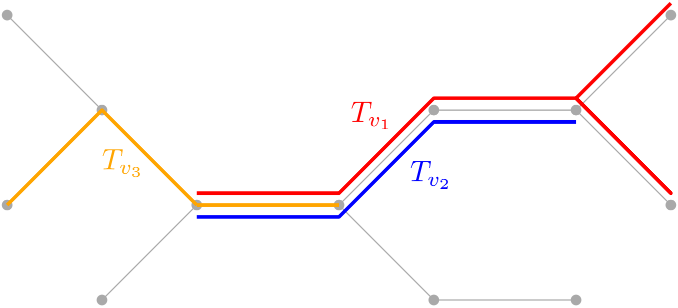

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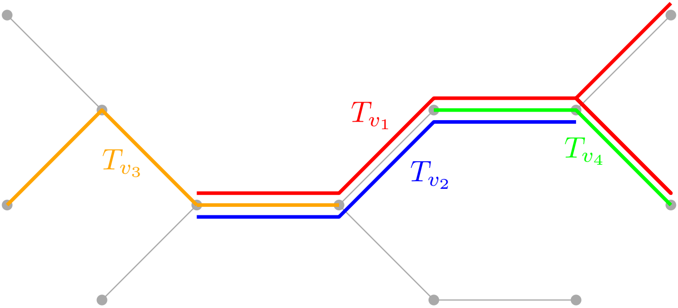

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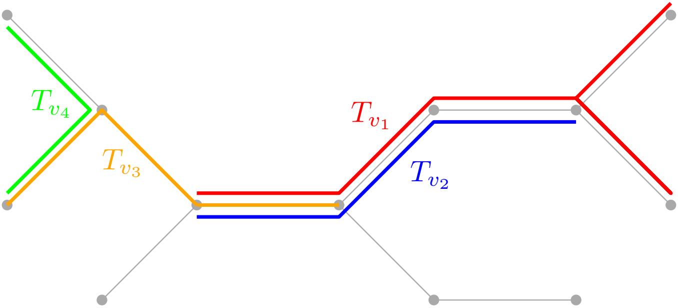

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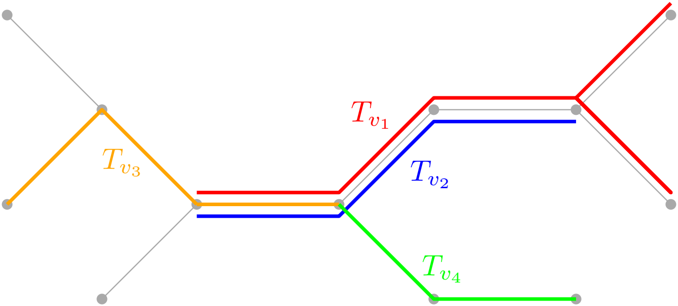

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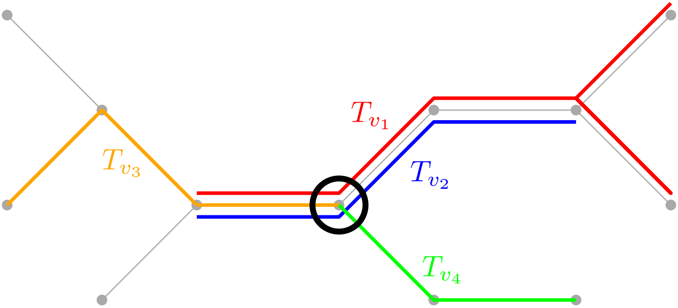

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3. In many practical scenarios, it turns out that the treewidth of the associated graph is small (programming languages, road networks, ...).

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- Two simple algorithms
- Courcelle's theorem
- Introduction to parameterized complexity

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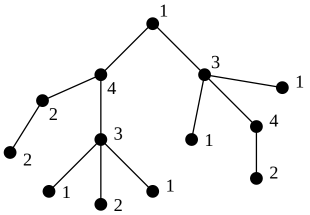
Weighted Independent Set on trees

[slides borrowed from Christophe Paul]



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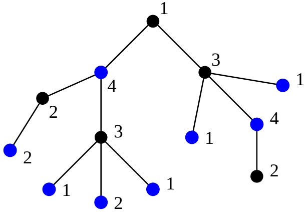

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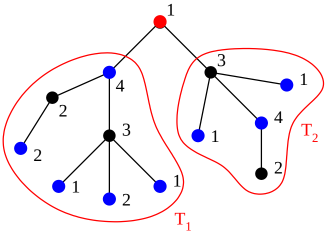

Observations:

1. Every vertex of a tree is a separator.
2. The union of independent sets of distinct connected components is an independent set.

Weighted Independent Set on trees

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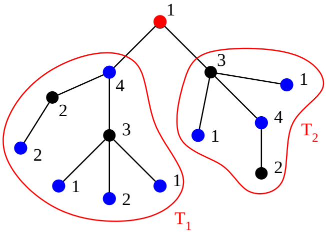

Let x be the root of T, $x_1 \dots x_\ell$ its children, $T_1, \dots T_\ell$ subtrees of $T - x$:

- $wIS(T, x)$: maximum weighted independent set containing x.
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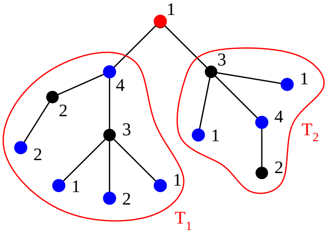

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$$\begin{cases} wIS(T,x) = \omega(x) + \sum_{i \in [\ell]} wIS(T_i, \overline{x_i}) \\ \end{cases}$$

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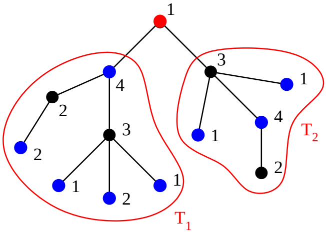

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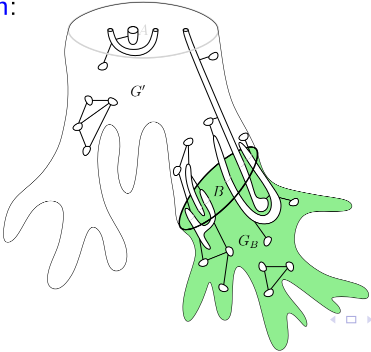
- The way that these partial solutions are defined depends on each particular problem:



[Figure by Valentin Garnero]

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- V_t: all vertices of G appearing in bags that are descendants of t.
- $G_t = G[V_t]$.

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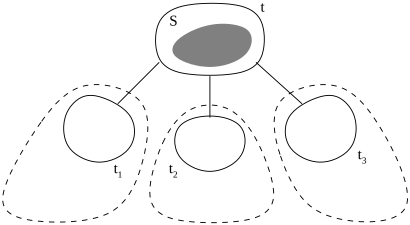

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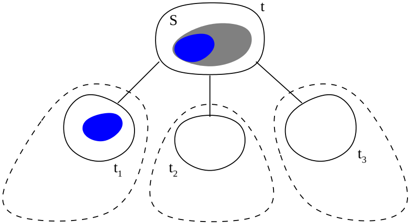

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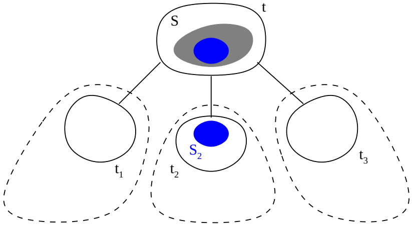

Lemma If $S \subseteq X_t$ and $S_j = S \cap X_{t_j}$, then $|IS(S,t) \cap V_{t_j}| = |IS(S_j,t_j)|$.

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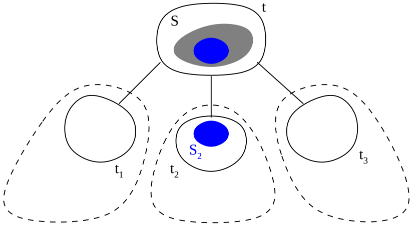

Lemma: If $S \subseteq X_t$ and $S_j = S \cap X_{t_j}$, then $|IS(S,t) \cap V_{t_j}| = |IS(S_j, t_j)|$.

For contradiction: suppose $IS(S,t) \cap V_{t_j}$ is not maximum in G_{t_j}.

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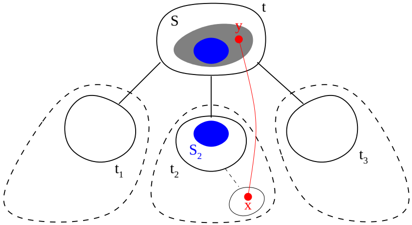

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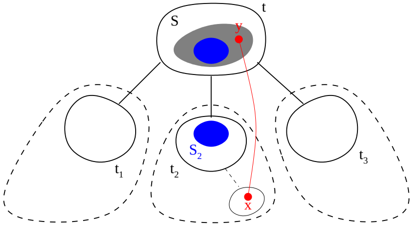

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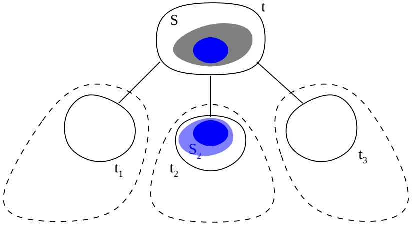
Contradiction! X_{t_j} is not a separator.

INDEPENDENT SET on tree decompositions

Idea of the dynamic programming algorithm:



How to compute $|IS(S,t)|$ from $|IS(S_j^i, t_j)|$, $\forall j \in [\ell]$, $\forall S_j^i \subseteq X_{t_j}$:

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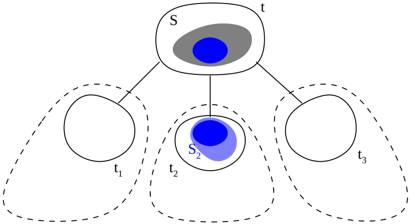

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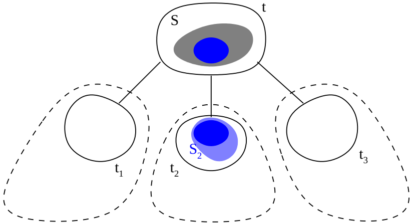

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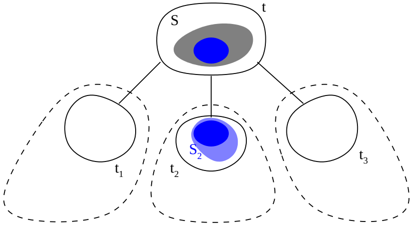

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Independent Set on tree decompositions

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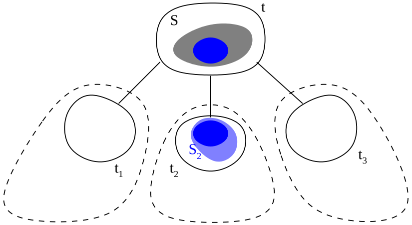

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$$|IS(S,t)| = \left\{ \sum_{i\in[\ell]} \max \begin{array}{l} |S| + \\ \{|IS(S_j^i, t_j)| - |S_j| : \\ S_j^i \cap X_t = S_j \ \wedge \ S_j \subseteq S_j^i \text{ independent}\} \end{array} \right.$$

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★ We have to add the time in order to compute a "good" tree decomposition of the input graph (we discuss this later).

Helpful tool: nice tree decompositions

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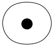

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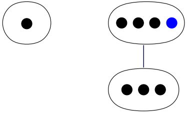

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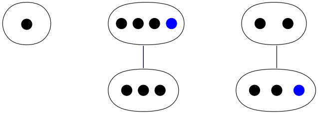

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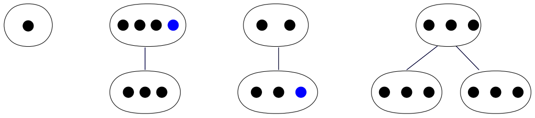

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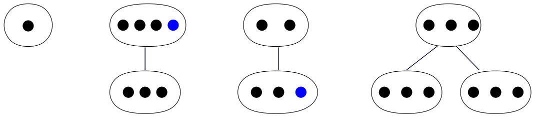

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Lemma

A tree decomposition $(T, \{X_t : t \in T\})$ of width k and x nodes of an n-vertex graph G can be transformed in time $\mathcal{O}(k^2 \cdot n)$ into a nice tree decomposition of G of width k and $k \cdot x$ nodes.

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$$|IS(S,t)| = \begin{cases} |IS(S,t')| & \text{if } v \notin S \\ |IS(S \setminus \{v\}, t')| + 1 & \text{if } v \in S \text{ and } S \text{ independent} \\ -\infty & \text{otherwise} \end{cases}$$

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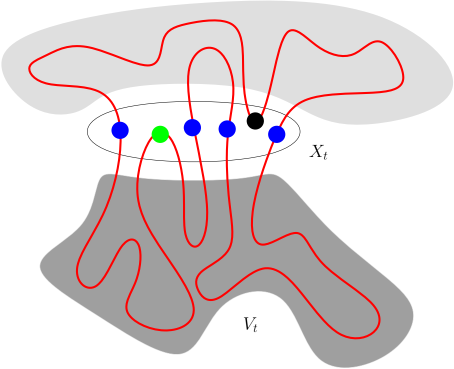
$$\boxed{\text{Complexity}: \mathcal{O}(2^k \cdot k^2 \cdot n)}$$

[slides borrowed from Christophe Paul]

Let $\mathcal{C}$ be a Hamiltonian cycle.

- Note that $\mathcal{C} \cap G[V_t]$ is a collection of paths.



HAMILTONIAN CYCLE on tree decompositions

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Let $\mathcal{C}$ be a Hamiltonian cycle.

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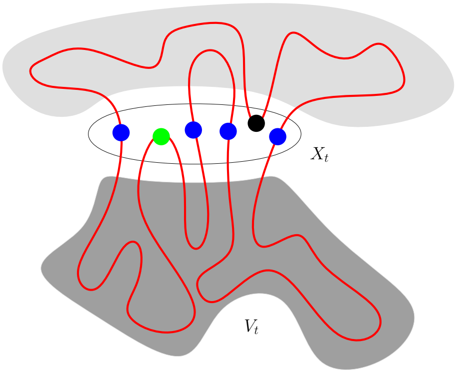

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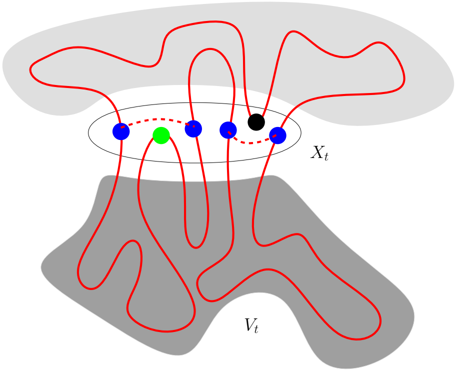


For every node t of the tree decomposition, we need to know if

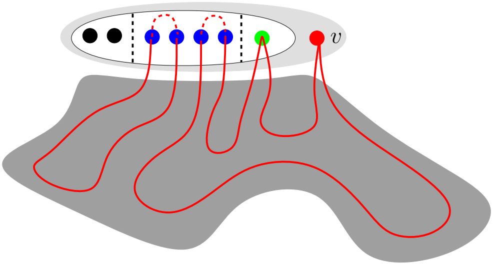
$$(X_t^0, X_t^1, X_t^2, M)$$

where M is a matching on X_t^1, corresponds to a partial solution.

skip

Forget node

Let t be a forget node and t' its child such that $X_t = X_{t'} \setminus \{v\}$.



Claim X_t is a separator $\Rightarrow$

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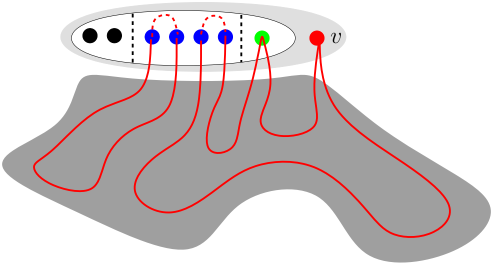

$\boxed{\text{Claim}}$ X_t is a separator $\Rightarrow$
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$$(X^0_{t'}, X^1_{t'}, X^2_{t'} \setminus \{v\}, M) \text{ is a partial solution for } t$$
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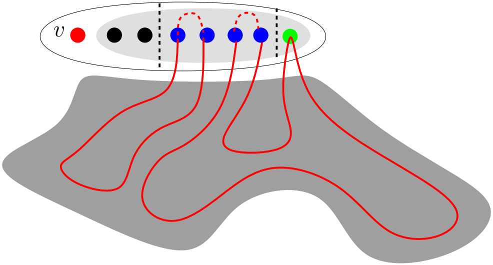

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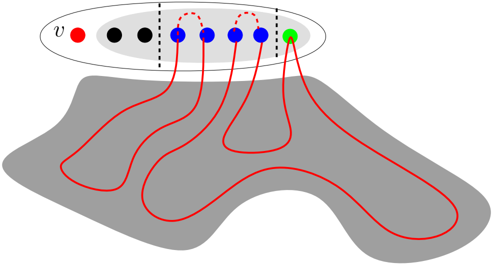

$$(X_{t'}^0 \cup \{v\}, X_{t'}^1, X_{t'}^2, M) \text{ is a partial solution for } t$$
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Let t be an introduce node and t' its child such that $X_t = X_{t'} \cup \{v\}$.

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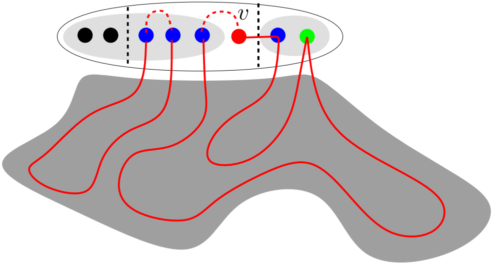

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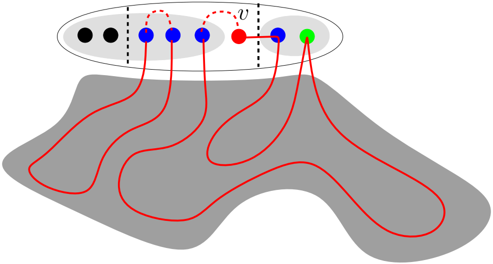

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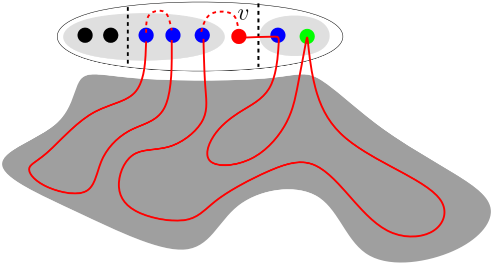

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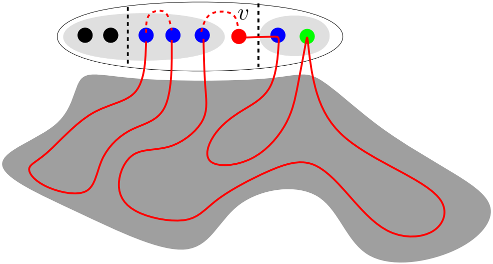

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$$(X_{t'}^0, X_{t'}^1 \cup \{v\} \setminus \{u\}, X_{t'}^2 \cup \{u\}, M') \text{ is a partial solution for } t$$
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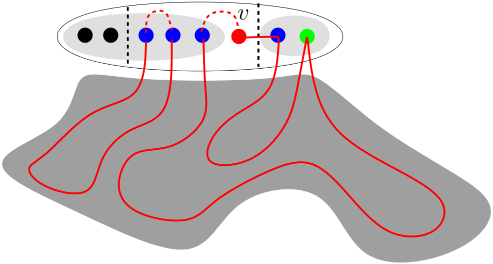

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- a vertex $u \in X_{t'}^1$ becomes internal $\Rightarrow$ $u \in X_t^2$.
- or a vertex $w \in X_{t'}^0$ becomes extremity of a path $\Rightarrow$ $w \in X_t^1$ (similar).

Introduce node (3)

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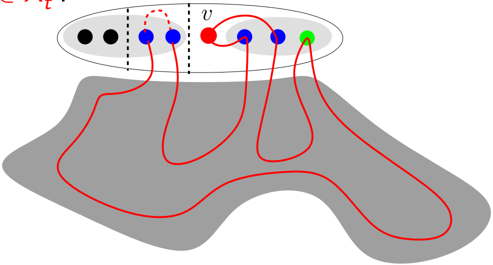

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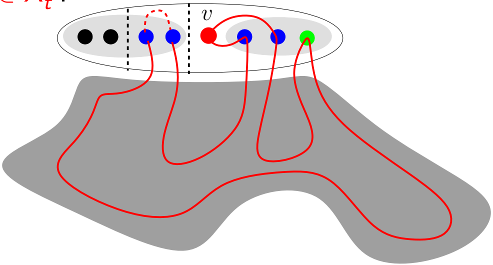

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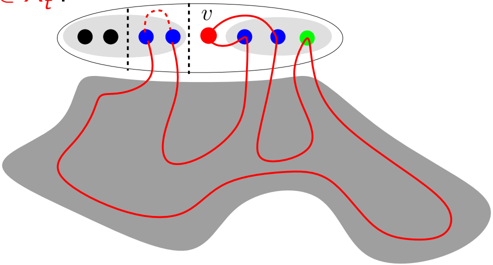

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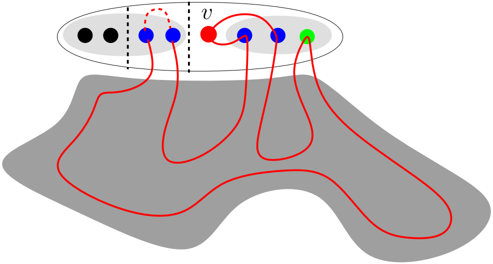

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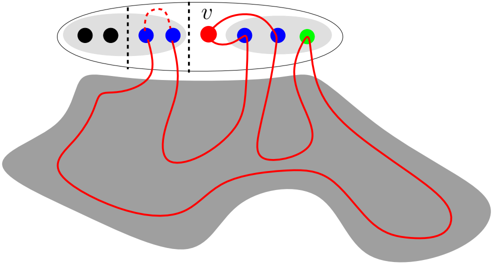

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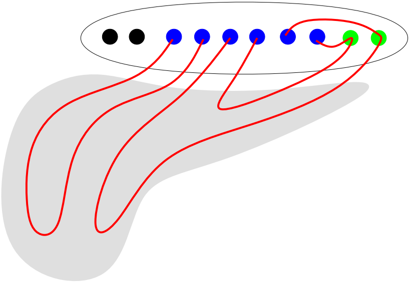

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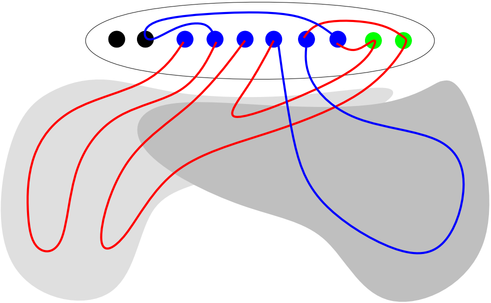

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Can this approach be generalized to more problems?

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In parameterized complexity: FPT parameterized by treewidth.

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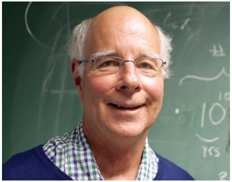
$\boxed{\text{Idea}}$ Measure the complexity of an algorithm in terms of the input size and an additional parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established and very active area.

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- k-Vertex Cover: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \leq k$, containing at least an endpoint of every edge?
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These three problems are NP-hard, but are they equally hard?

They behave quite differently...

- k-Vertex Cover: Solvable in time $\mathcal{O}(2^k \cdot (m+n))$
- k-Clique: Solvable in time $\mathcal{O}(k^2 \cdot n^k)$
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skip

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Working hypothesis of parameterized complexity: $\boxed{k\text{-Clique is not FPT}}$

(in classical complexity: 3-SAT cannot be solved in poly-time)

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W[1]-hard problem: $\exists$ parameterized reduction from k-CLIQUE to it.

W[2]-hard problem: $\exists$ param. reduction from k-DOMINATING SET to it.

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W[1]-hard problem: $\exists$ parameterized reduction from k-CLIQUE to it.

W[2]-hard problem: $\exists$ param. reduction from k-DOMINATING SET to it.

W[i]-hard: strong evidence of not being FPT.

How to transfer hardness among parameterized problems?

Let $A, B \subseteq \Sigma^* \times \mathbb{N}$ be two parameterized problems.

A parameterized reduction from A to B is an algorithm such that:

$\boxed{\text{Instance } (x, k) \text{ of } A}$ $\xrightarrow{\text{time } f(k) \cdot |x|^{\mathcal{O}(1)}}$ $\boxed{\text{Instance } (x', k') \text{ of } B}$

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Back to treewidth: only good news?

Theorem (Courcelle. 1990)

Every problem expressible in MSO_2 can be solved in time $f(\mathrm{tw}) \cdot n$ on graphs on n vertices and treewidth at most tw.

In parameterized complexity: FPT parameterized by treewidth.

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   The vast majority, but not all of them:

   - LIST COLORING is W[1]-hard parameterized by treewidth.

     [Fellows, Fomin, Lokshtanov, Rosamond, Saurabh, Szeider, Thomassen. 2007]

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2. Most natural problems (Vertex Cover, Dominating Set, ...) do not admit polynomial kernels parameterized by treewidth.

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Theorem (Robertson and Seymour. 1993)

For every $k \geq 0$ and graph G, the treewidth of G is at least k if and only if G contains a bramble of order at least $k+1$.

Another dual notion to treewidth: linkedness

[slides borrowed from Christophe Paul]

- Two sets $Y, Z \subseteq V(G)$, with $|Y| = |Z|$, are separable if there is a set $S \subseteq V(G)$ with $|S| < |Y|$ and such that $G - S$ contains no path between $Y \setminus S$ and $Z \setminus S$.

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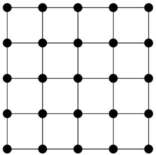

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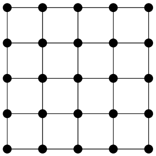

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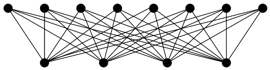

$K_{2k,k}$ is also k-linked

Highly linked graphs have large treewidth

Lemma

If G contains a $(k+1)$-linked set X with $|X| \geqslant 3k$, then $\mathrm{tw}(G) \geq k$.

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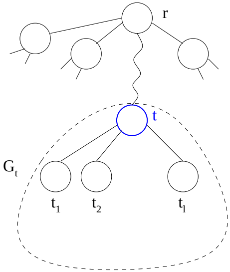

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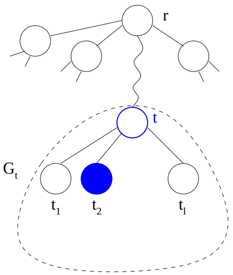

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If $\exists i \in [\ell]$ such that $|V_{t_i} \cap X| \geqslant k$, then we can choose $Y \subseteq V_{t_i} \cap X$, $|Y| = k$ and $Z \subseteq (V \setminus V_{t_i}) \cap X$, $|Z| = k$.

But $S = X_{t_i} \cap X_t$ separates Y and Z and $|S| \leq k-1$.

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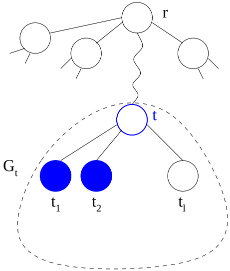

Let t be a "lowest" node with $|V_t \cap X| > 2k$.

Otherwise, let $W = V_{t_1} \cup \cdots \cup V_{t_i}$ with $|W \cap X| > k$ and $|(W \setminus V_{t_j}) \cap X| < k$ for $1 \leq j \leq i$.

$Y \subseteq W \cap X$, $|Y| = k+1$ and $Z \subseteq (V \setminus W) \cap X$, $|Z| = k+1$.

But $S = X_t$ separates Y from Z and $|S| \leqslant k$.

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Remark If X is not k-linked we can find, within the same running time, two separable subsets $Y, Z \subseteq X$.

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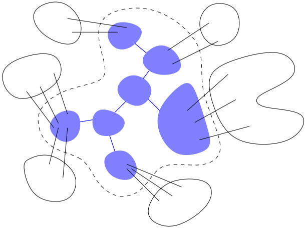

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- We add vertices to a set U in a greedy way, until $U = V(G)$.

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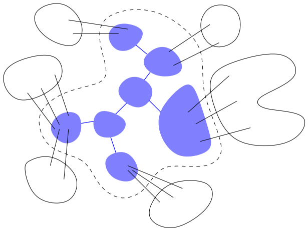

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4-approximation of Robertson and Seymour

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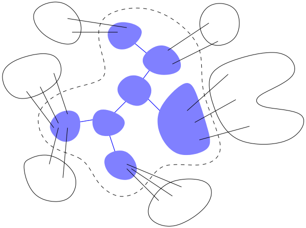

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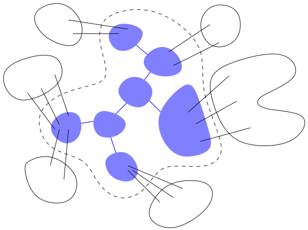

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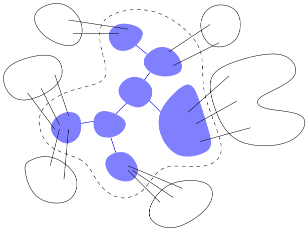

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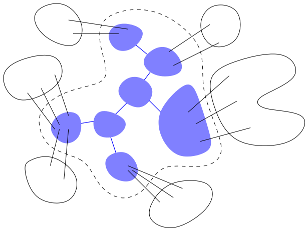

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Initially, we start with U being any set of $3k$ vertices.

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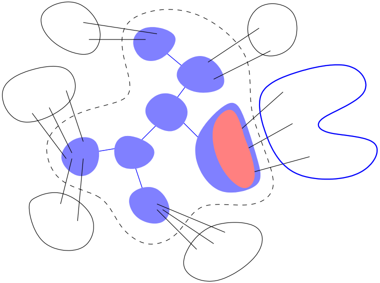

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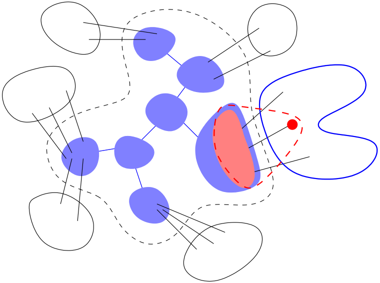

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- If $|X| < 3k$: we add a node t' neighbor of t such that $X_{t'} = \{x\} \cup X$, with $x \in C$ being a neighbor of X_t.

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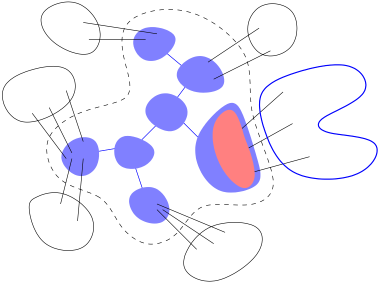

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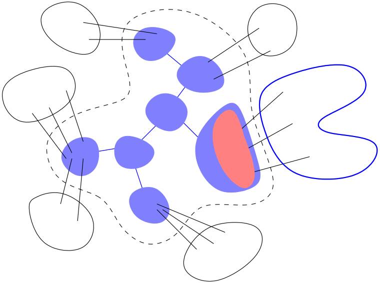

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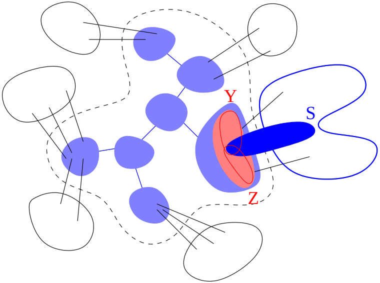

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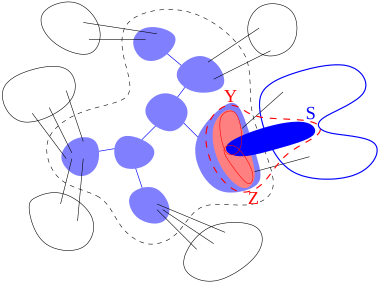

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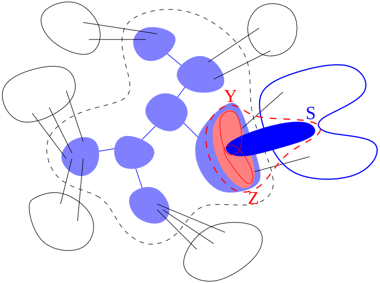
- If $|X| = 3k$: we test whether X is $(k+1)$-linked in time $f(k) \cdot n^{\mathcal{O}(1)}$:
  1. If X is $(k+1)$-linked, then $tw(G) \geq k$, and we stop.
  2. Otherwise, we find sets Y, Z, S with $|S| < |Y| = |Z| \leq k+1$ and such that S separates Y and Z.
     We create a node t' neighbor of t s.t. $X_{t'} = (S \cap C) \cup X$.

4-approximation of Robertson and Seymour (2)



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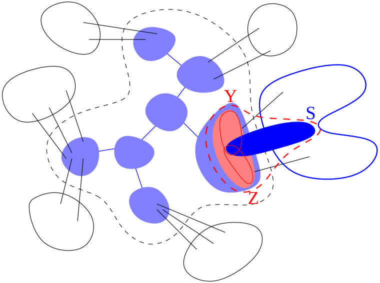

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Gràcies!