Some algorithmic applications of twin-width

Rémi Watrigant (LIP, Lyon)

Results mainly from:

*Twin-width III*, É. Bonnet, C. Geniet, E.J. Kim, S. Thomassé, R. W.

arxiv.org/abs/2007.14161

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Outline:

- **Maximum Independent Set**
- **Minimum Coloring**
- **Minimum Dominating Set**
Maximum Independent Set (MIS)

Theorem [Tww I]

Given a FO formula $\varphi$ and a $n$-vertex graph $G$ with a $d$-sequence of $G$, one can decide $G \models \varphi$ in time $f(|\varphi|, d)n$ for some computable function $f$.
Maximum Independent Set (MIS)

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"$\alpha(G) \geq k$" is equivalent to:

$$\exists x_1 \exists x_2 \cdots \exists x_k \bigwedge_{1 \leq i < j \leq k} \neg (x_i = x_j) \land \neg E(x_i, x_j) \land \neg E(x_j, x_i)$$

$\Rightarrow$ Deciding MIS is FPT in $k$ and $d := tww(G)$
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$\Rightarrow$ Deciding MIS is FPT in $k$ and $d := \text{tww}(G)$

But the function $f$ is a tower of exponentials 😞

$\rightarrow$ Now: $O(k^2 d^{2k} n)$ for MIS
Maximum Independent Set (MIS)

Before twin-width: cographs: twin-decomposition

\[ G_n \rightarrow G_{n-1} \rightarrow \ldots \rightarrow G_{i+1} \rightarrow G_i \rightarrow \ldots \rightarrow G_1 \]
Maximum Independent Set (MIS)
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\[ G_n \rightarrow G_{n-1} \rightarrow \ldots \rightarrow G_{i+1} \rightarrow G_i \rightarrow \ldots \rightarrow G_1 \]

Solving MIS:
- for \( i = n, \ldots, 1 \), for each \( u \in V(G_i) \), compute

\[ OPT(u) := OPT(G[u(G)]) \]

\[ \rightarrow \text{initialization ok} \]
\[ \rightarrow \text{in } G_1: \ OPT(u) = OPT(V(G)) \]
**Maximum Independent Set (MIS)**

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- when contracting \( u, v \) into \( z \):
Maximum Independent Set (MIS)

With a $d$-contraction sequence:

\[ G_n \rightarrow G_{n-1} \rightarrow \ldots \rightarrow G_{i+1} \rightarrow G_i \rightarrow \ldots \rightarrow G_1 \]

- each $G_i$ is a trigraph : $(V_i, E_i, R_i)$

Solving MIS:

for $i = n, \ldots, 1$

for each $T \subseteq V(G_i)$ connected red induced subgraph of size $\leq k$

Compute:

\[ \text{OPT}(T) := \text{OPT} \text{ of } G[\bigcup_{u \in T} u(G)] \]

intersecting each $u(G)$, for all $u \in T$

We might have $\text{OPT}(T) = \text{nil}$ (great figure by ´Edouard)
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(great figure by Édouard)
A graph with $n$ vertices and maximum degree $d$ has at most $d^{2k} n$ connected induced subgraphs of $\leq k$ vertices
Maximum Independent Set (MIS)

Lemma [folklore]

A graph with \( n \) vertices and maximum degree \( d \) has at most \( d^{2k} n \) connected induced subgraphs of \( \leq k \) vertices

\[ G_{i+1} \rightarrow G_i \]

\[ u, v \quad z \]

Let \( T \) be a \( CRIS_{\leq k} \) in \( G_i \)

How to compute \( OPT(T) \)?
**Maximum Independent Set (MIS)**

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A graph with $n$ vertices and maximum degree $d$ has at most $d^{2k}n$ connected induced subgraphs of $\leq k$ vertices

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How to compute $OPT(T)$?

- if $z \notin T$, we take $OPT(T)$ from $G_{i+1}$
Maximum Independent Set (MIS)

Lemma [folklore]

A graph with \( n \) vertices and maximum degree \( d \) has at most \( d^{2k} n \) connected induced subgraphs of \( \leq k \) vertices

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\[
u, v \rightarrow z
\]

Let \( T \) be a \( CRIS_{\leq k} \) in \( G_i \)

How to compute \( OPT(T) \)?

- if \( z \not\in T \), we take \( OPT(T) \) from \( G_{i+1} \)
- if \( z \in T \). How will \( OPT \) intersect \( z(G) \)?

\[
\begin{align*}
\text{OPT intersects only } u(G) & \quad \rightarrow T'_1 := T \setminus \{z\} \cup \{u\} \\
\text{OPT intersects only } v(G) & \quad \rightarrow T'_2 := T \setminus \{z\} \cup \{v\} \\
\text{OPT intersects both } u(G), v(G) & \quad \rightarrow T'_3 := T \setminus \{z\} \cup \{u, v\}
\end{align*}
\]

Construct a solution for each \( T'_\ell \) and take the best as \( OPT(T) \)
Maximum Independent Set (MIS)

What is $T'_\ell$ in $G_{i+1}$?
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example: OPT intersects only $v(G)$
What is $T'_\ell$ in $G_{i+1}$?
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Each $T'_\ell$ has $\leq d$ connected components in $G_{i+1}$

$$T_1, \ldots, T_q$$

which are all $CRIS_{\leq k}$ in $G_{i+1} \rightarrow$ take their $OPT(T_x)$
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- if:
  - there is a black edge between two $T_x, T_y$
  - or
    - $OPT(T_x)$ is $nil$ for some $x$
  → discard $T'_\ell$

- otherwise: take $OPT(T_1) \cup \cdots \cup OPT(T_q)$
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Then:

- if all $T'_1, T'_1, T'_3$ are discarded, $OPT(T)$ gets nil
- otherwise: take the best
Maximum Independent Set (MIS)

Running time:

- $n$ steps in the sequence
- at each step:
  - enumerate all $CRIS_{\leq k}$: $d^{2k} n$
  - look for a black edge between red c.c.: $k^2$

Theorem

Given $k \in \mathbb{N}$ and $G$ on $n$ vertices coming with a $d$-sequence, we can solve MIS in time $O(k^2 d^{2k} n^2)$

Same running time for:

- Maximum Clique
- Minimum Dominating Set
- $r$-Scattered Set
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Given $k \in \mathbb{N}$ and $G$ on $n$ vertices coming with a $d$-sequence, we can solve MIS in time $O(k^2d^{2k}n^2)$ $O(k^2d^{2k}n) = 2^{O_d(k)}n$

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Maximum Independent Set (MIS)

Generalizations:

- weighted version in $2^{O_d(k \log k)n}$
- **Induced Subgraph Isomorphism** in $2^{O_d(k \log k)n}$
  (generalizes the result for $H$-minor free [Pilipczuk, Siebertz 2019])

Lower bound (for MIS):

given a $O(1)$-sequence, no $2^{o(n/\log n)}$ algorithm unless ETH (subcubic graphs + $(2 \log n)$-subdivision)

Open questions:

- runs in poly-time in "number of connected red induced subgraphs" → graph classes admitting sequences with small number of such things?
- → does general graphs have contraction sequences with $O(cn)$ such things for some $c < 2$?
- → what about other properties than "bounded red degree"?
**Maximum Independent Set (MIS)**

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Open questions:
- runs in poly-time in “number of connected red induced subgraphs”
  → graph classes admitting sequences with small number of such things?
  → does general graphs have contraction sequences with $O(c^n)$ such things for some $c < 2$?
  → what about other properties than “bounded red degree”?
Outline:

- **Maximum Independent Set**
- **Minimum Coloring** (χ-boundedness)
- **Minimum Dominating Set**
Coloring ($\chi$-boundedness)

- for any graph $G$, it holds that $\chi(G) \geq \omega(G)$
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- a graph class $\mathcal{G}$ is $\chi$-bounded if there exists a function $f$ such that

  $$\forall G \in \mathcal{G} \quad \chi(G) \leq f(\omega(G))$$
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**Theorem [Tww III]**

For any graph $G$ of twin-width $\leq d$, we have $\chi(G) \leq (d + 2)^{\omega(G) - 1}$

If a $d$-sequence is given, we can find such a coloring in polynomial-time.
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**Theorem [Tww III]**

For any graph $G$ of twin-width $\leq d$, we have $\chi(G) \leq (d + 2)\omega(G) - 1$

If a $d$-sequence is given, we can find such a coloring in polynomial-time.

Works by induction on $\omega(G)$. Let’s prove the base case $\omega(G) = 2$, that is:

Given a triangle-free graph $G$ and a $d$-sequence of it, one can find in polynomial-time a $(d + 2)$-coloring of $G$. 

Coloring ($\chi$-boundedness)

Consider the $d$-sequence backward:

$$G_1 \rightarrow G_2 \rightarrow \ldots \rightarrow G_i \rightarrow G_{i+1} \rightarrow \ldots \rightarrow G_n$$

Observation 1 when $z$ splits into $u$, $v$:

$$N_E i \cup R_i (z) = N_E i + 1 \cup R_i + 1 (u, v)$$

Observation 2 (for triangle-free graphs only)

In the triangle-free case:

if $z$ is incident to a black edge, then $z (G)$ is an independent set
COLORING ($\chi$-boundedness)

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**Coloring (χ-boundedness)**

\[ G_1 \rightarrow G_2 \rightarrow \ldots \rightarrow G_i \rightarrow G_{i+1} \rightarrow \ldots \rightarrow G_n \]

"proper coloring" = with respect to \( E_i \cup R_i \)

- Assume \( G_i \) is properly \((d+2)\)-colored
  - \( z \) splits into \( u, v \)
- by Obs. 1, we can give to \( u \) the same color as \( z \)
  How do we color \( v \)?
**Coloring ($\chi$-boundedness)**

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    - if \( uv \notin E_{i+1} \cup R_{i+1} \), then give to \( v \) the same color as \( u \)
COLORING ($\chi$-boundedness)

$G_1 \rightarrow G_2 \rightarrow \ldots \rightarrow G_i \rightarrow G_{i+1} \rightarrow \ldots \rightarrow G_n$

“proper coloring” = with respect to $E_i \cup R_i$

- Assume $G_i$ is properly $(d + 2)$-colored
  - $z$ splits into $u, v$

- by Obs. 1, we can give to $u$ the same color as $z$

  How do we color $v$?
  - if $uv \notin E_{i+1} \cup R_{i+1}$, then give to $v$ the same color as $u$
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This is a proper \((d+2)\)-coloring of \( G_{i+1} \). Proof:
- *proper* by Obs. 1
**Coloring ($\chi$-boundedness)**

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This is a proper $(d + 2)$-coloring of $G_{i+1}$. Proof:

- *proper* by Obs. 1
- $d + 2$ colors:
  - if $z$ was incident to a black edge, then $uv \notin E_{i+1} \cup R_{i+1}$ (Obs. 2)
**COLORING (χ-boundedness)**

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This is a proper \((d + 2)\)-coloring of \( G_{i+1} \). Proof:

- **proper** by Obs. 1
- **\(d + 2\) colors**:
  - if \( z \) was incident to a black edge, then \( uv \notin E_{i+1} \cup R_{i+1} \) (Obs. 2)
  - otherwise, \( z \) had only \( \leq d \) (red) neighbors, so \( v \) has \( \leq d + 1 \) black/red neighbors
We have just seen $K_3$-free graphs coming with a $d$-sequence can be $(d + 2)$-colored in polynomial-time.

Generalization to $K_t$-free graphs, by induction on $t$: 
**Coloring** ($\chi$-boundedness)

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$K_3$-free graphs coming with a $d$-sequence can be $(d + 2)$-colored in polynomial-time.

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- now **Observation 2** becomes: *if $z$ is incident to a black edge, then $z(G)$ is $K_{t-1}$-free*
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- now **Observation 2** becomes: *if $z$ is incident to a black edge, then $z(G)$ is $K_{t-1}$-free*

  → we get by induction a coloring of $z(G)$ with $(d + 2)^{t-3}$ colors
Related work/open question:

- provides an “elementary” proof of “bounded rank-width classes are \( \chi \)-bounded” [Dvořák, Král’, 2012]

- bounded clique-width classes are \textit{polynomially} \( \chi \)-bounded [Bonamy, Pilipczuk, 2020]

\[ \rightarrow \text{are bounded twin-width graphs polynomially } \chi \text{-bounded?} \]
Outline:

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- **Minimum Coloring** ($\chi$-boundedness)
- **Minimum Dominating Set**
**Minimum Dominating Set**

**Versatile tree of $d$-contractions [Tww II]**

Up to a small degradation on the twin-width value $d$ of a graph:
- at each step of the sequence: there exist $\frac{|V(G_i)|}{s}$ disjoint pairs of vertices that we can contract
- all trigraphs of the tree have red degree $\leq d'$

→ can be computed in poly-time (given a $d$-sequence)
→ $s$ and $d'$ are functions of $d$ only
**Minimum Dominating Set**

**Linear program:**

\[
\text{minimize } \sum_{x \in V} w(x) \\
\text{s.t. } \sum_{y \in N[x]} w(x) \geq 1 \quad \text{for all } x \in V \\
0 \leq w(x) \leq 1 \quad \text{for all } x \in V
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Minimum Dominating Set

Linear program:

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\begin{align*}
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\end{align*}
\]

Let \( \gamma^*(G) \) be the optimal value of the LP
let \( w^* \) be its associated solution

We will prove the following:
Given an \( s \)-versatile tree of \( d \)-contractions, one can compute in polynomial-time a dominating set \( D \) of size \( \leq 2s(d + 1)\gamma^*(G) \)
**Minimum Dominating Set**

Using the $s$-versatile tree of $d$-contractions, we construct a $d$-sequence

**Contraction Rule**

At each step, choose a pair $(u, v)$ such that $w^*(u(G)), w^*(v(G)) < \frac{1}{2(d+1)}$

stop the sequence when there is no such pair

$G_n \rightarrow G_{n-1} \rightarrow \ldots \rightarrow G_{\text{stuck}}$

Let $n_{\text{stuck}}$ be the number of vertices in $G_{\text{stuck}}$
Minimum Dominating Set

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Let $n_{stuck}$ be the number of vertices in $G_{stuck}$

**Observation 1**

$n_{stuck} \leq 2s(d+1)\gamma^*(G)$
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Let $n_{stuck}$ be the number of vertices in $G_{stuck}$

**Observation 1**

$n_{stuck} \leq 2s(d+1)\gamma^*(G)$

**Proof:**

- In $G_{stuck}$, there are $\geq \frac{n_{stuck}}{s}$ disjoint pairs of $d$-contractions
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G_n \rightarrow G_{n-1} \rightarrow \ldots \rightarrow G_{\text{stuck}}
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Let $n_{\text{stuck}}$ be the number of vertices in $G_{\text{stuck}}$

### Observation 1

\[n_{\text{stuck}} \leq 2s(d+1)\gamma^*(G)\]

**Proof:**

- in $G_{\text{stuck}}$, there are $\geq \frac{n_{\text{stuck}}}{s}$ disjoint pairs of $d$-contractions
  - Contraction rule $\Rightarrow$ at least $\frac{n_{\text{stuck}}}{s}$ parts have weight $\geq \frac{1}{2(d+1)}$
Minimum Dominating Set

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Let $n_{\text{stuck}}$ be the number of vertices in $G_{\text{stuck}}$

**Observation 1**

$$n_{\text{stuck}} \leq 2s(d+1)\gamma^*(G)$$

Proof:

- in $G_{\text{stuck}}$, there are $\geq \frac{n_{\text{stuck}}}{s}$ disjoint pairs of $d$-contractions
  - Contraction rule $\Rightarrow$ at least $\frac{n_{\text{stuck}}}{s}$ parts have weight $\geq \frac{1}{2(d+1)}$
  - $\sum_{u \in V(G_{\text{stuck}})} w^*(u(G)) = \gamma^*(G)$
End of the algorithm:
Pick one arbitrary vertex from each $u \in V(G_{stuck}) \rightarrow$ solution $D$

- $|D| \leq 2s(d + 1)\gamma^*(G)$ by Obs 1
Minimum Dominating Set

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Pick one arbitrary vertex from each \( u \in V(G_{stuck}) \) \( \rightarrow \) solution \( D \)

\( |D| \leq 2s(d + 1) \gamma^*(G) \) by Obs 1

\( D \) is a dominating set of \( G \)

Proof: let \( u \in V(G_{stuck}) \), show that \( u(G) \) is dominated
Minimum Dominating Set

End of the algorithm:
Pick one arbitrary vertex from each $u \in V(G_{stuck}) \rightarrow$ solution $D$

- $|D| \leq 2s(d + 1)\gamma^*(G)$ by Obs 1
- $D$ is a dominating set of $G$
  Proof: let $u \in V(G_{stuck})$, show that $u(G)$ is dominated
    - if $u$ is incident to a black edge: done
**End of the algorithm:**

Pick one arbitrary vertex from each $u \in V(G_{stuck}) \rightarrow$ solution $D$

- $|D| \leq 2s(d + 1)\gamma^*(G)$ by Obs 1
- $D$ is a dominating set of $G$

Proof: let $u \in V(G_{stuck})$, show that $u(G)$ is dominated
  
  ▶ if $u$ is incident to a black edge: done
  
  ▶ otherwise: only $\leq d$ red neighbors
    
    for $y \in u(G)$, let $v_1, \ldots, v_q$ be the bags with at least one edge with $y$
    
    Claim: one of $u(G), v_1(G), \ldots, v_q(G)$ is a singleton:
    
    $$w^*(u) + \sum_{i=1}^{q} w^*(v_i) \geq 1$$

    One of them must have weight $\geq \frac{1}{d+1}$
    
    $\rightarrow$ must be a singleton by our **Contraction Rule**
Minimum Dominating Set

Related work/open questions:

- There is a PTAS in minor-closed classes [Cabello, Gajser, 2015]

OPEN:
- $c$-approximation in bounded tww graphs ($c$ independent of tww)?
- PTAS in bounded tww graphs?
Minimum Dominating Set

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Maximum Independent Set

- any $c$-approximation implies a PTAS in bounded tww graphs (iterated lexicographic product preserves tww)

- PTAS?
Questions?