Balanced separators and (layered) treewidth

Nicolas Bousquet

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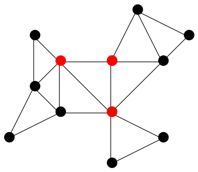
Every bag of a tree decomposition is a separator.



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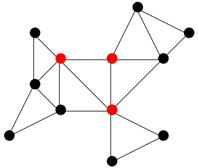

Question :

Can we say better ?

Balanced Separators

S is a balanced separator if there exists a partition A, B of $G \setminus S$ such that :

- $\max(|A|, |B|) \leq \frac{2n}{3}$ and,
- A, B are anticomplete.

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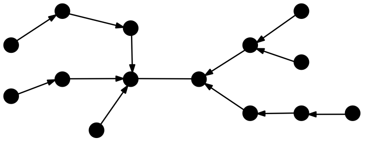
Proof :

- Orient the edge $u \rightarrow v$ if $G \setminus u$ contains $\geq \frac{1}{2}$ of the vertices.



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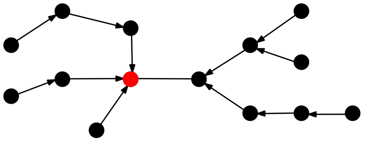
- The orientation admits a sink.



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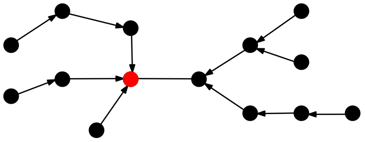
- A sink vertex satisfies the lemma.



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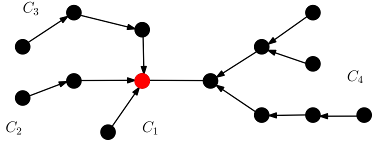

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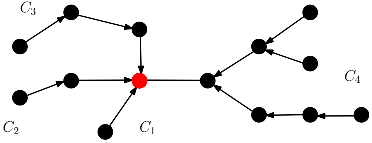

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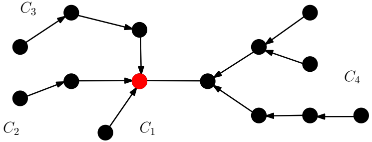

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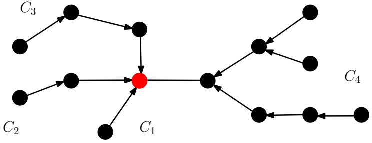

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Every graph of treewidth at most k has a balanced separator of size $k+1$.

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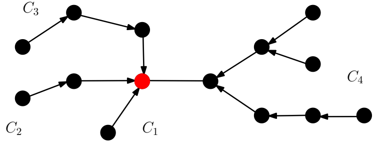

Theorem

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Proof : Replace nodes by bags.

Super balanced separators

A separator S is super balanced if A, B have size at most $n/2$.

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If G has a balanced separator of size $\leq k$ then G has a super balanced separator of size $\leq ck$.

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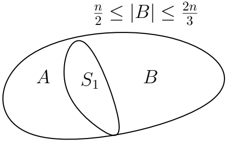

Gauche	Dr.	?	Sep.
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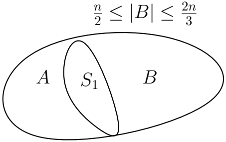

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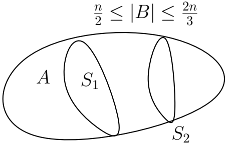

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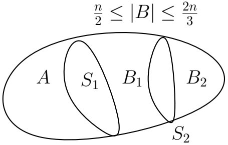

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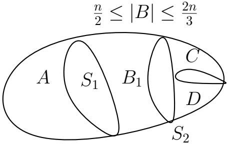

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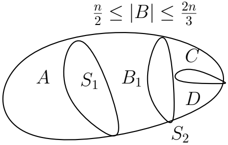

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Total size of the separator $\leq \sum_{i=1}^{\log n} (2/3)^i k = c \cdot k$.

Questions

- When does it exist (small) balanced separators ?
- Why are we looking for (small) balanced separators ?
- When can't we find (small) balanced separators ?

Separator and treewidth II

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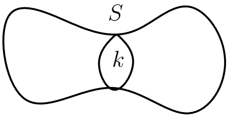

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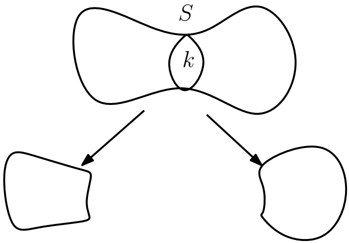

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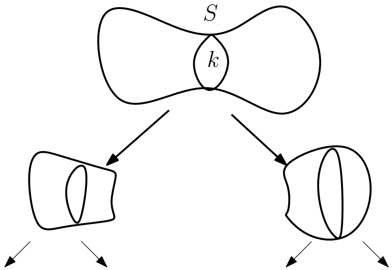

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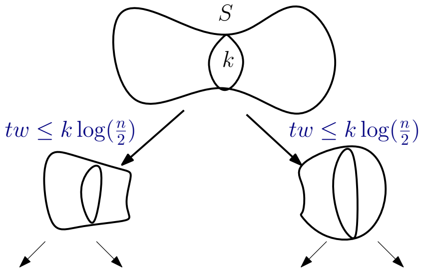

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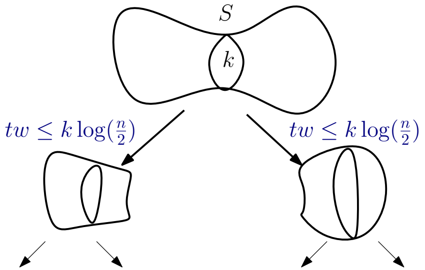

$tw(G) \leq k + k\log(n/2) \leq k \cdot (\log 2 + \log(n/2) \leq k \log n$

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Theorem (Lipton, Tarjan)

Planar graphs have balanced separators of size $O(\sqrt{n})$.

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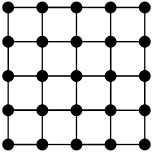

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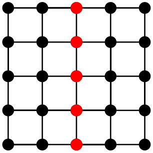

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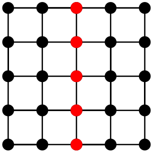

Theorem (Alon, Seymour, Thomas)

Every K_t-minor free graph has a balanced separator of size $O(t^{3/2}\sqrt{n})$.

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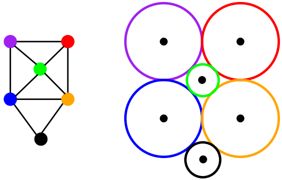
Proof of Har-Peled '11 :

Theorem (Koebe, Andreev, Thurston '36)

G is planar iff it is the contact graph of disks on the sphere.



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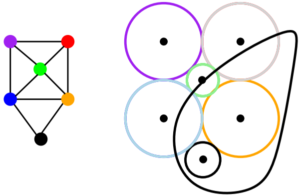

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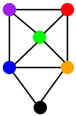

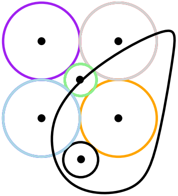

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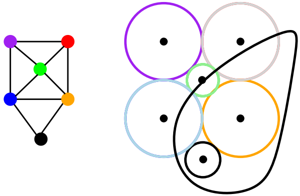

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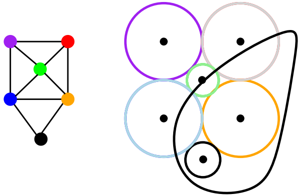

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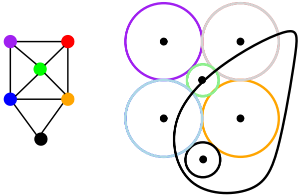

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$\Rightarrow$ S_r separate the "interior" from the "exterior".
To conclude, we want to prove :

- S_r is balanced (no too large component on each side).
- S_r is not too big (the expected size of S_r is $O(\sqrt{n})$).

S is balanced

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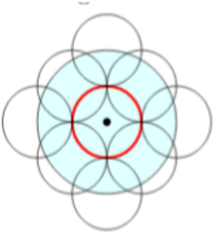

Fig. Har-Peled

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$\Rightarrow$ By minimality of the ball, $\mathcal{P}'$ has size at most $9n/10$.

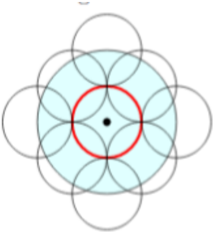

Fig. Har-Peled

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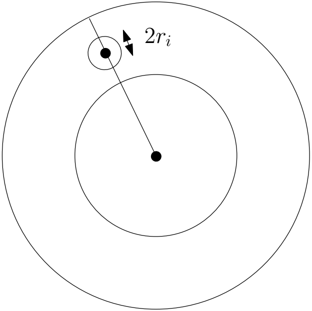

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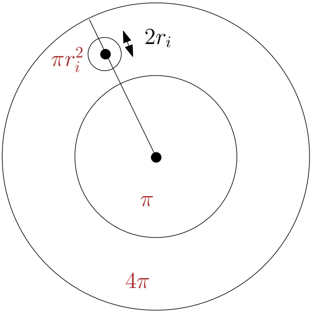
- Total area is 4π.



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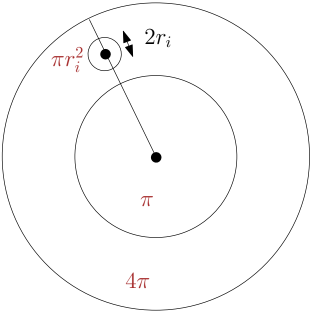
- Total area is 4π.
- Using Cauchy-Scharwz :
  $\mathbb{E}(S) = \sum_i^n \mathbb{P}(D(x,r) \cap D_i \neq \emptyset)$
  $\mathbb{E}(S) \leq \sum_{i=1}^n 2 \cdot r_i$
  $\mathbb{E}(S) \leq \sqrt{(\sum_{i=1}^n 4) \cdot (\sum_{i=1}^n r_i^2)}$
  $\mathbb{E}(S) = O(\sqrt{n})$.



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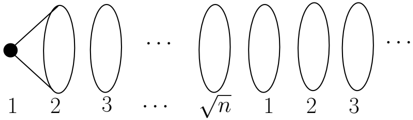

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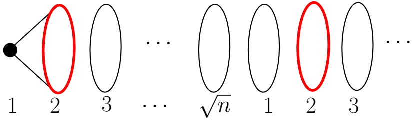

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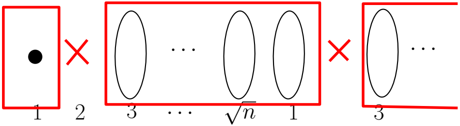

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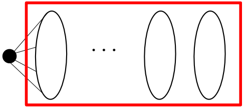

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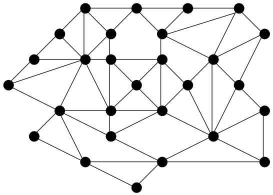

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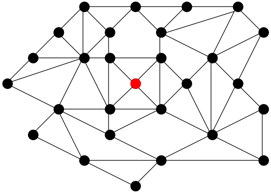

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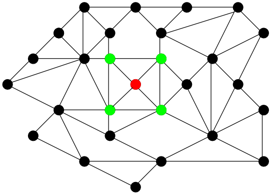

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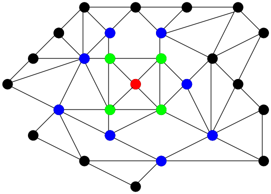

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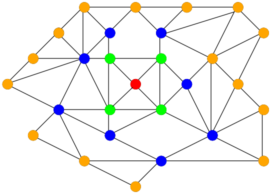

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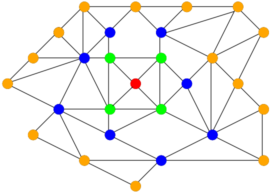

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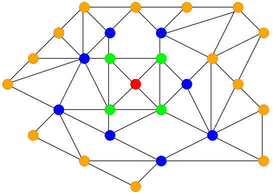

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Problem : How to do it ?

Layered treewidth

A layering of G is a partition $V_1, \ldots, V_t$ of V such that every edge lies in a layer or between two consecutive layers.

Layered width$=$ maximum for $i \in \{1, \ldots, t\}$ of the treewidth of $(T, G[V_i])$.

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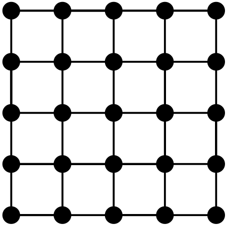
Example

Grids have layered treewidth 2.



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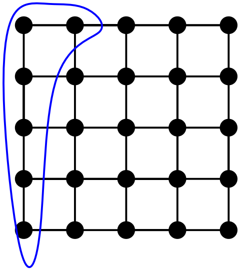

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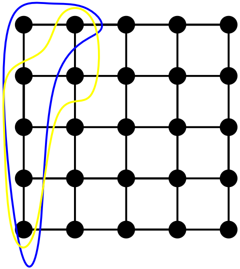

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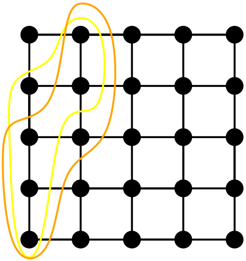

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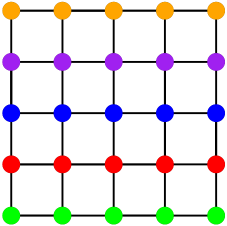

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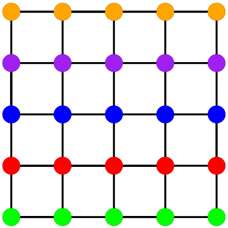

Theorem (Dujmović, Morin, Wood '17)

Every planar graph has layered treewidth at most 3

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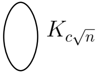

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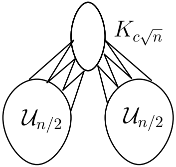

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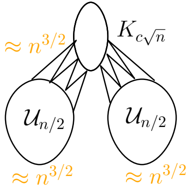

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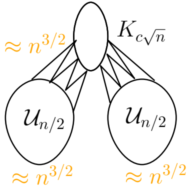

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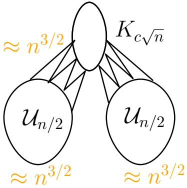

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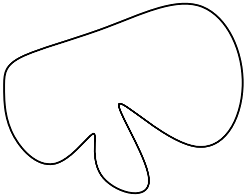

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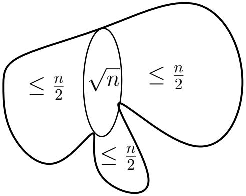

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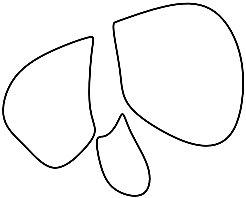

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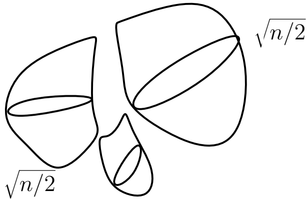

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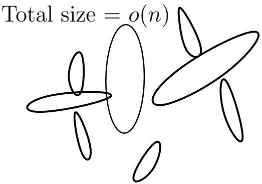

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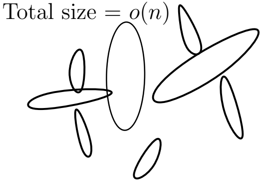

$\rightarrow$ A careful counting ensures that only $o(n)$ vertices have been removed.

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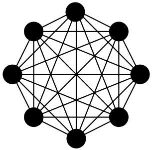

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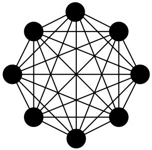

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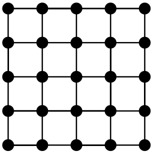

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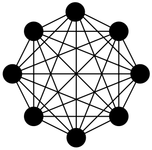

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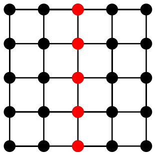

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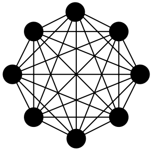

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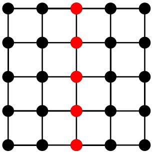

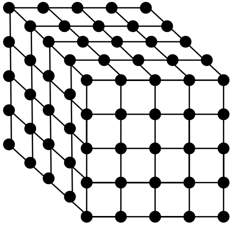

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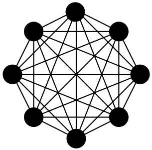

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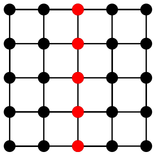

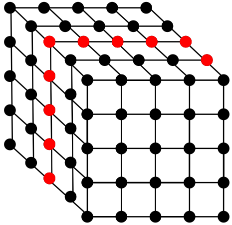

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Theorem

The following are equivalent

- G is a c-expander for $c > 0$.
- G has linear treewidth.
- G has no sublinear balanced separator.

Remark :

A graph of maximum degree 2 is not an expander.

Building expanders ?

Remark :

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H a "small graph" of size D and degree d which is an expander.

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"Replace" every vertex $v \in V(G)$ by a "cloud" of size D which is connected in a "dirty" way to the original neighbors of v in G.

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Thanks for your attention !