Introduction to twin-width

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based on joint works with Colin Geniet, Eun Jung Kim,
Stéphan Thomassé, and Rémi Watrigant

ENS Lyon, LIP

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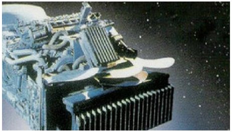
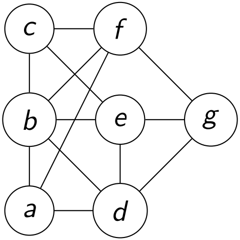
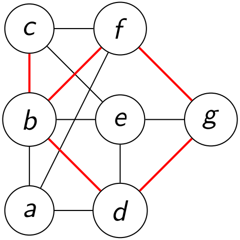


Graphs



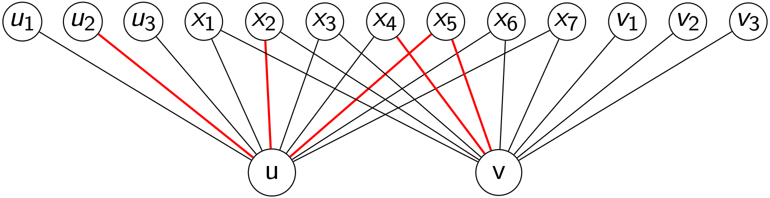
Two outcomes between a pair of vertices:
edge or non-edge

Trigraphs



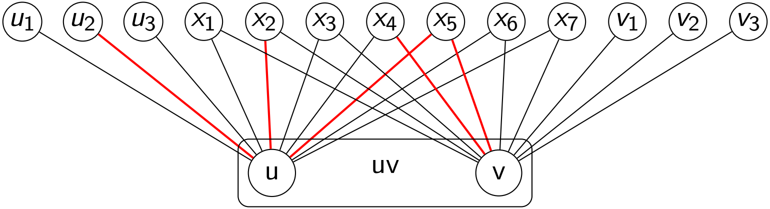
Three outcomes between a pair of vertices:
edge, or non-edge, or red edge (error edge)

Contractions in trigraphs



Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



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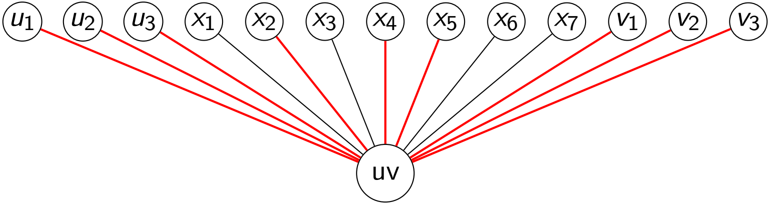
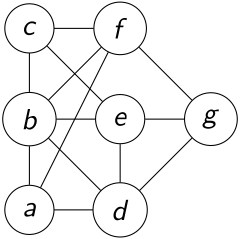


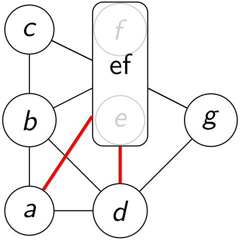
edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Contraction sequence



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



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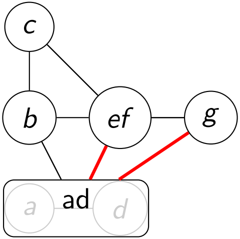

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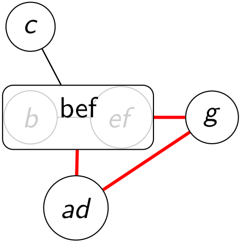

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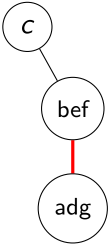

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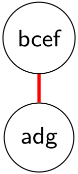

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Contraction sequence

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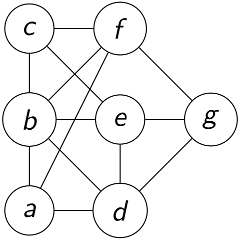
Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.



Maximum red degree $= 0$

overall maximum red degree $= \mathbf{0}$

Twin-width

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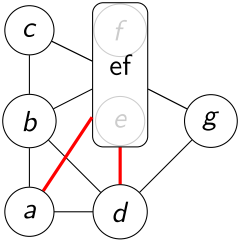

Maximum red degree $= 2$

overall maximum red degree = 2

Twin-width

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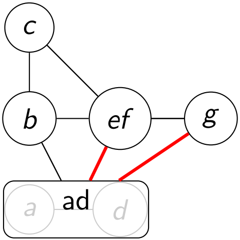

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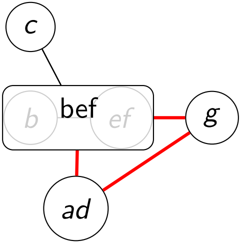

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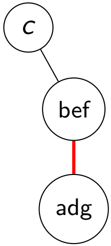

Maximum red degree $= 1$

overall maximum red degree = 2

Twin-width

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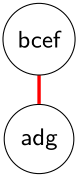

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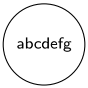


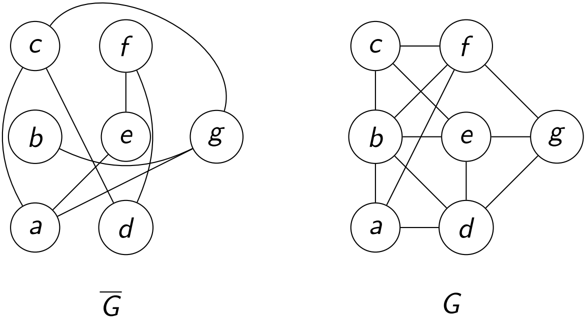
Maximum red degree $= 0$

overall maximum red degree = 2

Simple operations preserving small twin-width

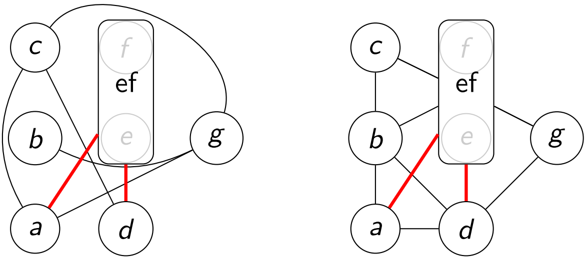
- complementation: remains the same
- taking induced subgraphs: may only decrease
- adding one vertex linked arbitrarily: at most “doubles”
- substitution, lexicographic product: max of the twin-widths

Complementation



$$\mathrm{tww}(\overline{G}) = \mathrm{tww}(G)$$

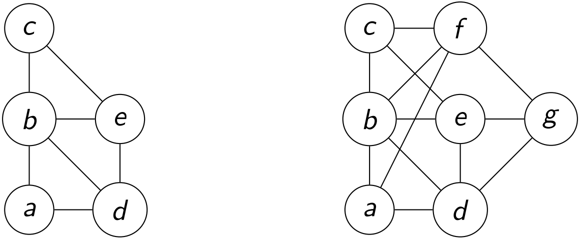
Complementation



$\overline{G_6}$ G_6

$$\mathrm{tww}(\overline{G}) = \mathrm{tww}(G)$$

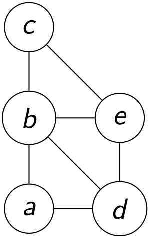
Induced subgraph



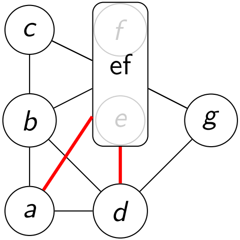
H $\qquad$ G

$$\mathrm{tww}(H) \leqslant \mathrm{tww}(G)$$

Induced subgraph

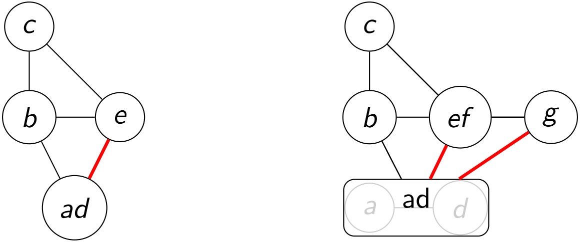


H



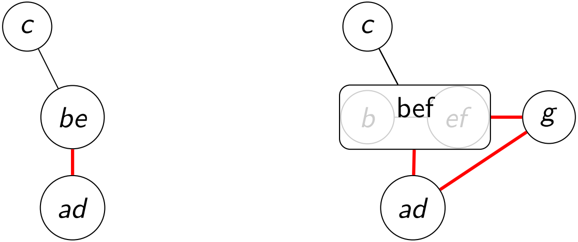
Ignore absent vertices

Induced subgraph



Mimic the contractions otherwise

Induced subgraph



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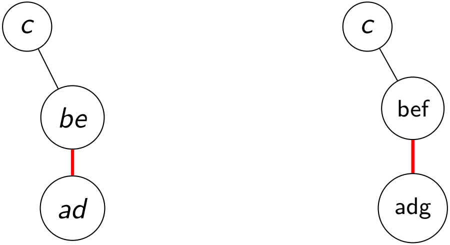

Mimic the contractions otherwise

Induced subgraph

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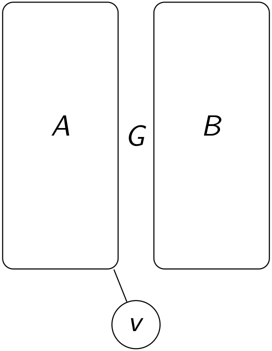
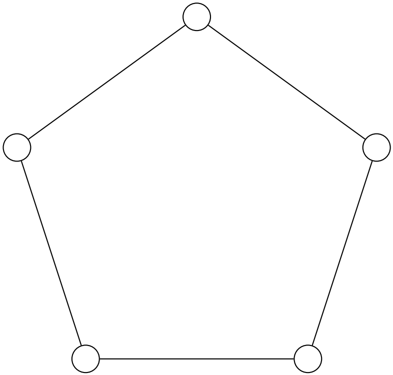
Adding one apex v



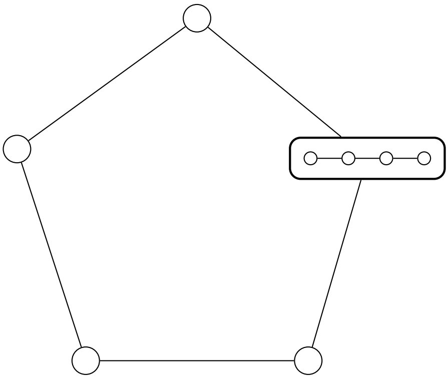
Ignore the contractions of $X \subseteq A$ with $Y \subseteq B$

Substitution and lexicographic product



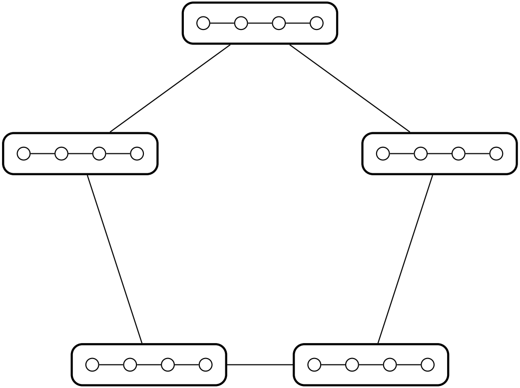
$G = C_5$

Substitution and lexicographic product



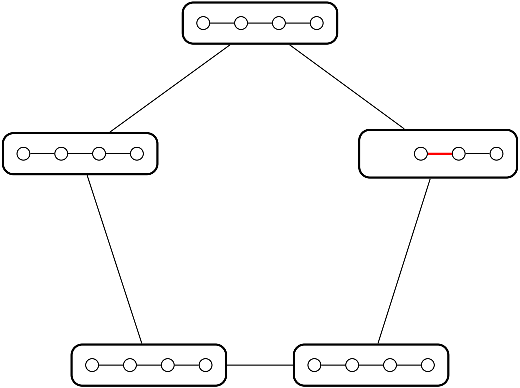
$G = C_5$, $H = P_4$, substitution $G[v \leftarrow H]$

Substitution and lexicographic product



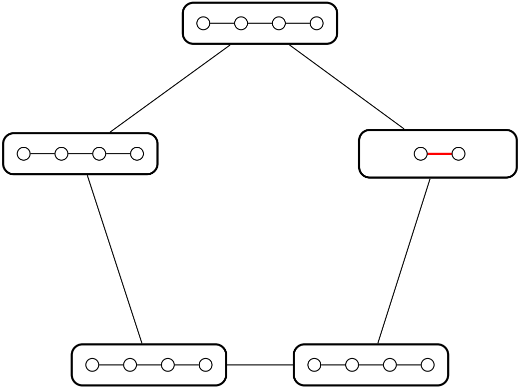
$G = C_5$, $H = P_4$, lexicographic product $G[H]$

Substitution and lexicographic product



More generally any modular decomposition

Substitution and lexicographic product



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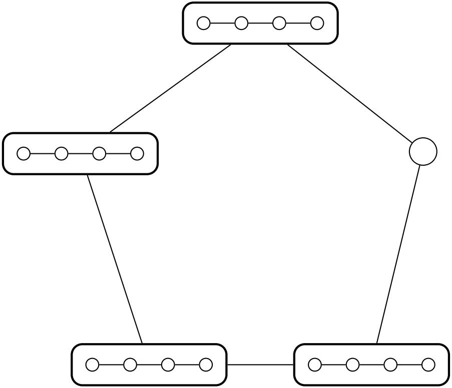
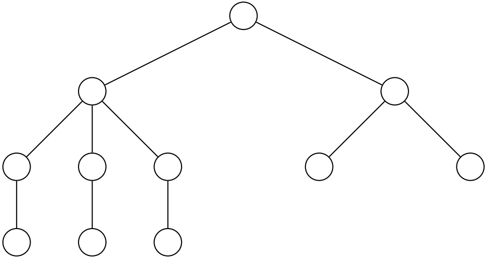


$$\mathrm{tww}(G[H]) = \max(\mathrm{tww}(G), \mathrm{tww}(H))$$

Classes with bounded twin-width

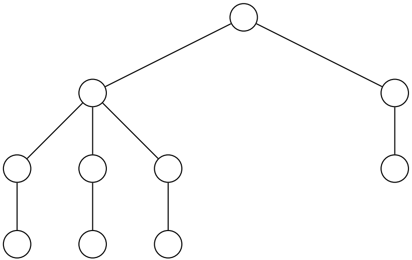
- cographs = twin-width 0
- trees, bounded treewidth, clique-width/rank-width
- grids
- . . .

Trees



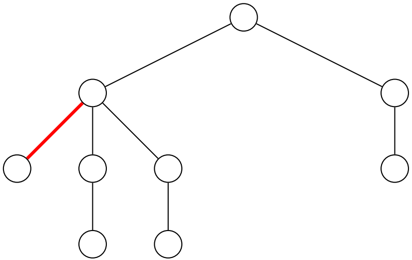
If possible, contract two twin leaves

Trees



If not, contract a deepest leaf with its parent

Trees



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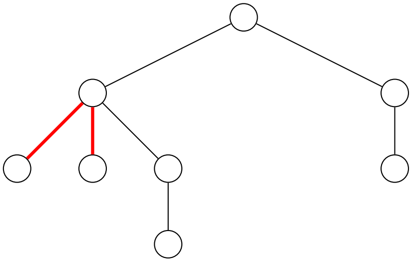

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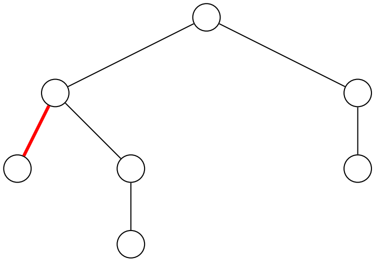
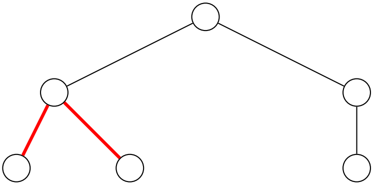


Cannot create a red degree-3 vertex

Trees



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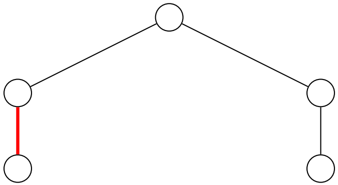

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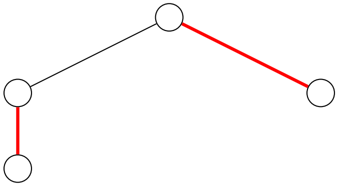

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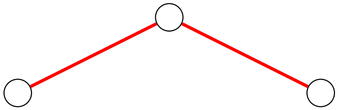

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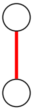
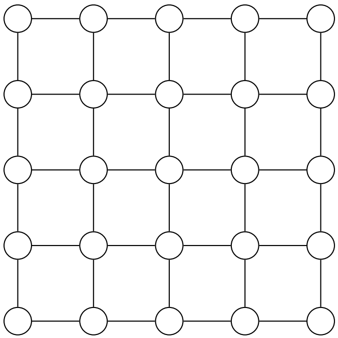


Cannot create a red degree-3 vertex

Trees

Generalization to bounded *treewidth* and even bounded *rank-width*

Grids



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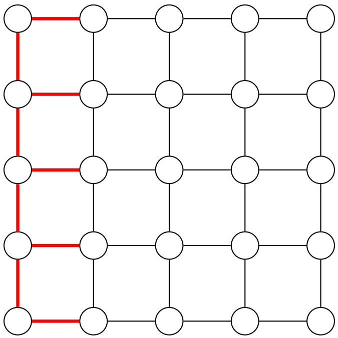

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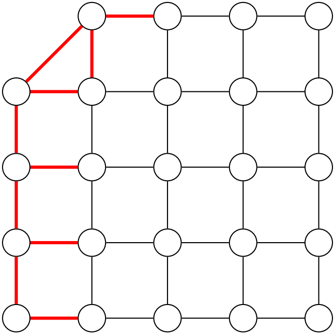

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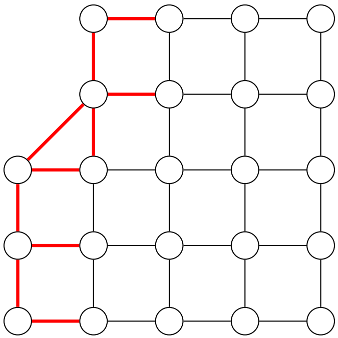

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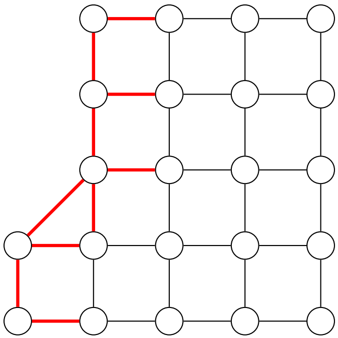

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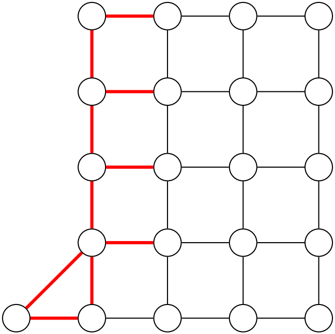

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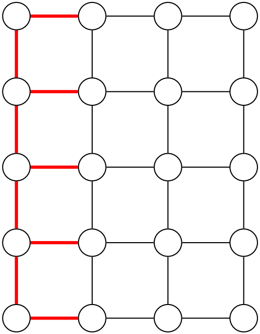
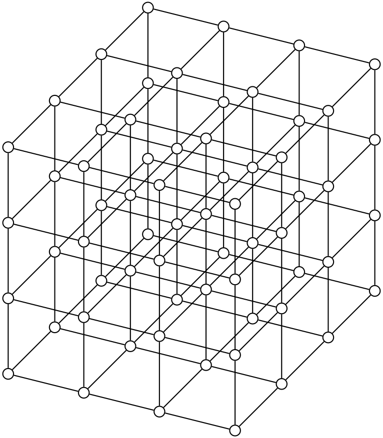


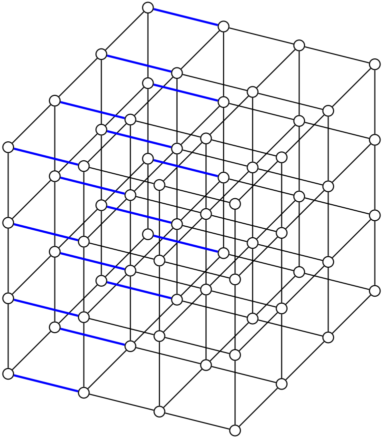
4-sequence for planar grids

3-dimensional grids



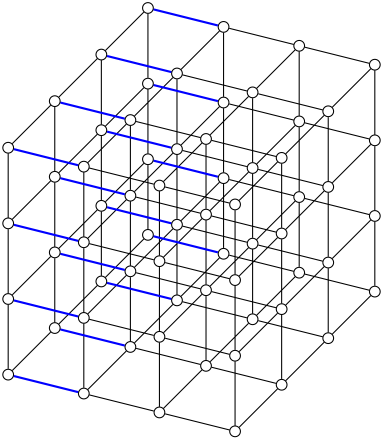
Contains arbitrary large clique minors

3-dimensional grids



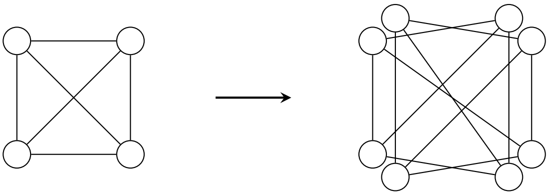
Contract the blue edges in any order $\rightarrow$ 12-sequence

3-dimensional grids



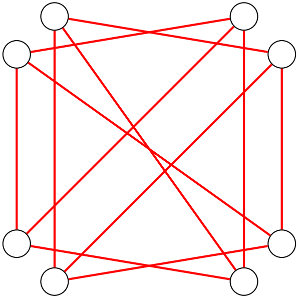
The d-dimensional grid has twin-width $\leqslant 4d$ (even $3d$)

2-lifts, expanders with bounded twin-width



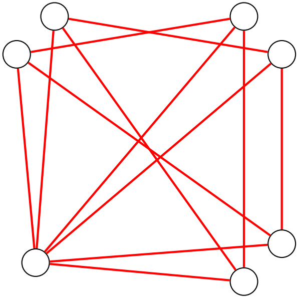
split each vertex in 2, replace each edge by 1 of the 2 matchings

2-lifts, expanders with bounded twin-width



Iterated 2-lifts of K_4 have twin-width at most 6

2-lifts, expanders with bounded twin-width



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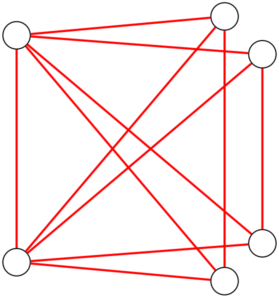

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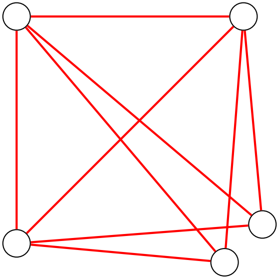
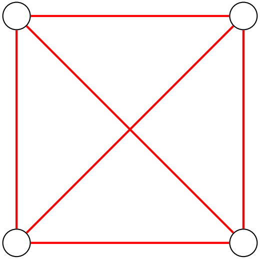


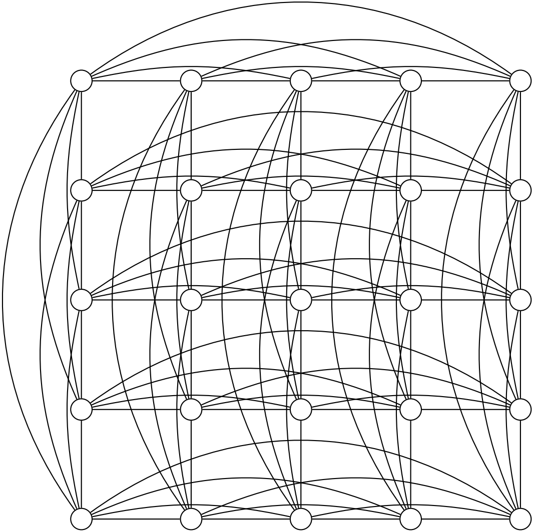
Iterated 2-lifts of K_4 have twin-width at most 6

2-lifts, expanders with bounded twin-width



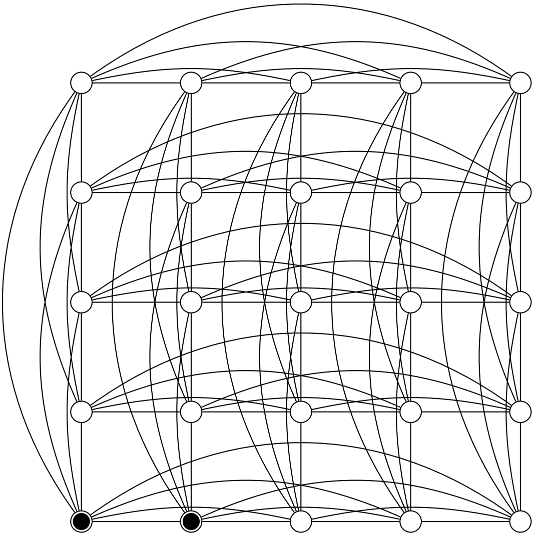
Iterated 2-lifts of K_4 have twin-width at most 6
but no balanced separators of size $o(n)$

First example of unbounded twin-width



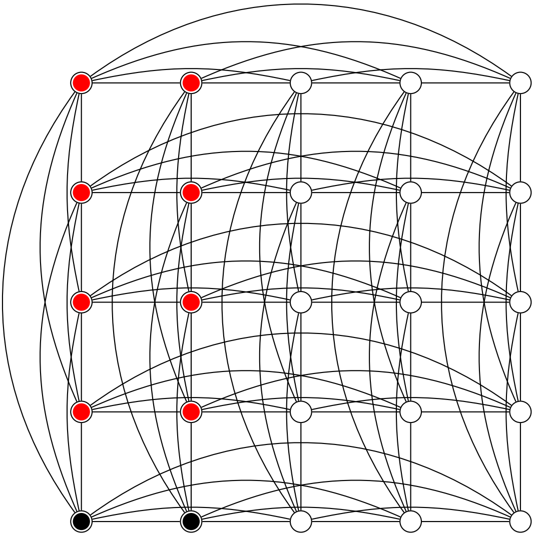
Line graph of a biclique a.k.a. rook graph

First example of unbounded twin-width



No pair of near twins

First example of unbounded twin-width



No pair of near twins

Universal bipartite graph

No $O(1)$-contraction sequence:
twin-width is *not* an iterated identification of near twins.

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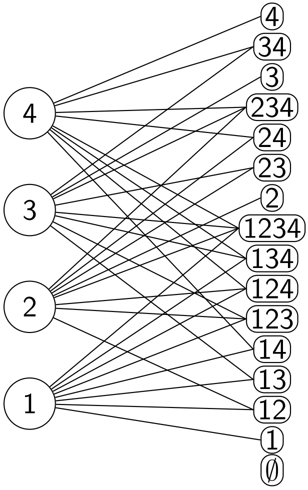

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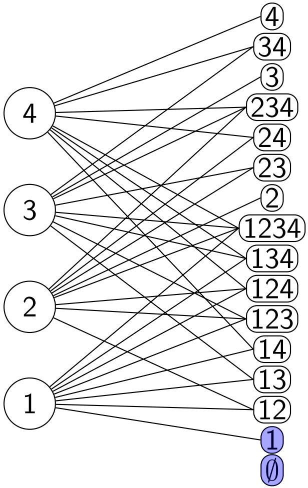

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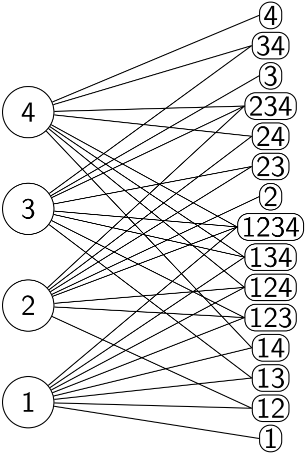

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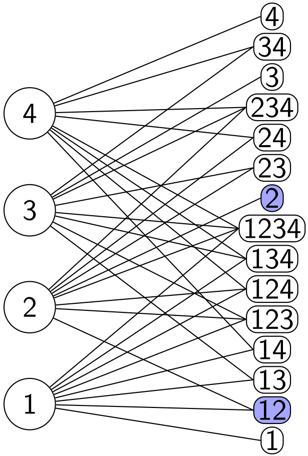

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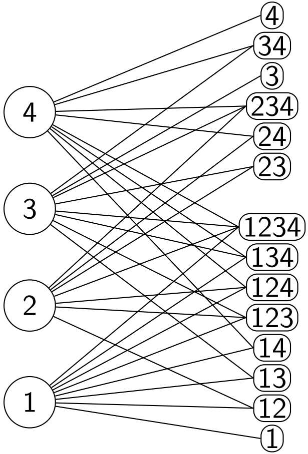

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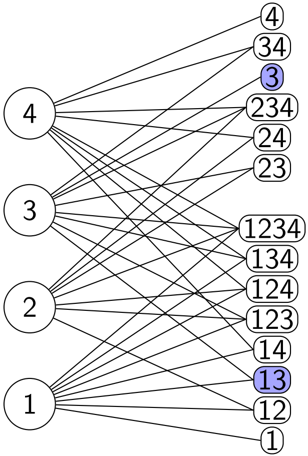

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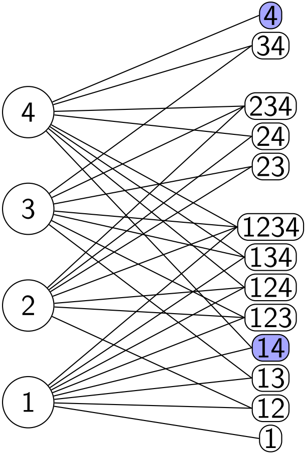

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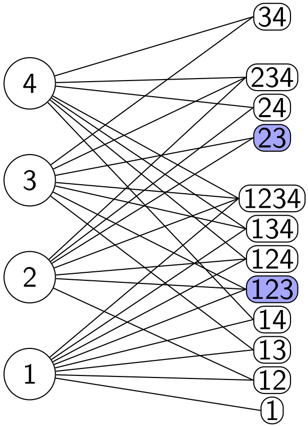

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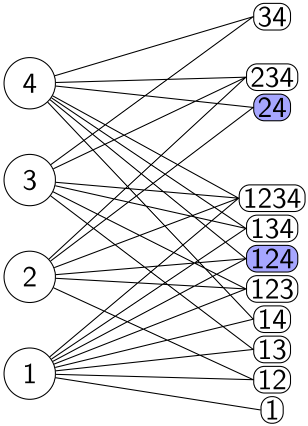

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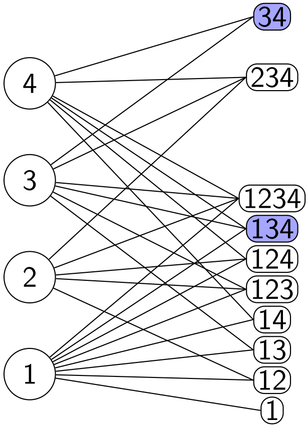

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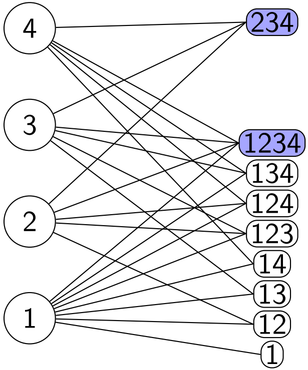

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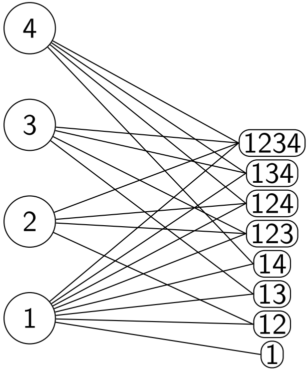

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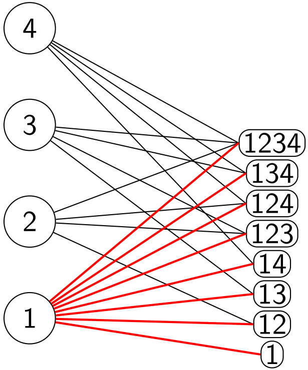
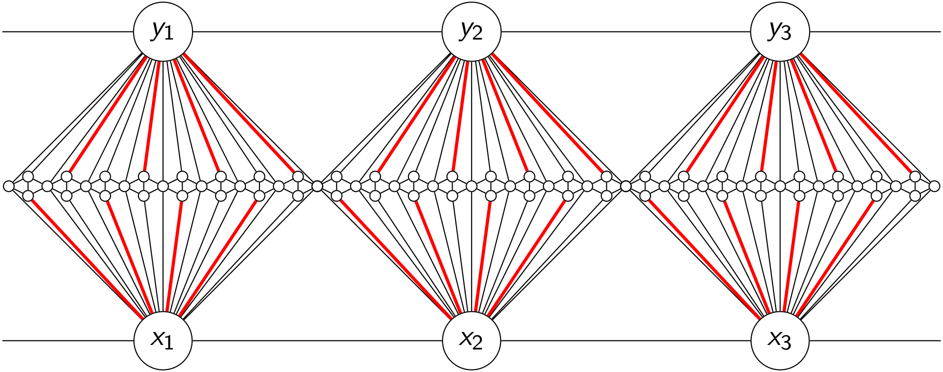


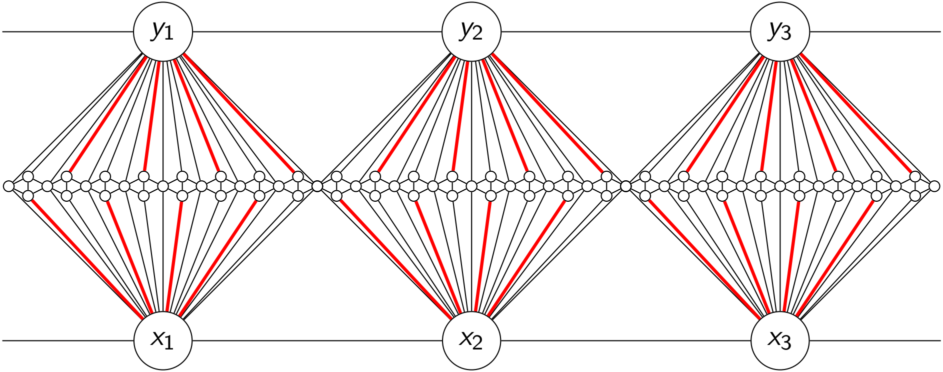
Planar graphs?

Planar graphs?



For every d, a planar trigraph without planar d-contraction

Planar graphs?



For every d, a planar trigraph without planar d-contraction

More powerfool tool needed

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Encode a bipartite graph (or, if symmetric, any graph)

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Contraction of two columns (similar with two rows)

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & \color{red}{r} & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & \color{red}{r} & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & \color{red}{r} & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

How is the twin-width (re)defined?

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

How to tune it for non-bipartite graph?

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$
\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]
$$

Maximum number of non-constant zones per column or row part
= **error value**

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$
\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]
$$

Maximum number of non-constant zones per column or row part

…until there are a single row part and column part

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Twin-width as maximum error value
of a contraction/division sequence

Grid minor

t-grid minor: $t \times t$-division where every cell is non-empty
Non-empty cell: contains at least one 1 entry

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

4-grid minor

Grid minor

t-grid minor: $t \times t$-division where every cell is non-empty
Non-empty cell: contains at least one 1 entry

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

4-grid minor

A matrix is said t-**grid free** if it does not have a t-grid minor

Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

Mixed minor

Mixed cell: not horizontal nor vertical

$$
\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]
$$

3-mixed minor

Every mixed cell is witnessed by a 2×2 square = **corner**

Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

3-mixed minor

A matrix is said t-**mixed free** if it does not have a t-mixed minor

Mixed value

$$\begin{array}{c c}
\begin{array}{c} R_4 \\ R_3 \\ \\ R_2 \\ \\ R_1 \\ \\ \end{array} & \left[\begin{array}{cc|ccc|c|cc}
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}\right] \\
 & \qquad\qquad C_2
\end{array}$$

$\approx$ (maximum) number of cells with a corner per row/column part

Mixed value

$$\begin{array}{c} R_4 \\ \\ R_3 \\ \\ R_2 \\ \\ R_1 \\ \end{array} \left[\begin{array}{cc|ccc|c|cc} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

C_2

But we add the number of *boundaries* containing a corner

Mixed value

$$
\begin{array}{c c}
R_4 & \\
R_3 & \\
\cup & \\
R_2 & \\
R_1 & \\
\end{array}
\left[\begin{array}{cc|ccc|c|cc}
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
$$

$$C_2$$

$\therefore$ merging row parts do not increase mixed value of column part

Twin-width and mixed freeness

Theorem

*If G admits **a** t-mixed free adjacency matrix, then* $\text{tww}(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

Theorem

If $\exists \sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

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If $\exists\sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
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\end{array}\right]$$

Merge consecutive parts greedily

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0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
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$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

Question

For every k, is there a c_k such that every $n \times m$ $0,1$-matrix with at least c_k 1 per row and column admits a k-grid minor?

Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

Conjecture (reformulation of Füredi-Hajnal conjecture '92)

For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Stanley-Wilf conjecture / Marcus-Tardos theorem

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For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Conjecture (Stanley-Wilf conjecture '80s)

Any proper permutation class contains only $2^{O(n)}$ n-permutations.

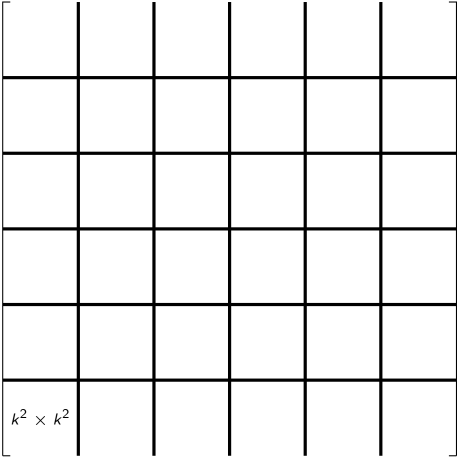
Klazar showed Füredi-Hajnal $\Rightarrow$ Stanley-Wilf in 2000
Marcus and Tardos showed Füredi-Hajnal in 2004

Marcus-Tardos one-page inductive proof

$$M = \begin{bmatrix} \; \end{bmatrix}$$

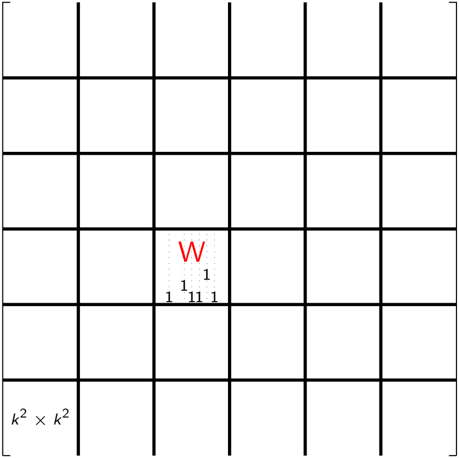
Let M be an $n \times n$ $0,1$-matrix without k-grid minor

Marcus-Tardos one-page inductive proof

$M =$ 

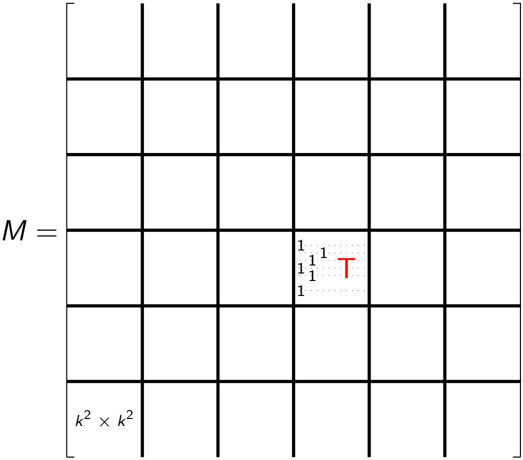
Draw a regular $\frac{n}{k^2} \times \frac{n}{k^2}$ division on top of M

Marcus-Tardos one-page inductive proof

$M =$ 

A cell is *wide* if it has at least k columns with a 1

Marcus-Tardos one-page inductive proof



A cell is *tall* if it has at least k rows with a 1

Marcus-Tardos one-page inductive proof

$$M = \left[\begin{array}{c|c|c|c|c|c}
 & & W & & & \\ \hline
 & & W & & & \\ \hline
 & & & & & \\ \hline
 & & W & & & \\ \hline
 & & & & & \\ \hline
k^2 \times k^2 & & & & &
\end{array}\right]$$

There are less than $k\binom{k^2}{k}$ wide cells per column part. Why?

Marcus-Tardos one-page inductive proof

$$M = \left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & \color{red}{\mathrm{T}} & & \color{red}{\mathrm{T}} & & \color{red}{\mathrm{T}} \\ \hline & & & & & \\ \hline {\scriptstyle k^2 \times k^2} & & & & & \end{array}\right]$$

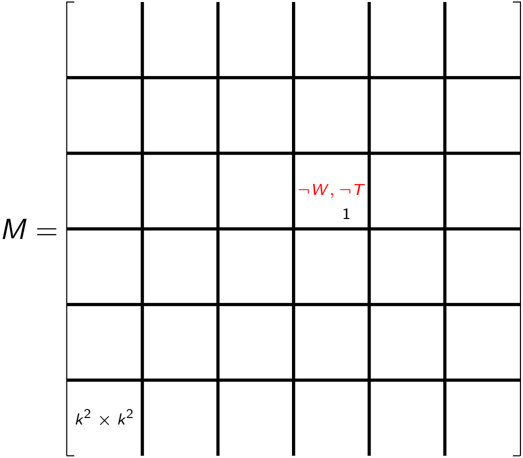
There are less than $k\binom{k^2}{k}$ tall cells per row part

Marcus-Tardos one-page inductive proof

$$M = \left[\begin{array}{c|c|c|c|c|c}
 & & W & & & \\ \hline
 & W & W & & & T \\ \hline
 & & & & & \\ \hline
 & T & W & T & & T \\ \hline
 & & T & & & \\ \hline
k^2 \times k^2 & & & & & W
\end{array}\right]$$

In W and T, at most $2 \cdot \frac{n}{k^2} \cdot k\binom{k^2}{k} \cdot k^4 = 2k^3\binom{k^2}{k} n$ entries 1

Marcus-Tardos one-page inductive proof



There are at most $(k-1)^2 c_k \frac{n}{k^2}$ remaining 1. Why?

Marcus-Tardos one-page inductive proof

$$M = \left[\begin{array}{c|c|c|c|c|c}
 & & \mathrm{W} & & & \\ \hline
 & \mathrm{W} & \mathrm{W} & & & \mathrm{T} \\ \hline
 & & & \begin{matrix}\neg W, \neg T \\ 1\end{matrix} & & \\ \hline
 & \mathrm{T} & \mathrm{W} & \mathrm{T} & & \mathrm{T} \\ \hline
 & & \mathrm{T} & & & \\ \hline
k^2 \times k^2 & & & & & \mathrm{W}
\end{array}\right]$$

Choose $c_k = 2k^4\binom{k^2}{k}$ so that $(k-1)^2 c_k \frac{n}{k^2} + 2k^3\binom{k^2}{k} n \leqslant c_k n$

Twin-width and mixed freeness

Theorem

If $\exists\sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$

$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

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$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

Impossible!

Twin-width and mixed freeness

Theorem

If $\exists \sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$
Step 2: find a contraction sequence with error value $g(t)$

$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Refinement of $\mathcal{D}_i$ where each part coincides on the non-mixed cells

Twin-width and mixed freeness

Theorem

If $\exists\sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

Theorem

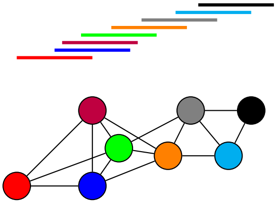
If $\exists \sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Now to bound the twin-width of a class $\mathcal{C}$:

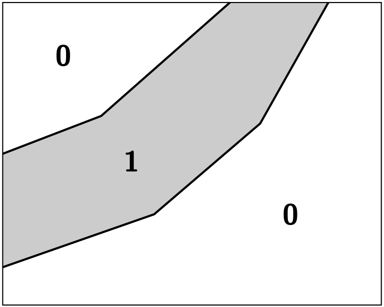
1) Find a *good* vertex-ordering procedure
2) Argue that, in this order, a t-mixed minor would conflict with $\mathcal{C}$

Unit interval graphs

Intersection graph of unit segments on the real line

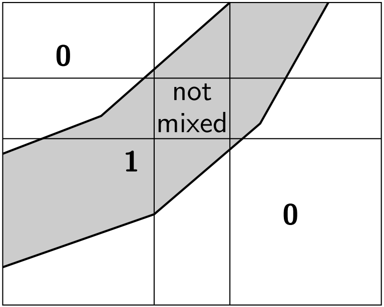


Unit interval graphs



order by left endpoints

Unit interval graphs



No 3-by-3 grid has all 9 cells crossed by two non-decreasing curves

Graph minors

Formed by **vertex deletion**, **edge deletion**, and **edge contraction**

A graph G is *H-minor free* if H is not a minor of G

A graph class is *H-minor free* if all its graphs are

Graph minors

Formed by **vertex deletion**, **edge deletion**, and **edge contraction**

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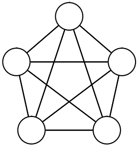
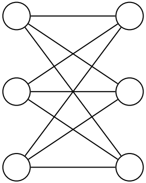
Planar graphs are exactly the graphs without K_5 or $K_{3,3}$ as a minor



K_5

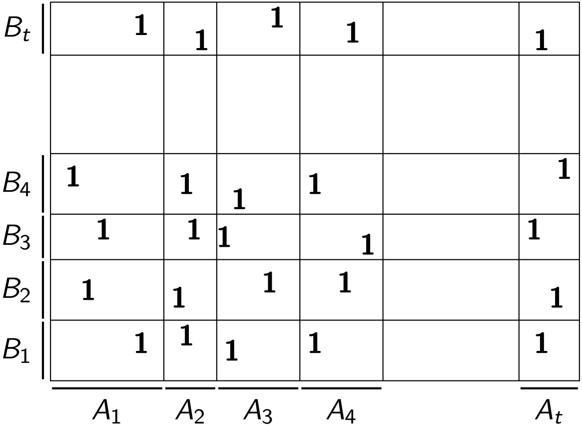


$K_{3,3}$

Bounded twin-width – K_t-minor free graphs

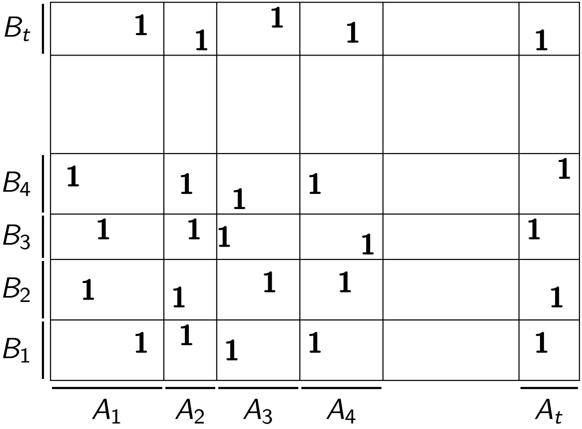
Given a hamiltonian path, we would just use this order

Bounded twin-width – K_t-minor free graphs



Contracting the $2t$ subpaths yields a $K_{t,t}$-minor, hence a K_t-minor

Bounded twin-width – K_t-minor free graphs



Instead we use a specially crafted lex-DFS discovery order

Theorem

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size (seen as digraphs),*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- $\Omega(\log n)$*-subdivisions of all the n-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from K_4,*
- *strong products of two bounded twin-width classes, one with bounded degree, etc.*

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Can we solve problems faster, given an $O(1)$-sequence?