Planar graphs have 1-string representations *

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Abstract

We prove that every planar graph is an intersection graph of strings in the plane, such that any two strings intersect at most once.

1 Introduction

A *string* $\mathbf{s}$ is a curve of the plane homeomorphic to a segment. A string $\mathbf{s}$ has two ends, the points of $\mathbf{s}$ that are not ends of $\mathbf{s}$ are *internal points* of $\mathbf{s}$. Two strings $\mathbf{s_1}$ and $\mathbf{s_2}$ *cross* if they have a common point $\mathbf{p} \in \mathbf{s_1} \cap \mathbf{s_2}$ and if going around $\mathbf{p}$ we successively meet $\mathbf{s_1}$, $\mathbf{s_2}$, $\mathbf{s_1}$, and $\mathbf{s_2}$. This means that a tangent point is not a "crossing". In the following we consider string sets without tangent points.

In this paper, we consider intersection models for simple planar graphs (*i.e.*, planar graphs without loops or multiple edges). A *string representation* of a graph $G = (V, E)$ is a set Σ of strings in the plane such that every vertex $v \in V$ maps to a string $\mathbf{v} \in \Sigma$ and such that $uv \in E$ if and only if the strings $\mathbf{u}$ and $\mathbf{v}$ cross (at least once). Similarly, a *segment representation* of a graph G is a string representation of G in which the strings are segments.

These notions were introduced by Ehrlich *et al.* [3], who proved the following:

Theorem 1 *[3] Planar graphs have a string representation.*

In [9], Koebe proved that planar graphs are the contact graphs of disks in the plane. Note that in this model the curves bounding two adjacent disks are tangent. However by inflating these circles we obtain string representations for planar graphs. In his PhD thesis, Scheinerman [10] conjectures a stronger result:

Conjecture 1 *[10] Planar graphs have a segment representation.*

*An abstract of this paper appeared in the Proceedings of the eighteenth annual ACM-SIAM Symposium on Discrete algorithms (SODA 2007).

Hartman *et al.* [8] and de Fraysseix *et al.* [4] proved Conjecture 1 for bipartite planar graphs. Castro *et al.* [1] proved Conjecture 1 for triangle-free planar graphs. Recently de Fraysseix and Ossona de Mendez [6] extended this to planar graphs that have a 4-coloring in which every induced cycle of length 4 uses at most 3 colors. Observe that, since parallel segments never cross, a set of parallel segments in a segment representation of a graph induces a stable set of vertices. The construction in [4, 8] (resp. [1]) has the nice property that there are only 2 (resp. 3) possible slopes for the segments. So the construction induces a 2-coloring (resp. 3-coloring) of G. Note that Castro *et al.* do not prove the 3-colorability of triangle-free planar graphs, they use such coloring of the graphs (by Grötzsch's Theorem) in their construction. West [11] proposed a stronger version of Conjecture 1 in which only 4 slopes are allowed, thus using the fact that these graphs are 4-colorable.

Notice that two segments cross at at most one point, whereas in the construction of Theorem 1, strings may cross twice. Let us define a *1-string representation* as a string representation in which any two strings cross at most once. Thus the following theorem is a step towards Conjecture 1.

Theorem 2 *Planar graphs have a 1-string representation.*

Note that if we would allow and consider tangent points, this theorem would directly follow from Koebe's theorem. Theorem 2 answers an open problem of de Fraysseix and Ossona de Mendez [5]. In the same article they noticed that Theorem 2 implies that any planar multigraph has a string representation such that the number of crossings between two strings equals the number of edges between the two corresponding vertices.

In the next section we provide some definitions and prove that it is sufficient to prove this theorem for triangulations. Section 3 is devoted to the study of string representations of 4-connected triangulations. In this section we use a decomposition technique of 4-connected triangulations that is inspired on Whitney's work [12] and that was recently used by the second author [7]. Then in Section 4 we finally prove Theorem 2 for all triangulations.

2 Preliminaries

2.1 Restriction to triangulations

Lemma 1 *Every planar graph is an induced subgraph of some planar triangulation.*

Proof. Let G be a planar graph embedded in the plane (*i.e.* a plane graph). The graph $h(G)$ is obtained from G by adding in every face f of G a new vertex v_f adjacent to every vertex incident to f in G. Notice that $h(G)$ is also a plane graph and that G is an induced subgraph of $h(G)$. Moreover $h(G)$ is connected, $h(h(G))$ is 2-connected, and $h(h(h(G)))$ is a triangulation.

Note that we have to apply the h operator several times: if a facial walk goes through the same vertex several times, since multiples edges are not allowed, we obtain a non-triangular face. □

It is clear that a 1-string representation of a triangulation T induces a 1-string representation for any of its induced subgraphs. It is thus sufficient to prove Theorem 2 for triangulations.

2.2 String Representations

In a plane graph G, the unbounded face of G is called the *outer-face* and every other face of G is an *inner-face* of G. An *outer-vertex* (resp. *outer-edge*) of G is a vertex (resp. edge) of G incident to the outer-face. The other vertices (resp. edges) of G are *inner-vertices* (resp. *inner-edges*). The set of outer-vertices (resp. outer-edges, inner-vertices, and inner-edges) of G is denoted by $V_o(G)$ (resp. $E_o(G)$, $V_i(G)$, and $E_i(G)$). A *near-triangulation* is a plane graph in which all the inner-faces are triangles. An edge uv is a *chord* of some near-triangulation T if uv is an inner-edge linking two outer-vertices. From now on, we use the following notation: the strings corresponding to vertices of a graph G are denoted by bold letters, *i.e.*, for any $v \in V(G)$ we denote its corresponding string by $\mathbf{v}$. We need that in a 1-string representation of a plane graph G, each face of G corresponds to some topological region of the string representation.

Definition 1 *Let $G = (V, E)$ be a plane graph with a 1-string representation Σ. Given a face abc of G, consider a triplet (a, b, c) of its incident vertices. An* (a, b, c)-region $\mathbf{abc}$ *is a region of the plane homeomorphic to a disk such that (see Figure 1):*

- *for any vertex $v \neq a$, b, and c we have $\mathbf{abc} \cap \mathbf{v} = \emptyset$ (i.e., $\mathbf{abc}$ intersects only with $\mathbf{a}, \mathbf{b}, \mathbf{c}$),*
- *$\mathbf{abc} \cap \mathbf{a} \cap \mathbf{b} = \emptyset$, $\mathbf{abc} \cap \mathbf{b} \cap \mathbf{c} = \emptyset$, and $\mathbf{abc} \cap \mathbf{c} \cap \mathbf{a} = \emptyset$ (i.e., $\mathbf{a}, \mathbf{b}, \mathbf{c}$ intersect outside $\mathbf{abc}$),*
- *both $\mathbf{abc} \cap \mathbf{b}$ and $\mathbf{abc} \cap \mathbf{c}$ are connected,*
- *the boundary of $\mathbf{abc}$ successively crosses (clockwise or anticlockwise) $\mathbf{a}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}$, $\mathbf{c}$.*

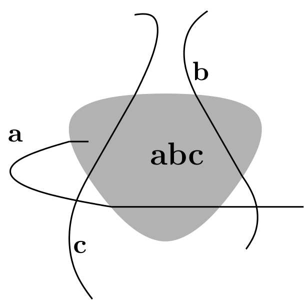

Figure 1: An (a, b, c)-region $\mathbf{abc}$.

Note that according to this definition $\mathbf{abc} \cap \mathbf{a}$ has two components and one end of $\mathbf{a}$ is in $\mathbf{abc}$. Note that the order in the triplet (a, b, c) matters: a region τ of the plane cannot be an (a, b, c)-region and a (c, b, a)-region for example. A region $\mathbf{abc}$ of the plane is an $\{a, b, c\}$-*region* if it is either an (a, b, c)-region, an (a, c, b)-region, a (b, a, c)-region, a (b, c, a)-region, a (c, a, b)-region, or a (c, b, a)-region. When the vertices a, b, and c are not mentioned, we call such a region a *face-region*.

Definition 2 *A* strong 1-string representation *(S-representation, for short) of a near-triangulation T is a pair (Σ, R) such that:*

(1) Σ *is a 1-string representation of* T*,*

(2) R *is a set of disjoint face-regions such that for every inner-face abc of* T*,* R *contains an* $\{a, b, c\}$*-region.*

A partial strong 1-string representation *(PS-representation, for short) of a near-triangulation* T *is a triplet* (Σ, R, F) *in which* $F \subseteq E(T)$ *and such that* (Σ, R) *is a strong 1-string representation of* T *without the crossings corresponding to the edges of* F*.*

In a PS-representation (Σ, R, F) of T, note that Σ is a 1-string representation of $T \setminus F$ and that each inner-face of T has a corresponding face-region in R.

2.3 Special Triangulations

In a near-triangulation T, a *separating 3-cycle* C is a cycle of length 3 such that some vertices of T lie inside C whereas other vertices lie outside. It is well known that a triangulation is 4-connected if and only if it contains no separating 3-cycle. In [12], Whitney considered a special family of near-triangulations, it is why we call them W-triangulations.

Definition 3 *A* W-triangulation *is a 2-connected near-triangulation containing no separating 3-cycle.*

In particular, any 4-connected triangulation is a W-triangulation. Note that since a W-triangulation has no cut vertex, its outer-edges induce a cycle. The following lemma gives a sufficient condition for a subgraph of a W-triangulation T to be a W-triangulation.

Lemma 2 *Let* T *be a W-triangulation and consider a cycle* C *of* T*. The subgraph induced by the vertices lying on and inside* C *is a W-triangulation.*

Proof. Consider the near-triangulation T' inside some cycle C of T. By definition, T has no separating 3-cycle and consequently T' does not have any separating 3-cycle. Since T' is clearly connected and has more than two vertices, we prove that it is 2-connected by showing that it does not contain any cut vertex.

Since the cycle C delimits the outer-face of T', any vertex $v \in V(T')$ appears at most once on the outer face. Since the outerface appears at most once around v and since all its other incident faces are triangles, T' contains a path linking all the neighbors of v. This implies that $T' \setminus v$ is connected and thus T' has no cut vertex. $\square$

Definition 4 *A W-triangulation* T *is* 3-bounded *if the outer-boundary of* T *is the union of three paths* $(a_1, \ldots, a_p)$*,* $(b_1, \ldots, b_q)$*, and* $(c_1, \ldots, c_r)$ *that satisfy the following conditions (see Figure 2):*

- $a_1 = c_r$*,* $b_1 = a_p$ *and* $c_1 = b_q$*.*
- *the paths are non-trivial, i.e.,* $p \geq 2$*,* $q \geq 2$ *and* $r \geq 2$*.*
- *there exists no chord* a_ia_j*,* b_ib_j *or* c_ic_j*.*

Such a 3-boundary *of* T *will be denoted by* $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$*.*

In the following, we will use the order on the three paths and their directions, *i.e.*, $(a_1, \ldots,\ a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ will be different from $(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$-$(a_1, \ldots, a_p)$ and $(a_p, \ldots, a_1)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_1)$.

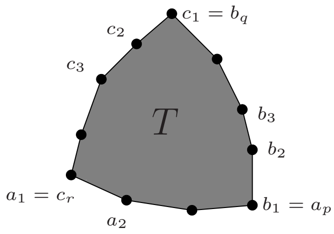

Figure 2: 3-boundary of T.

3 Proof for 4-connected triangulations.

The following property describes the shape of a PS-representation of a 3-bounded W-triangulation.

Property 1 *Consider a 3-bounded W-triangulation T with a 3-boundary $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. The W-triangulation T has Property 1 if T has a PS-representation (Σ, R, F) contained inside a region τ of the plane homeomorphic to the disk that satisfies the following properties (see Figure 3):*

(a) $F = E_o(T) \backslash \{a_1a_2\}$ (i.e., the missing crossings correspond to the outer edges, except a_1a_2),

(b) on the boundary of τ we successively have the ends of $\mathbf{a_2}, \mathbf{a_3}, \ldots, \mathbf{a_p}, \mathbf{b_1}, \ldots, \mathbf{b_q}, \mathbf{c_1}, \ldots, \mathbf{c_r}$.

If going clockwise (resp. anticlockwise) around the boundary of τ, we cross the strings in the order described in (b), we say that the PS-representation is *clockwise* (resp. *anticlockwise*). Note that by an axial symmetry, one can obtain a clockwise PS-representation from an anticlockwise PS-representation, and vice versa. Observe that since $a_p = b_1$, $b_q = c_1$, and $c_r = a_1$, both ends of $\mathbf{b_1}$ and $\mathbf{c_1}$ lie on the boundary of τ, but it is not the case for $\mathbf{a_1}$ or any other string (*i.e.*, all the strings appearing on the boundary of τ have an end inside τ except $\mathbf{b_1}$ and $\mathbf{c_1}$).

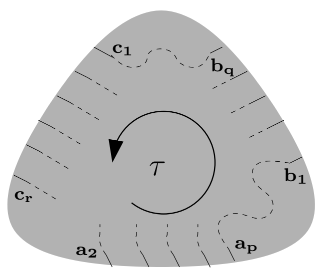

Figure 3: Property 1

Before proving that each 3-bounded W-triangulation has Property 1, we give some definitions and we present Property 2. Consider a 3-bounded W-triangulation $T \neq K_3$ whose boundary is $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ and such that T does not contain any chord a_ib_j or a_ic_j. Let $D \subseteq V_i(T)$ be the set of inner-vertices of T that are adjacent to some vertex a_i with $i > 1$ (the black vertices on the left of Figure 4). Since T has at least 4 vertices, no separating 3-cycle, and no chord a_ia_j, a_ib_j, or a_ic_j, then a_1 and a_2 (resp. b_1 and b_2) have exactly one common neighbor in $V_i(T)$ that will be denoted a (resp. d_1).

Since there is no chord a_ia_j, a_ib_j, or a_ic_j, for each vertex a_i with $i \in [2, p-1]$, all the neighbors of a_i (resp. a_p) except a_{i-1} and a_{i+1} (resp. a_{p-1} and b_2) are in D. Since for each $i \in [2, p]$, there is a path linking the neighbors of a_i in D, and since the vertices a_i and a_{i+1} have a common neighbor in D, then the set D induces a connected graph. Since a is in D, the set $D \cup \{a_1\}$ also induces a connected graph.

Definition 5 *The* adjacent path *of T with respect to the 3-boundary $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ is the shortest path linking d_1 and a_1 in $T[D \cup \{a_1\}]$ (the graph induced by $D \cup \{a_1\}$). This path will be denoted $(d_1, d_2, \ldots, d_s, a_1)$.*

Observation 1 *There exists neither an edge d_id_j with $2 \leq i+1 < j \leq s$, nor an edge a_1d_i with $1 \leq i < s$. Otherwise, $(d_1, d_2, \ldots d_s, a_1)$ would not be the shortest path between d_1 and a_1.*

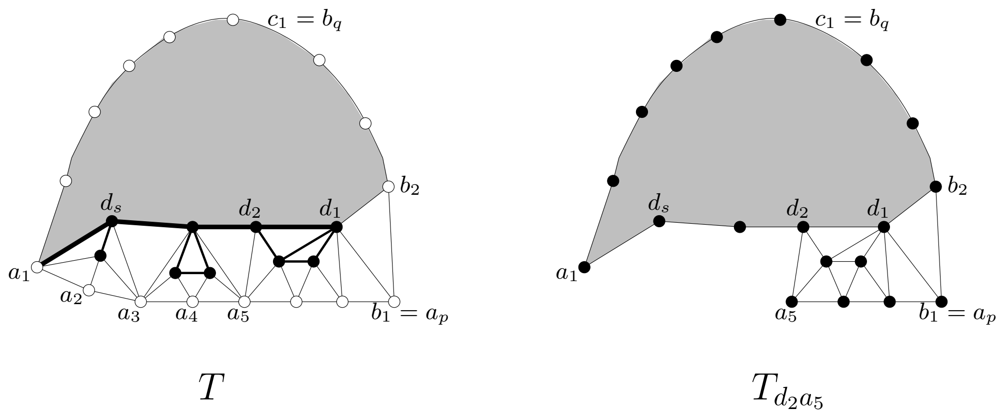

Figure 4: the adjacent path of T and the graph $T_{d_2a_5}$.

Definition 6 *For each edge $d_xa_y \in E(T)$ with $x \in [1, s]$ and $y \in [2, p]$, the graph $T_{d_xa_y}$ is the graph lying inside the cycle $C = (a_1, d_s, \ldots, d_x, a_y, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_r)$ (see Figure 4).*

Note that since $D \subseteq V_i(T)$, C is a cycle and by Lemma 2, $T_{d_xa_y}$ is a W-triangulation. The following property describes the shape of a PS-representation of $T_{d_xa_y}$.

Property 2 *Consider a 3-bounded W-triangulation T with a 3-boundary $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ that does not have any chord a_ib_j or a_ic_j and let $(d_1, d_2, \ldots, d_s, a_1)$ be its adjacent path. Consider an edge $d_xa_y \in E(T)$ with $y > 1$. The W-triangulation $T_{d_xa_y}$ has Property 2 if $T_{d_xa_y}$ has a PS-representation (Σ, R, F) satisfying the following properties (see Figure 5):*

(a) $F = E_o(G) \setminus \{d_xa_y\}$,

(b) Every string $\mathbf{v} \in \Sigma \setminus \{\mathbf{d_x}, \mathbf{a_y}\}$ is contained in a region τ of the plane homeomorphic to the disk. Furthermore $\mathbf{d_x}$ and $\mathbf{a_y}$ have their ends in τ (or on the boundary of τ) but they cross each other outside τ.

(c) *each face-region of R is contained inside τ,*

(d) *on the boundary of τ we successively have the ends of* $\mathbf{a_y}, \ldots, \mathbf{a_p}, \mathbf{b_1}, \ldots, \mathbf{b_q}, \mathbf{c_1}, \ldots, \mathbf{c_r}$, $\mathbf{a_1}, \mathbf{d_s}, \ldots, \mathbf{d_{x+1}}$, *and then we successively have internal points of* $\mathbf{d_x}, \mathbf{a_y}, \mathbf{d_x}$, *and* $\mathbf{a_y}$.

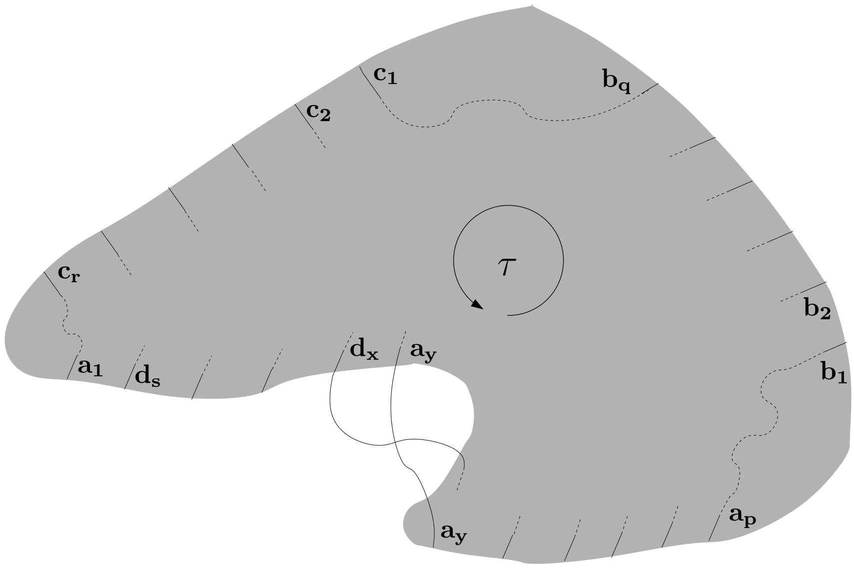

Figure 5: Property 2.

Here again, if going clockwise (resp. anticlockwise) around the boundary of τ, we cross the strings in the order described in (d), we say that the PS-representation is *clockwise* (resp. *anticlockwise*). In the proof of Theorem 2, we only use Property 1. However, in order to prove Property 1, we use Property 2. We prove these two properties by doing a "crossed" induction.

Proof of Property 1 and Property 2

We prove, by induction on $m \geq 3$, that the following two statements hold:

- Property 1 holds if T has at most m edges.
- Property 2 holds if $T_{d_x a_y}$ has at most m edges.

The initial case, $m = 3$, is easy to prove since there is only one W-triangulation having at most 3 edges, K_3. For Property 1, we have to consider all the possible 3-boundaries of K_3. All these 3-boundaries are equivalent, so let $V(K_3) = \{a, b, c\}$ and consider the 3-boundary (a, b)-(b, c)-(c, a). In Figure 6 there is a PS-representation (Σ, R, F) of K_3 with $F = \{bc, ac\}$ that fulfills Property 1. For Property 2, since a W-triangulation $T_{d_x a_y}$ has at least 4 vertices, a_1, b_1, c_1, and d_1, we have $T_{d_x a_y} \neq K_3$ and there is no W-triangulation $T_{d_x a_y}$ with at most 3 edges. So by vacuity, Property 2 holds for $T_{d_x a_y}$ with at most 3 edges.

The induction step applies to both Property 1 and Property 2. This means that we prove Property 1 (resp. Property 2) for the W-triangulations T (resp. $T_{d_x a_y}$) with m edges using both Property 1 and Property 2 on W-triangulations with less than m edges. We first prove the induction for Property 1.

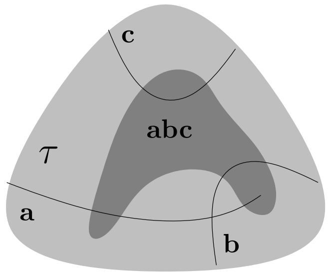

Figure 6: Initial case for Property 1.

Case 1: Proof of Property 1 for a W-triangulation T with m edges. Let $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ be the 3-boundary of T considered. We distinguish different cases according to the existence of a chord a_ib_j or a_ic_j in T. We successively consider the case where there is a chord a_1b_i, with $1 < i < q$, the case where there is a chord a_ib_j, with $1 < i < p$ and $1 < j \leq q$, and the case where there is a chord a_ic_j, with $1 < i \leq p$ and $1 < j < r$. We then finish with the case where there is no chord a_ib_j, with $1 \leq i \leq p$ and $1 \leq j \leq q$ (by definition of 3-boundary, T has no chord a_1b_q, a_ib_1, or a_pb_j), and no chord a_ic_j, with $1 \leq i \leq p$ and $1 \leq j \leq r$ (by definition of 3-boundary, T has no chord a_pc_1, a_ic_r, or a_1c_j).

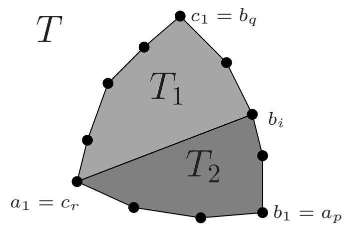

Figure 7: Case 1.1: Chord a_1b_i.

Case 1.1: There is a chord a_1b_i, with $1 < i < q$ (see Figure 7). Let T_1 (resp. T_2) be the subgraph of T that lies inside the cycle $(a_1, b_i, \ldots, b_q, c_2, \ldots, c_r)$ (resp. $(a_1, a_2, \ldots, a_p, b_2, \ldots, b_i, a_1)$). By Lemma 2, T_1 and T_2 are W-triangulations. Since T has no chord a_xa_y, b_xb_y, or c_xc_y, (b_i, a_1)-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_i)$ (resp. $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_i)$-(b_ia_1)) is a 3-boundary of T_1 (resp. T_2). Furthermore, since $a_1a_2 \notin E(T_1)$ and $c_1c_2 \notin E(T_2)$, T_1 and T_2 have less edges than T and Property 1 holds for T_1 and T_2 with the mentioned 3-boundaries. Let (Σ_1, R_1, F_1) (resp. (Σ_2, R_2, F_2)) be a clockwise (resp. anticlockwise) PS-representation contained in the region τ_1 (resp. τ_2) obtained for T_1 (resp. T_2) with $F_1 = E_o(T_1) \setminus \{a_1b_i\}$ (resp. $F_2 = E_o(T_2) \setminus \{a_1a_2\}$). In Figure 8 we show how to associate these two representations to obtain (Σ, R, F), an anticlockwise PS-representation of T contained in τ. Note that the two strings $\mathbf{a_1}$ (resp. $\mathbf{b_i}$) from Σ_1 and Σ_2 have been linked.

We easily verify that (Σ, R, F) satisfies Property 1:

- Σ is a string representation of $T \setminus F$ with $F = E_o(T) \setminus \{a_1a_2\}$. Indeed, since $V(T_1) \cup V(T_2) = V(T)$ and $V(T_1) \cap V(T_2) = \{a_1, b_i\}$, every vertex $v \in V(T)$ has exactly one

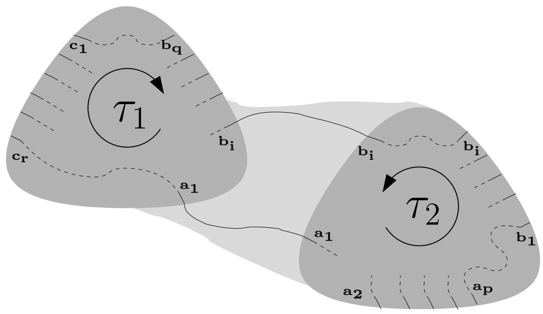

Figure 8: Case 1.1: (Σ, R, F).

string in Σ. Furthermore, since $(E(T_1) \setminus F_1) \cup (E(T_2) \setminus F_2) = E(T) \setminus F$, Σ is a string representation of $T \setminus F$.

- Σ is a 1-string representation. The only edge that belongs to both T_1 and T_2 is $a_1 b_i$. Since $\mathbf{a_1}$ and $\mathbf{b_i}$ cross each other in Σ_1 ($a_1 b_i \notin F_1$) but not in Σ_2 ($a_1 b_i \in F_2$), $\mathbf{a_1}$ and $\mathbf{b_i}$ cross exactly once in Σ.
- (Σ, R) is "strong": Each inner-face of T is an inner-face in T_1 or T_2 and the regions τ_1 and τ_2 are disjoint (so the face-regions in τ_1 are disjoint from the face-regions in τ_2).

Finally we see in Figure 8 that point (b) of Property 1 is satisfied.

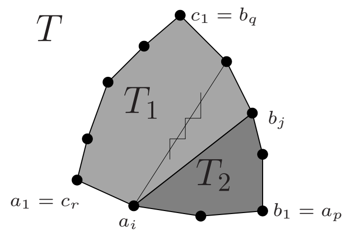

Figure 9: Case 1.2: Chord $a_i b_j$.

Case 1.2: There is a chord $a_i b_j$, with $1 < i < p$ and $1 < j \leq q$ (see Figure 9). If there are several chords $a_i b_j$, we consider one that maximizes j, *i.e.*, there is no chord $a_i b_k$ with $j < k \leq q$. Let T_1 (resp. T_2) be the subgraph of T that lies inside the cycle $(a_1, a_2, \ldots, a_i, b_j, \ldots, b_q, c_2, \ldots, c_r)$ (resp. $(a_i, \ldots, a_p, b_2, \ldots, b_j, a_i)$). By Lemma 2, T_1 and T_2 are W-triangulations. Since T has no chord $a_x a_y$, $b_x b_y$, $c_x c_y$, or $a_i b_k$ with $k > j$, $(a_1, \ldots, a_i)$-$(a_i, b_j, \ldots, b_q)$-$(c_1, \ldots, c_r)$ (resp. (a_i, b_j)-$(b_j, \ldots, b_1)$-$(a_p, \ldots, a_i)$) is a 3-boundary of T_1 (resp. T_2). Furthermore, since $b_1 b_2 \notin E(T_1)$ and $a_1 a_2 \notin E(T_2)$, T_1 and T_2 have less edges than T and Property 1 holds for T_1 and T_2 with the mentioned 3-boundaries. Let (Σ_1, R_1, F_1) (resp. (Σ_2, R_2, F_2)) be an anticlockwise (resp. clockwise) PS-representation contained in the region τ_1 (resp. τ_2) obtained for T_1 (resp. T_2), with $F_1 = E_o(T_1) \setminus \{a_1 a_2\}$ (resp. $F_2 = E_o(T_2) \setminus \{a_i b_j\}$). In Figure 10 we show how to associate these two representations to obtain (Σ, R, F), an anticlockwise PS-representation of T contained in τ. Note that in this construction the two strings $\mathbf{a_i}$ (resp. $\mathbf{b_j}$) from Σ_1 and Σ_2 have been linked.

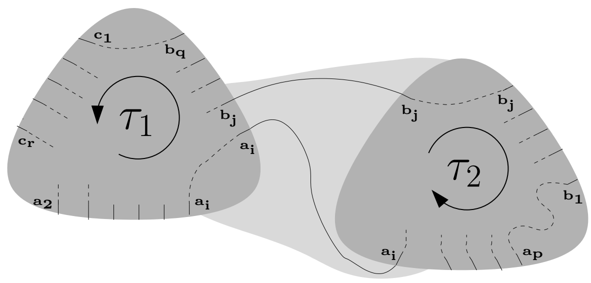

Figure 10: Case 1.2: (Σ, R, F).

As in Case 1.1, we easily verify that (Σ, R, F) satisfies Property 1.

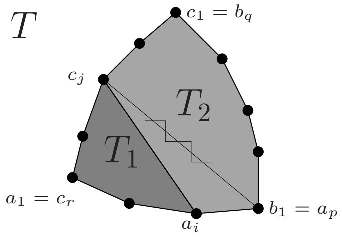

Figure 11: Case 1.3: Chord a_ic_j.

Case 1.3: There is a chord a_ic_j, with $1 < i \leq p$ and $1 < j < r$ (see Figure 11). If there are several chords a_ic_j, we consider one which maximizes i, *i.e.*, there is no chord a_kc_j with $i < k \leq p$. Let T_1 (resp. T_2) be the subgraph of T that lies inside the cycle $(a_1, a_2, \ldots, a_i, c_j, \ldots, c_r)$ (resp. $(c_j, a_i, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_j)$). By Lemma 2, T_1 and T_2 are W-triangulations. Since T has no chord a_xa_y, b_xb_y, c_xc_y or a_kc_j with $k > i$, $(a_1, \ldots, a_i)$-(a_i, c_j)-$(c_j, \ldots, c_r)$ (resp. $(c_j, a_i, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_j)$) is a 3-boundary of T_1 (resp. T_2). Furthermore, since $b_1b_2 \notin E(T_1)$ and $a_1a_2 \notin E(T_2)$, T_1 and T_2 have less edges than T and Property 1 holds for T_1 and T_2 with the mentioned 3-boundaries. Let (Σ_1, R_1, F_1) (resp. (Σ_2, R_2, F_2)) be an anticlockwise PS-representation contained in the region τ_1 (resp. τ_2) obtained for T_1 (resp. T_2), with $F_1 = E_o(T_1) \setminus \{a_1a_2\}$ (resp. $F_2 = E_o(T_2) \setminus \{c_ja_i\}$). In Figure 12 we show how to associate these two representations to obtain (Σ, R, F), an anticlockwise PS-representation of T contained in τ. Note that in this construction the two strings $\mathbf{a_i}$ (resp. $\mathbf{c_j}$) from Σ_1 and Σ_2 have been linked.

As in Case 1.1, we easily verify that (Σ, R, F) satisfies Property 1.

Case 1.4: There is no chord a_ib_j, with $1 \leq i \leq p$ and $1 \leq j \leq q$, and no chord a_ic_j, with $1 \leq i \leq p$ and $1 \leq j \leq r$ (see Figure 13). In this case we consider the adjacent path $(d_1, \ldots, d_s, a_1)$ (see Figure 4) of T with respect to its 3-boundary, $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. Consider the edge d_sa_y, with $1 < y \leq p$ and which minimizes y. This edge exists since, by definition of the adjacent path, d_s is adjacent to some vertex a_y with $y > 1$. The W-triangulation $T_{d_sa_y}$ having less edges than T ($a_1a_2 \notin E(T_{d_sa_y})$), Property 2 holds for

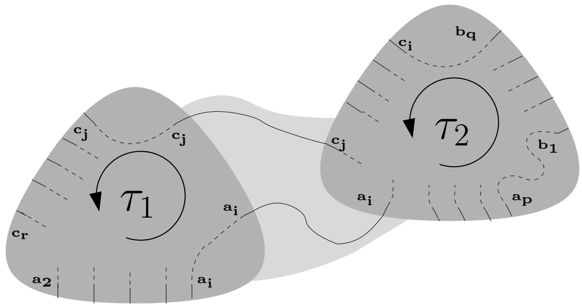

Figure 12: Case 1.3: (Σ, R, F).

$T_{d_s a_y}$. Let (Σ', R', F') be an anticlockwise PS-representation almost contained in the region τ' obtained for $T_{d_s a_y}$, with $F' = E_o(T_{d_s a_y}) \setminus \{d_s a_y\}$.

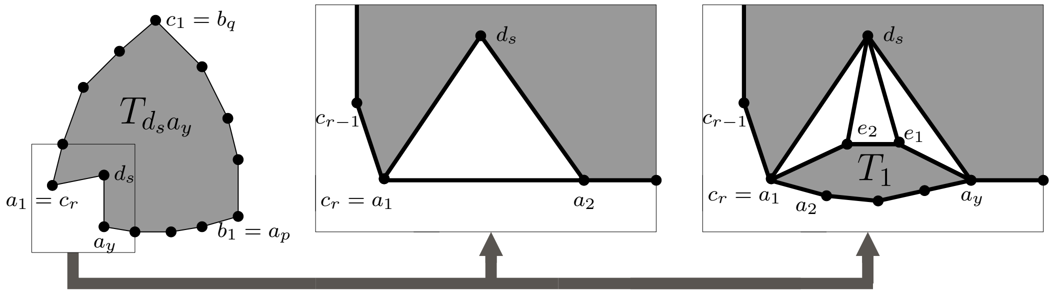

Figure 13: Case 1.4: No chord $a_i b_j$ or $a_i c_j$.

Now we distinguish two cases according to the position of a_y: either $y = 2$ (Case 1.4.1), or $y > 2$ (Case 1.4.2).

Case 1.4.1: $y = 2$. In Figure 14, starting from (Σ', R', F'), we show how to extend the string $\mathbf{a_1} \in \Sigma'$ (in order to cross $\mathbf{d_s}$ and $\mathbf{a_2}$) and how to draw the (a_1, a_2, d_s)-region $\mathbf{a_1 a_2 d_s}$ to obtain (Σ, R, F), an anticlockwise PS-representation of T contained in a region τ.

One can verify on Figure 14 that (Σ, R, F) satisfies Property 1.

Case 1.4.2: $y > 2$. Let us denote $e_1, e_2, \ldots, e_t$ the neighbors of d_s strictly inside the cycle $(d_s, a_1, a_2, \ldots, a_y, d_s)$, going "from right to left" (see Figure 13). By minimality of y we have $e_i \neq a_j$, for all $1 \leq i \leq t$ and $1 \leq j \leq y$.

Let T_1 be the subgraph of T that lies inside the cycle $(a_1, \ldots, a_y, e_1, \ldots, e_t, a_1)$. By Lemma 2, T_1 is a W-triangulation. Since the W-triangulation T has no separating 3-cycle (d_s, a_1, e_i), (d_s, a_y, e_i) or (d_s, e_i, e_j), there exists no chord $a_1 e_i$, $a_y e_i$ or $e_i e_j$ in T_1. So (a_2, a_1)-$(a_1, e_t, \ldots, e_1, a_y)$-$(a_y, \ldots, a_2)$ is a 3-boundary of T_1. Finally, since T_1 has less edges than T ($a_1 d_s \notin E(T_1)$), Property 1 holds for T_1 with respect to the mentioned 3-boundary. Let (Σ_1, R_1, F_1) be a clockwise PS-representation contained in the region τ_1 obtained for T_1, with $F_1 = E_0(T_1) \setminus \{a_2 a_1\}$.

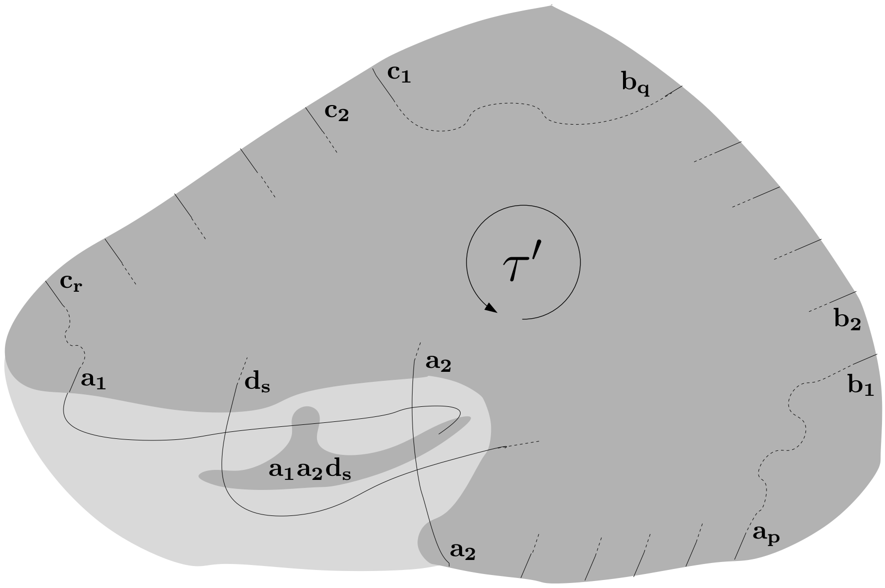

Figure 14: Case 1.4.1.

In Figure 15, starting from (Σ', R', F') and (Σ_1, R_1, F_1), we show how to join the strings $\mathbf{a_1}$ (resp. $\mathbf{a_y}$) of Σ' and Σ_1, how to extend the strings $\mathbf{e_i}$, for $1 \leq i \leq t$, and how to draw the face-regions $\mathbf{a_y e_1 d_s}$, $\mathbf{e_t a_1 d_s}$, and $\mathbf{e_i e_{i-1} d_s}$, for $2 \leq i \leq t$, in order to obtain (Σ, R, F), an anticlockwise PS-representation of T contained in a region τ.

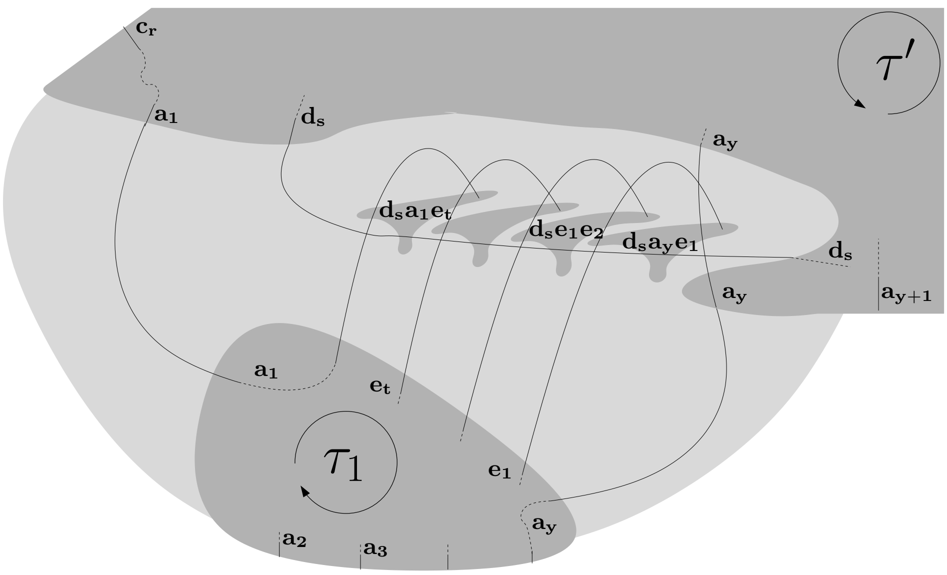

Figure 15: Case 1.4.2.

We verify that (Σ, R, F) satisfies Property 1:

- Σ is a string representation of $T \setminus F$ with $F = E_o(T) \setminus \{a_1 a_2\}$. Indeed, since $V(T_{d_s a_y}) \cup$

$V(T_1) = V(T)$ and $V(T_{d_s a_y}) \cap V(T_1) = \{a_1, a_y\}$, every vertex $v \in V(T)$ has exactly one string in Σ. Furthermore, since $E(T) \setminus F = (E(T_{d_s a_y}) \setminus F') \cup (E(T_1) \setminus F_1) \cup \{a_y e_1, e_t a_1, d_s a_1\} \cup \{e_i e_{i-1} \mid i \in [2,t]\} \cup \{d_s e_i \mid i \in [1,t]\}$, Σ is a string representation of $T \setminus F$.

- Σ is a 1-string representation. Indeed $T_{d_s a_y}$ and T_1 do not have common edges, and the new crossings added correspond to edges missing in both $E(T_{d_s a_y}) \setminus F'$ and $E(T_1) \setminus F_1$.

- (Σ, R) is "strong": The only inner-faces of T not in $T_{d_s a_y}$ nor in T_1 are the faces $d_s a_y e_1$, $d_s a_1 e_t$ and $d_s e_i e_{i+1}$, with $1 \leq i < t$. These faces correspond to the new face-regions.

Finally we see in Figure 15 that point (b) of Property 1 is satisfied.

So Property 1 holds for any W-triangulation T with m edges and this concludes the proof of Case 1.

Case 2: Proof of Property 2 for a W-triangulation $T_{d_x a_y}$ with m edges. Recall that the W-triangulation $T_{d_x a_y}$ is a subgraph of a W-triangulation T with 3-boundary $(a_1, \dots, a_p)$-$(b_1, \dots, b_q)$-$(c_1, \dots, c_r)$. Moreover, T has no chord $a_i b_j$ or $a_i c_j$ and its adjacent path is $(d_1, \dots, d_s, a_1)$, with $s \geq 1$. We distinguish the case where $d_x a_y = d_1 a_p$ and the case where $d_x a_y \neq d_1 a_p$.

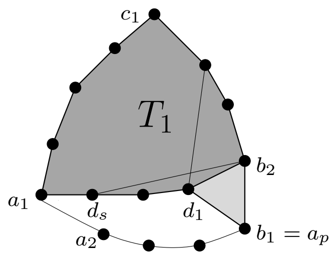

Figure 16: Case 2.1: $T_{d_x a_y} = T_{d_1 a_p}$.

Case 2.1: $d_x a_y = d_1 a_p$ (see Figure 16). Let T_1 be the subgraph of $T_{d_1 a_p}$ that lies inside the cycle $(a_1, d_s, \dots, d_1, b_2, \dots, b_q, c_2, \dots, c_r)$. By Lemma 2, T_1 is a W-triangulation. This W-triangulation has no chord $b_i b_j$, $c_i c_j$, $d_i d_j$, or $a_1 d_j$. We consider two cases according to the existence of an edge $d_1 b_i$ with $2 < i \leq q$.

- If T_1 has no chord $d_1 b_i$ then $(d_1, b_2, \dots, b_q)$-$(c_1, \dots, c_r)$-$(a_1, d_s, \dots, d_1)$ is a 3-boundary of T_1.

- If T_1 has a chord $d_1 b_i$, with $2 < i \leq q$, note that $q > 2$ and that there cannot be a chord $b_2 a_1$ or $b_2 d_j$, with $1 < j \leq s$ (this would violate the planarity of $T_{d_x a_y}$, see Figure 16) So in this case, $(b_2, d_1, \dots, d_s, a_1)$-$(c_r, \dots, c_1)$-$(b_q, \dots, b_2)$ is a 3-boundary of T_1.

Finally, since T_1 is a W-triangulation with less edges than $T_{d_1 a_p}$ ($b_1 b_2 \notin E(T_1)$), Property 1 holds for T_1 with respect to at least one of the two mentioned 3-boundaries. Whichever 3-boundary we consider, we obtain a PS-representation (Σ_1, R_1, F_1) of T_1 contained in a region τ_1, with the same following characteristics:

- $F_1 = E_o(T) \setminus \{d_1 b_2\}$,
- in the boundary of τ_1 we successively meet the ends of $\mathbf{d_1}, \ldots, \mathbf{d_s}, \mathbf{a_1}, \mathbf{c_r}, \ldots, \mathbf{c_1}, \mathbf{b_q}, \ldots, \mathbf{b_2}$ (clockwise or anticlockwise).

In Figure 17 we modify (Σ_1, R_1, F_1), by extending the strings $\mathbf{d_1}$ and $\mathbf{b_2}$ and by adding a new string $\mathbf{a_p}$ and a new face-region $\mathbf{d_1 b_2 a_p}$. This leads to (Σ, R, F), a PS-representation of $T_{d_1 a_p}$ contained in a region τ.

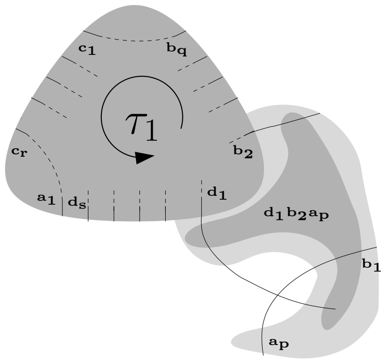

Figure 17: Case 2.1: (Σ, R, F).

We verify that (Σ, R, F) satisfies Property 2:

- Σ is a 1-string representation of $T_{d_1 a_p} \setminus F$: Indeed, $E(T_{d_1 a_p}) \setminus F$ is the disjoint union of $E(T_1) \setminus F_1$ and $\{a_p d_1\}$.
- (Σ, R) is “strong”: The only inner-face of $T_{d_1 a_p}$ that is not an inner-face of T_1 is $d_1 a_p b_2$, which corresponds to the new face-region $\mathbf{d_1 a_p b_2}$.

Finally we see in Figure 17 that the other points of Property 2 are satisfied.

Case 2.2: $T_{d_x a_y} \neq T_{d_1 a_p}$**.** In this case we consider an edge $d_z a_w \in E(T_{d_x a_y})$ such that $d_z a_w \neq d_x a_y$. Among all the possible edges $d_z a_w$ we choose the one that first maximizes z and then minimizes w. Such an edge necessarily exists and actually one can see that $d_z = d_x$ or $d_z = d_{x-1}$. Indeed, if $d_x = d_1$ there is at least one edge $d_1 a_w$ with $w > y$, the edge $d_1 a_p$. If $x > 1$, it is clear by definition of the adjacent path that the vertex d_{x-1} is adjacent to at least one vertex a_w with $w \geq y$.

By Lemma 2, $T_{d_z a_w}$ is a W-triangulation. Since $d_x a_y \notin E(T_{d_z a_w})$, the W-triangulation $T_{d_z a_w}$ has less edges than $T_{d_x a_y}$, and so Property 2 holds for $T_{d_z a_w}$. Let (Σ', R', F') be an anticlockwise PS-representation almost contained in the region τ' obtained for $T_{d_z a_w}$, with $F' = E_o(T_{d_z a_w}) \setminus \{d_z a_w\}$.

We distinguish 4 cases according to the edge $d_z a_w$. When $z = x$ we consider the case where $w = y + 1$ and the case where $w > y + 1$. When $z = x - 1$ we consider the case where $w = y$ and the case where $w > y$.

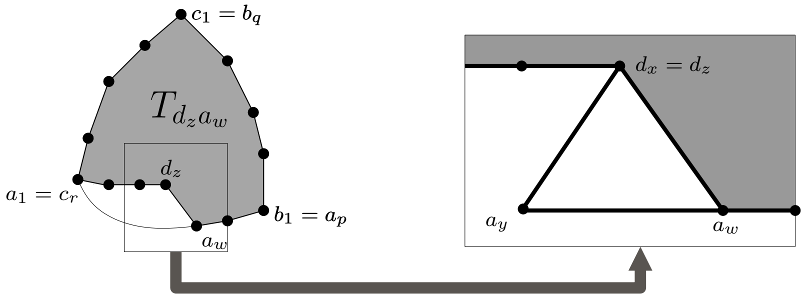

Figure 18: Case 2.2.1: $z = x$ and $w = y + 1$.

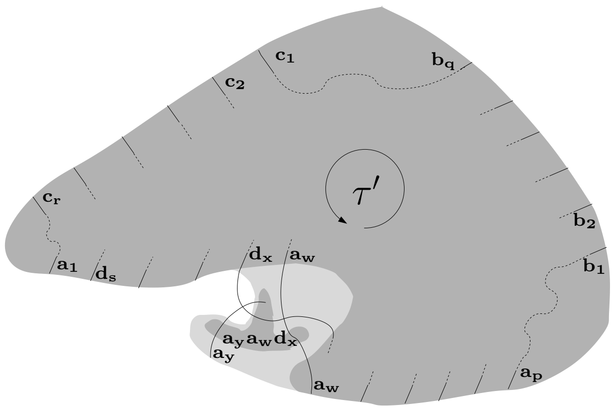

Figure 19: Case 2.2.1: (Σ, R, F).

Case 2.2.1: $T_{d_x a_y} \neq T_{d_1 a_p}$**,** $z = x$ **and** $w = y + 1$ **(see Figure 18).** In Figure 19 we modify (Σ', R', F'), by adding a new string $\mathbf{a_y}$ and a new face-region $\mathbf{a_y a_w d_x}$. This leads to (Σ, R, F), an anticlockwise PS-representation of $T_{d_x a_y}$ almost contained in a region τ.

We verify that (Σ, R, F) satisfies Property 2:

- Σ is a 1-string representation of $T_{d_x a_y} \setminus F$: Indeed, $E(T_{d_x a_y}) \setminus F$ is the disjoint union of $E(T_{d_z a_w}) \setminus F'$ and $\{d_x a_y\}$.
- (Σ, R) is "strong": The only inner-face of $T_{d_x a_y}$ that is not an inner-face of $T_{d_z a_w}$ is $d_x a_y a_w$, which corresponds to the new face-region $\mathbf{d_x a_y a_w}$.

Finally we see in Figure 19 that the other points of Property 2 are satisfied.

Case 2.2.2: $z = x - 1$ **and** $w = y$ **(see Figure 20).** In Figure 21, we modify (Σ', R', F') by extending the string $\mathbf{d_x}$ and by adding a new face-region $\mathbf{d_x d_z a_y}$. This leads to (Σ, R, F), an anticlockwise PS-representation of $T_{d_x a_y}$ almost contained in a region τ.

We verify that (Σ, R, F) satisfies Property 2:

- Σ is a 1-string representation of $T_{d_x a_y} \setminus F$: Indeed, $E(T_{d_x a_y}) \setminus F$ is the disjoint union of $E(T_{d_z a_w}) \setminus F'$ and $\{d_x d_z, d_x a_y\}$.

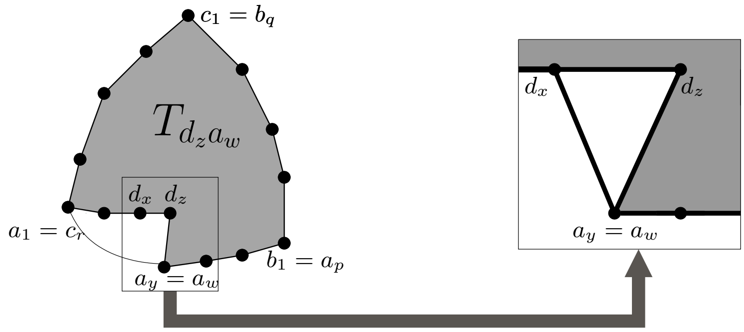

Figure 20: Case 2.2.2: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x - 1$ and $w = y$.

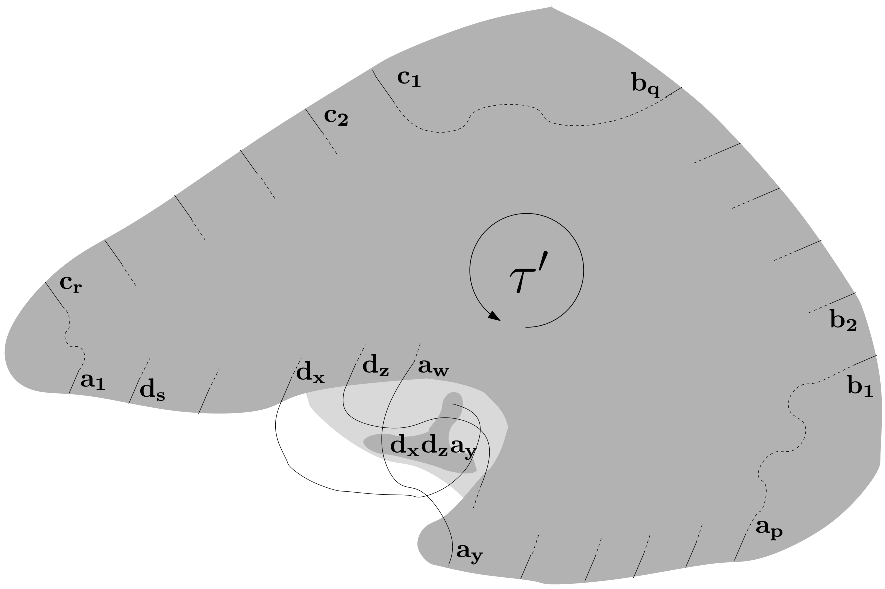

Figure 21: Case 2.2.2: (Σ, R, F).

- (Σ, R) is "strong": The only inner-face of $T_{d_x a_y}$ that is not an inner-face of $T_{d_z a_w}$ is $d_x d_z a_y$, which corresponds to the new face-region $\mathbf{d_x d_z a_y}$.

Finally we see in Figure 21 that the other points of Property 2 are satisfied.

Case 2.2.3: $z = x$ **and** $w > y + 1$ **(see Figure 22).** Let us denote $e_1, e_2, \ldots, e_t$ the neighbors of d_x strictly inside the cycle $(d_x, a_y, \ldots, a_w, d_x)$, going "from right to left" (see Figure 22). Since there is no chord $a_i a_j$ we have $t \geq 1$. Furthermore by minimality of w we have $e_i \neq a_j$, for all $1 \leq i \leq t$ and $y \leq j \leq w$. Let T_1 be the subgraph of $T_{d_x a_y}$ that lies inside the cycle $(a_y, \ldots, a_w, e_1, \ldots, e_t, a_y)$. By Lemma 2, T_1 is a W-triangulation. Since the W-triangulation $T_{d_x a_y}$ has no separating 3-cycle (d_x, a_w, e_i) or (d_x, e_i, e_j), there exists no chord $a_w e_i$ or $e_i e_j$ in T_1. With the fact that $t \geq 1$, we know that (e_t, a_y)-$(a_y, \ldots, a_w)$-$(a_w, e_1, \ldots, e_t)$ is a 3-boundary of T_1. Finally, since T_1 has less edges than $T_{d_x a_y}$ $(d_x a_y \notin E(T_1))$, Property 1 holds for T_1 with respect to the mentioned 3-boundary. Let (Σ_1, R_1, F_1) be an anticlockwise

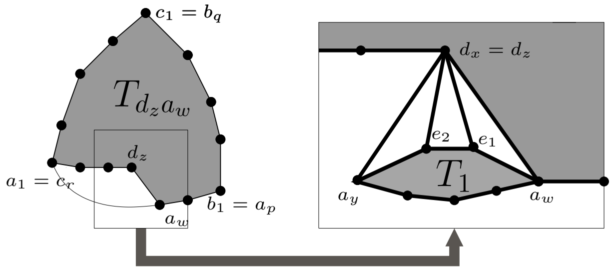

Figure 22: Case 2.2.3: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x$ and $w > y + 1$.

PS-representation contained in the region τ_1 obtained for T_1, with $F_1 = E_0(T_1) \setminus \{e_t a_y\}$.

In Figure 23, starting from (Σ', R', F') and (Σ_1, R_1, F_1), we show how to join the strings $\mathbf{a_w}$ of Σ' and Σ_1, how to extend the string $\mathbf{a_y}$ and the strings $\mathbf{e_i}$, for $1 \leq i \leq t$, and how to draw the face-regions $\mathbf{a_y e_t d_x}$, $\mathbf{e_1 a_w d_x}$, and $\mathbf{e_i e_{i-1} d_x}$, for $1 < i \leq t$, in order to obtain (Σ, R, F), an anticlockwise PS-representation of $T_{d_x a_y}$ contained in a region τ.

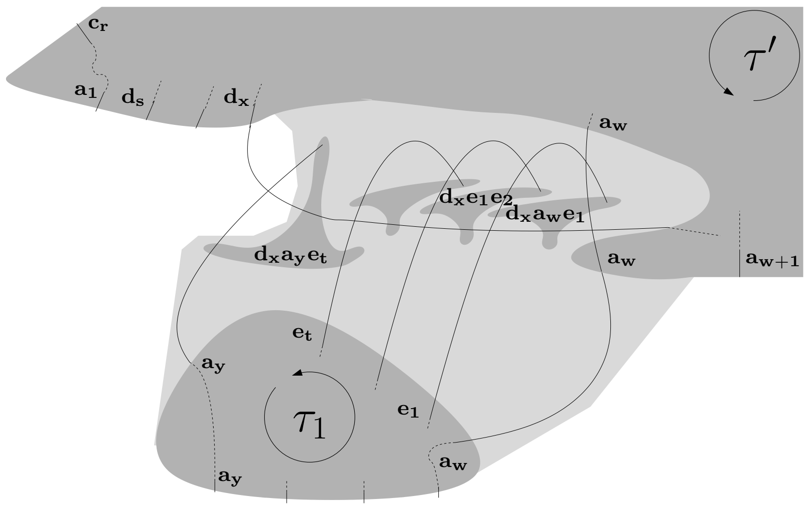

Figure 23: Case 2.2.3: (Σ, R, F).

We verify that (Σ, R, F) satisfies Property 2:

- Σ is a 1-string representation of $T_{d_x a_y} \setminus F$ with $F = E_o(T_{d_x a_y}) \setminus \{d_x a_y\}$: Indeed, $E(T_{d_x a_y}) \setminus F$ is the disjoint union of $E(T_{d_z a_w}) \setminus F'$, $E(T_1) \setminus F_1$, and $\{a_w e_1, d_x a_y\} \cup \{e_i e_{i-1} \mid i \in [2, t]\} \cup \{d_x e_i \mid i \in [1, t]\}$.
- (Σ, R) is "strong": The only inner-faces of $T_{d_x a_y}$ that are not inner-faces in $T_{d_z a_w}$ or T_1 are $d_x a_y e_t$, $d_x a_w e_1$, and the faces $d_x e_i e_{i-1}$, for $2 \leq i \leq t$, which correspond to the new face-regions.

Finally we see in Figure 23 that the other points of Property 2 are satisfied.

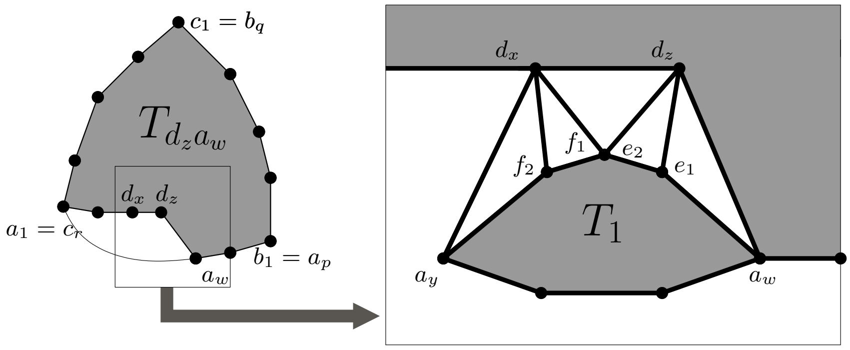

Figure 24: Case 2.2.4: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x - 1$ and $w > y$.

Case 2.2.4: $z = x - 1$ **and** $w > y$ **(see Figure 24).** Let us denote $e_1, e_2, \ldots, e_t$ the neighbors of d_z strictly inside the cycle $(d_z, d_x, a_y, \ldots, a_w, d_z)$, going "from right to left" (see Figure 24). By maximality of z, there is no edge $d_x a_w$, so $t \geq 1$. Let us denote $f_1, \ldots, f_u$ the neighbors of d_x strictly inside the cycle $(d_x, a_y, \ldots, a_w, d_z, d_x)$, going "from right to left" (see Figure 24). Note that $f_1 = e_t$ and that by minimality of w, there is no edge $d_z a_y$, so $u \geq 1$.

By minimality of w (resp. maximality of z) we have $e_i \neq a_j$ (resp. $f_i \neq a_j$), for all $1 \leq i \leq t$ (resp. $1 \leq i \leq u$) and $y \leq j \leq w$. Let T_1 be the subgraph of $T_{d_x a_y}$ that lies inside the cycle $(a_y, \ldots, a_w, e_1, \ldots, e_t, f_2, \ldots, f_u, a_y)$. By Lemma 2, T_1 is a W-triangulation. Since the W-triangulation $T_{d_x a_y}$ has no separating 3-cycle (d_z, a_w, e_i), (d_z, e_i, e_j), (d_x, f_i, f_j), or (d_x, f_i, a_y), there exists no chord $a_w e_i$, $e_i e_j$, $f_i f_j$, or $f_i a_y$ in T_1. With the fact that $t \geq 1$ and $u \geq 1$, we know that $(f_1, f_2, \ldots, f_u, a_y)$-$(a_y, \ldots, a_w)$-$(a_w, e_1, \ldots, e_t)$ is a 3-boundary of T_1. Finally, since T_1 has less edges than $T_{d_x a_y}$ ($d_x a_y \notin E(T_1)$), Property 1 holds for T_1 with respect to the mentioned 3-boundary. Let (Σ_1, R_1, F_1) be an anticlockwise PS-representation contained in the region τ_1 obtained for T_1, with $F_1 = E_0(T_1) \setminus \{f_1 f_2\}$.

In Figure 25, starting from (Σ', R', F') and (Σ_1, R_1, F_1), we show how to join the strings $\mathbf{a_w}$ of Σ' and Σ_1, how to extend the string $\mathbf{d_x}$, $\mathbf{a_y}$, the strings $\mathbf{e_i}$ for $1 \leq i \leq t$, and the strings $\mathbf{f_i}$ for $2 \leq i \leq u$, and how to draw the face-regions $\mathbf{d_z a_w e_1}$, $\mathbf{d_z e_i e_{i-1}}$, for $2 \leq i \leq t$, $\mathbf{d_z d_x e_t}$, $\mathbf{d_x f_i f_{i-1}}$, for $2 \leq i \leq u$, and $\mathbf{d_x a_y f_u}$ in order to obtain (Σ, R, F), an anticlockwise PS-representation of $T_{d_x a_y}$ almost contained in a region τ.

We verify that (Σ, R, F) satisfies Property 2:

- Σ is a 1-string representation of $T_{d_x a_y} \setminus F$ with $F = E_o(T_{d_x a_y}) \setminus \{d_x a_y\}$: Indeed, $E(T_{d_x a_y}) \setminus F$ is the disjoint union of $E(T_{d_z a_w}) \setminus F'$, $E(T_1) \setminus F_1$, and $\{d_x a_y, d_x d_z, a_w e_1, a_y f_u\} \cup \{d_z e_i \mid i \in [1, t]\} \cup \{d_x f_i \mid i \in [1, u]\} \cup \{e_i e_{i-1} \mid i \in [2, t]\} \cup \{f_i f_{i-1} \mid i \in [2, u]\}$.

- (Σ, R) is "strong": The only inner-faces of $T_{d_x a_y}$ that are not inner-faces in $T_{d_z a_w}$ or T_1 are $d_z a_w e_1$, $d_z e_i e_{i-1}$ for $2 \leq i \leq t$, $d_z d_x e_t$, $d_x f_i f_{i-1}$ for $2 \leq i \leq u$, and $d_x a_y f_u$, which correspond to the new face-regions.

Finally we see in Figure 25 that the other points of Property 2 are satisfied. So, Property 2 holds for any W-triangulation $T_{d_x a_y}$ with m edges and this completes the proofs of Property 1 and Property 2. $\square$

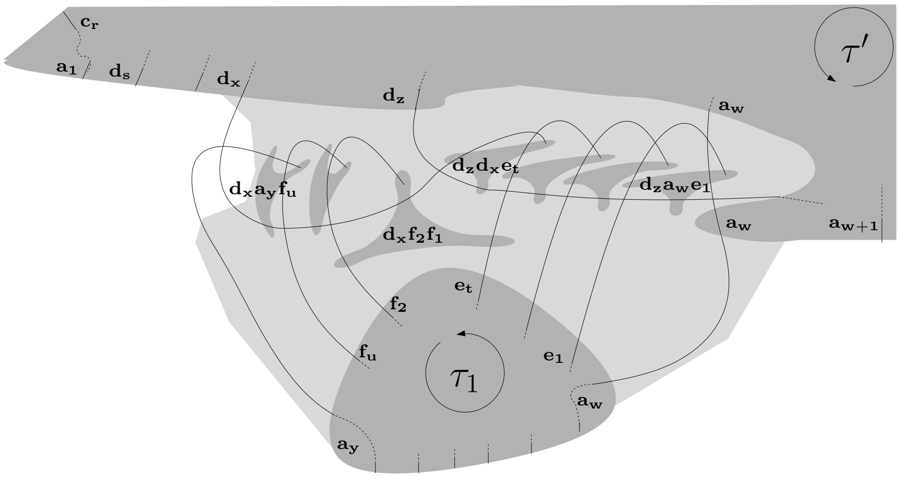

Figure 25: Case 2.2.4: (Σ, R, F).

4 Proof in the general case

Theorem 3 *Every triangulation T admits an S-representation* (Σ, R).

Proof. We prove this result by induction on the number of separating 3-cycles. Note that any triangulation T is 3-connected, and that if T has no separating 3-cycle, then T is 4-connected and is a W-triangulation. Consequently, if T is a 4-connected triangulation whose outer-vertices are a, b, and c, then T is a W-triangulation 3-bounded by (a, b)-(b, c)-(c, a). By Property 1, T admits a PS-representation (Σ, R, F), with $F = \{bc, ca\}$, that is contained in a region τ. Furthermore, in the boundary of τ we successively meet the ends of $\mathbf{b}, \mathbf{b}, \mathbf{c}, \mathbf{c}, \mathbf{a}$. To obtain an S-representation of T, it is sufficient to extend $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ outside of τ so that $\mathbf{c}$ crosses $\mathbf{a}$ and $\mathbf{b}$, as depicted in Figure 26.

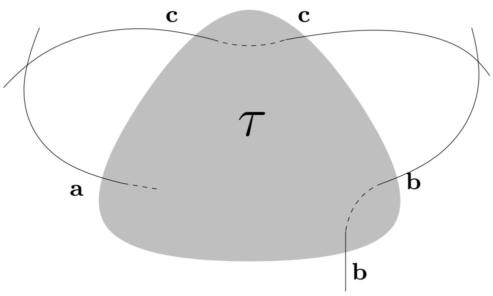

Figure 26: S-representation of T from (Σ, R, F).

Suppose now that T is a triangulation that contains at least one separating 3-cycle. Consider a separating 3-cycle (a, b, c) such that there is no other separating 3-cycle lying inside. This implies that the triangulation T' induced by the vertices on and inside (a, b, c) is 4-connected.

Let T_1 be the triangulation obtained by removing the vertices lying strictly inside (a, b, c). Let T_2 be the subgraph of T induced by the vertices lying strictly inside (a, b, c) (*i.e.*, $T_2 =$

$T' \setminus \{a, b, c\}$). In T_1, the cycle (a, b, c) is a face of the triangulation and is no more a separating 3-cycle. Thus T_1 has one separating cycle less than T, and so we have by induction hypothesis that T_1 admits an S-representation (Σ_1, R_1). This S-representation contains a face-region **abc** corresponding to the face abc. Without loss of generality, say that **abc** is an (a, b, c)-region, as depicted in Figure 27.

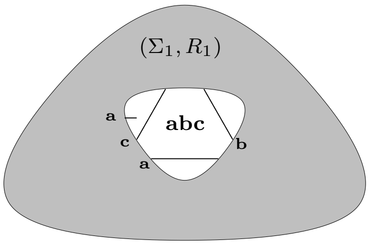

Figure 27: In the S-representation (Σ_1, R_1) of T_1, the (a, b, c)-region **abc**.

Since T' is a triangulation with at least four vertices, the neighbors of any vertex $v \in V(T')$ induce a cycle. Suppose that the vertex a (resp. b and c) has exactly one neighbor v that lies inside (a, b, c). Then there exists a cycle (b, v, c) (resp. (a, v, c) and (a, v, b)) in T' and consequently v is a neighbor of a, b, and c in T'. Suppose that there exists another vertex w in T', then w lies either inside the cycle (a, v, b), inside (a, v, c), or inside (b, v, c) and then one of these cycles is a separating 3-cycle. This is impossible by definition of (a, b, c). So we can distinguish two cases (see Figure 28), (A) the case where T_2 is a single vertex, and (B) the case where each of the vertices a, b, and c has at least two neighbors inside (a, b, c).

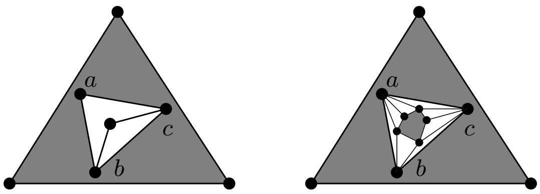

Figure 28: The cases (A) and (B).

Case (A): T_2 is a single vertex v. To obtain an S-representation (Σ, R) of T (see Figure 29), we add a string **v** in (Σ_1, R_1). Since $E(T) \setminus E(T_1) = \{va, vb, vc\}$ this string **v** crosses $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Moreover, we also define three disjoint face-regions $\mathbf{acv}, \mathbf{vbc}, \mathbf{vab}$ that correspond respectively to the faces acv, vbc, vab.

Since (Σ_1, R_1) is an S-representation of T_1 and since $\mathbf{v}, \mathbf{acv}, \mathbf{vbc}, \mathbf{vab}$ are drawn inside **abc**, it is clear that $(\Sigma \cup \{\mathbf{v}\}, (R \setminus \{\mathbf{abc}\}) \cup \{\mathbf{acv}, \mathbf{vbc}, \mathbf{vab}\})$ is an S-representation of T.

Case (B): Each of the vertices a, b, and c has at least two neighbors inside (a, b, c). There exists a cycle $(c, a_1, \ldots, a_p, b)$ (resp. $(a, b_1, \ldots, b_q, c)$ and $(b, c_1, \ldots, c_r, a)$) in T' whose vertices are exactly the neighbors of a (resp. b and c). We already know that $p > 1, q > 1, r > 1$ and that $a_p = b_1$, $b_q = c_1$, and $c_r = a_1$. Moreover, since b_1 and c (resp. c_1 and a, and a_1 and b) are the only two common neighbors of a and b (resp. b and c, and a and c) in T' (otherwise

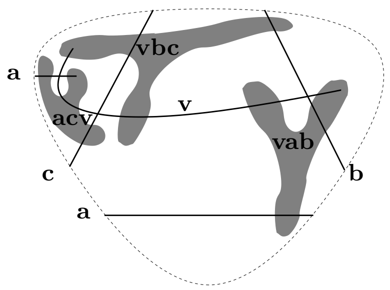

Figure 29: Case (A): Modifications inside **abc**.

there would be a separating 3-cycle) then $(a_1, \ldots, a_p = b_1, \ldots, b_q = c_1, \ldots, c_r = a_1)$ is a cycle. This implies from Lemma 2 that T_2 is a W-triangulation.

Suppose that there exists an edge $a_i a_j$ (resp. $b_i b_j$, $c_i c_j$) with $1 < i+1 < j \leq p$ (resp. $1 < i+1 < j \leq q$, $1 < i+1 < j \leq r$). Then, the cycle (a, a_i, a_j) (resp. (b, b_i, b_j), (c, c_i, c_j)) would be a separating 3-cycle of T'. Consequently, T_2 is 3-bounded by $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. With respect to this 3-boundary, T_2 has an anticlockwise PS-representation (Σ_2, R_2, F_2), with $F_2 = E_o \setminus \{a_1 a_2\}$ (c.f. Property 1). Let τ_2 be a region of **abc** containing this representation.

Since **abc** is an (a, b, c)-region, on its boundary we successively cross $\mathbf{a}, \mathbf{a}, \mathbf{b}, \mathbf{b}, \mathbf{c}, \mathbf{a}$ and $\mathbf{c}$ when going anticlockwise (by doing an axial symmetry if necessary).

In Figure 30, starting from (Σ_1, R_1) and (Σ_2, R_2) we obtain (Σ, R). We extend the strings $\mathbf{a_2}, \ldots, \mathbf{a_p}, \mathbf{b_1}, \ldots, \mathbf{b_q}, \mathbf{c_1}, \ldots, \mathbf{c_r}$ to obtain the crossings that correspond to the edges in the set $E(T) \setminus (E(T_1) \cup (E(T_2) \setminus F_2)) = \{aa_i \mid i \in [1,p]\} \cup \{bb_i \mid i \in [1,q]\} \cup \{cc_i \mid i \in [1,r]\} \cup \{a_i a_{i+1} \mid i \in [2,p-1]\} \cup \{b_i b_{i+1} \mid i \in [1,q-1]\} \cup \{c_i c_{i+1} \mid i \in [1,r-1]\}$. We also define face-regions for the faces in the set $\{abb_1, aca_1, bcc_1\} \cup \{aa_i a_{i+1} \mid i \in [1,p-1]\} \cup \{bb_i b_{i+1} \mid i \in [1,q-1]\} \cup \{cc_i c_{i+1} \mid i \in [1,r-1]\}$.

Since (Σ_1, R_1) is an S-representation of T_1 and (Σ_2, R_2, F_2) is a PS-representation of T_2, (Σ, R, F) is an S-representation of T.

- Σ is a 1-string representation of T: Indeed, we added all the crossings corresponding to the edges in $E(T) \setminus (E(T_1) \cup (E(T_2) \setminus F_2))$.
- (Σ, R) is "strong": Indeed, we added all the face-regions corresponding to the inner-faces of T that are neither in T_1 nor in T_2.

Consequently, every triangulation admits an S-representation, which proves Theorem 3 and then Theorem 2. □

5 Conclusion

The first and the second author recently improved the result presented in this article by proving Conjecture 1 [2]. For this they use the same decomposition of triangulation but their notion of face-region is quite different. One should also mention that their construction does not correspond to a stretching of the 1-string representation presented here.

Finally, an interesting question is whether the result presented here holds for other surfaces. For example, does any graph embedded on a surface $\mathbb{S}$ have a 1-string representation on $\mathbb{S}$?

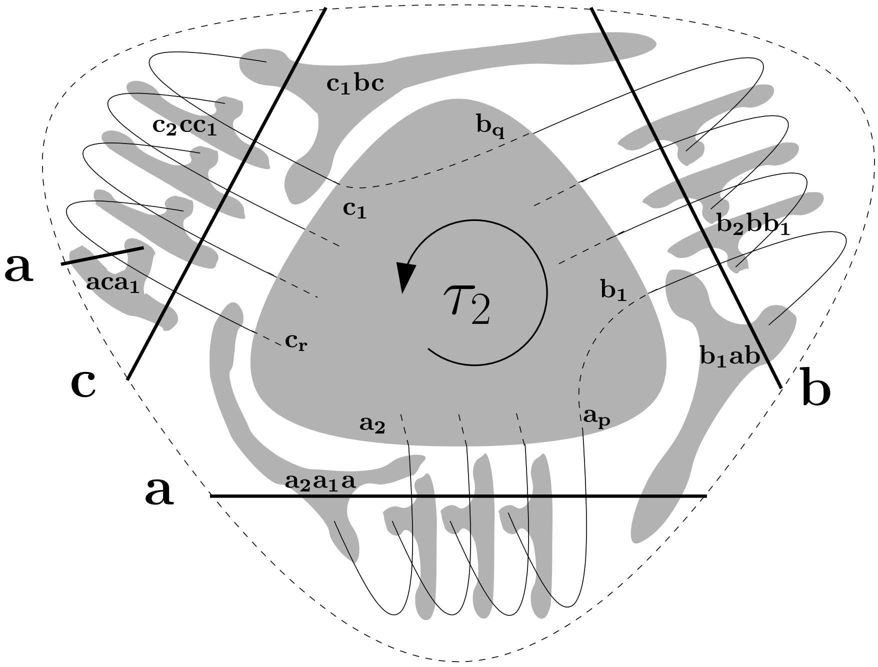

Figure 30: Case (B): Modifications inside **abc**.

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