Planar graphs have 1-string representations

Jérémie Chalopin$^1$ Daniel Gonçalves$^2$ Pascal Ochem$^3$

May 15, 2009

$^1$ LIF, CNRS et Aix-Marseille Université,
CMI, 39 rue Joliot-Curie 13453 Marseille Cedex 13, France.
$^2$ LIRMM, CNRS et Université Montpellier 2,
161 rue Ada 34392 Montpellier Cedex 05, France.
$^3$ LRI, CNRS et Université Paris-Sud,
Bât 490, 91405 Orsay Cedex, France

Abstract

We prove that every planar graph is an intersection graph of strings in the plane, such
that any two strings intersect at most once.

1 Introduction

A string $s$ is a curve of the plane homeomorphic to a segment. A string $s$ has two ends, the
points of $s$ that are not ends of $s$ are internal points of $s$. Two strings $s_1$ and $s_2$ cross if they
have a common point $p \in s_1 \cap s_2$ and if going around $p$ we successively meet $s_1$, $s_2$, $s_1$, and
$s_2$. This means that a tangent point is not a "crossing". In the following we consider string
sets without tangent points.

In this paper, we consider intersection models for simple planar graphs (i.e., planar graphs
without loops or multiple edges). A string representation of a graph $G = (V, E)$ is a set $\Sigma$ of
strings in the plane such that every vertex $v \in V$ maps to a string $v \in \Sigma$ and such that $uv \in E$
if and only if the strings $u$ and $v$ cross (at least once). Similarly, a segment representation of
a graph $G$ is a string representation of $G$ in which the strings are segments.

These notions were introduced by Ehrlich et al. [3], who proved the following:

Theorem 1 [3] Planar graphs have a string representation.

In [9], Koebbe proved that planar graphs are the contact graphs of disks in the plane.
Note that in this model the curves bounding two adjacent disks are tangent. However by
inflating these circles we obtain string representations for planar graphs. In his PhD thesis,
Scheinerman [10] conjectures a stronger result:

Conjecture 1 [10] Planar graphs have a segment representation.

*$^*$An abstract of this paper appeared in the Proceedings of the eighteenth annual ACM-SIAM Symposium
on Discrete algorithms (SODA 2007).
Hartman et al. [8] and de Fraysseix et al. [4] proved Conjecture 1 for bipartite planar graphs. Castro et al. [1] proved Conjecture 1 for triangle-free planar graphs. Recently de Fraysseix and Ossona de Mendez [6] extended this to planar graphs that have a 4-coloring in which every induced cycle of length 4 uses at most 3 colors. Observe that, since parallel segments never cross, a set of parallel segments in a segment representation of a graph induces a stable set of vertices. The construction in [4, 8] (resp. [1]) has the nice property that there are only 2 (resp. 3) possible slopes for the segments. So the construction induces a 2-coloring (resp. 3-coloring) of G. Note that Castro et al. do not prove the 3-colorability of triangle-free planar graphs, they use such coloring of the graphs (by Grötzsch’s Theorem) in their construction. West [11] proposed a stronger version of Conjecture 1 in which only 4 slopes are allowed, thus using the fact that these graphs are 4-colorable.

Notice that two segments cross at at most one point, whereas in the construction of Theorem 1, strings may cross twice. Let us define a 1-string representation as a string representation in which any two strings cross at most once. Thus the following theorem is a step towards Conjecture 1.

**Theorem 2** Planar graphs have a 1-string representation.

Note that if we would allow and consider tangent points, this theorem would directly follow from Koebe’s theorem. Theorem 2 answers an open problem of de Fraysseix and Ossona de Mendez [3]. In the same article they noticed that Theorem 2 implies that any planar multigraph has a string representation such that the number of crossings between two strings equals the number of edges between the two corresponding vertices.

In the next section we provide some definitions and prove that it is sufficient to prove this theorem for triangulations. Section 3 is devoted to the study of string representations of 4-connected triangulations. In this section we use a decomposition technique of 4-connected triangulations that is inspired on Whitney’s work [12] and that was recently used by the second author [7]. Then in Section 4 we finally prove Theorem 2 for all triangulations.

2 Preliminaries

2.1 Restriction to triangulations

**Lemma 1** Every planar graph is an induced subgraph of some planar triangulation.

**Proof.** Let G be a planar graph embedded in the plane (i.e. a plane graph). The graph \( h(G) \) is obtained from \( G \) by adding in every face \( f \) of \( G \) a new vertex \( v_f \) adjacent to every vertex incident to \( f \) in \( G \). Notice that \( h(G) \) is also a planar graph and that \( G \) is an induced subgraph of \( h(G) \). Moreover \( h(G) \) is connected, \( h(h(G)) \) is 2-connected, and \( h(h(h(G))) \) is a triangulation.

Note that we have to apply the \( h \) operator several times: if a facial walk goes through the same vertex several times, since multiples edges are not allowed, we obtain a non-triangular face.

It is clear that a 1-string representation of a triangulation \( T \) induces a 1-string representation for any of its induced subgraphs. It is thus sufficient to prove Theorem 2 for triangulations.
2.2 String Representations

In a plane graph $G$, the unbounded face of $G$ is called the outer-face and every other face of $G$ is an inner-face of $G$. An outer-vertex (resp. outer-edge) of $G$ is a vertex (resp. edge) of $G$ incident to the outer-face. The other vertices (resp. edges) of $G$ are inner-vertices (resp. inner-edges). The set of outer-vertices (resp. outer-edges, inner-vertices, and inner-edges) of $G$ is denoted by $V_{o}(G)$ (resp. $E_{o}(G)$, $V_i(G)$, and $E_i(G)$). A near-triangulation is a plane graph in which all the inner-faces are triangles. An edge $uv$ is a chord of some near-triangulation $T$ if $uv$ is an inner-edge linking two outer-vertices. From now on, we use the following notation: the strings corresponding to vertices of a graph $G$ are denoted by bold letters, i.e., for any $v \in V(G)$ we denote its corresponding string by $v$. We need that in a 1-string representation of a plane graph $G$, each face of $G$ corresponds to some topological region of the string representation.

**Definition 1** Let $G = (V, E)$ be a plane graph with a 1-string representation $\Sigma$. Given a face $abc$ of $G$, consider a triplet $(a, b, c)$ of its incident vertices. An $(a, b, c)$-region $abc$ is a region of the plane homeomorphic to a disk such that (see Figure 1):

- for any vertex $v \neq a, b, c$ we have $abc \cap v = \emptyset$ (i.e., $abc$ intersects only with $a, b, c$),
- $abc \cap a \cap b = \emptyset$, $abc \cap b \cap c = \emptyset$, and $abc \cap c \cap a = \emptyset$ (i.e., $a, b, c$ intersect outside $abc$),
- both $abc \cap b$ and $abc \cap c$ are connected,
- the boundary of $abc$ successively crosses (clockwise or anticlockwise) $a$, $a$, $b$, $b$, $c$, $a$, $c$.

![Figure 1: An (a, b, c)-region abc.](image)

Note that according to this definition $abc \cap a$ has two components and one end of $a$ is in $abc$. Note that the order in the triplet $(a, b, c)$ matters: a region $\tau$ of the plane cannot be an $(a, b, c)$-region and a $(c, b, a)$-region for example. A region $abc$ of the plane is an $\{a, b, c\}$-region if it is either an $(a, b, c)$-region, an $(a, c, b)$-region, $(b, a, c)$-region, $(b, c, a)$-region, $(c, b, a)$-region, or a $(c, b, a)$-region. When the vertices $a, b, c$ are not mentioned, we call such a region a face-region.

**Definition 2** A strong 1-string representation (S-representation, for short) of a near-triangulation $T$ is a pair $(\Sigma, R)$ such that:
(1) $\Sigma$ is a 1-string representation of $T$,

(2) $R$ is a set of disjoint face-regions such that for every inner-face $abc$ of $T$, $R$ contains an $(a, b, c)$-region.

A partial strong 1-string representation (PS-representation, for short) of a near-triangulation $T$ is a triplet $(\Sigma, R, F)$ in which $F \subseteq E(T)$ and such that $(\Sigma, R)$ is a strong 1-string representation of $T$ without the crossings corresponding to the edges of $F$.

In a PS-representation $(\Sigma, R, F)$ of $T$, note that $\Sigma$ is a 1-string representation of $T \setminus F$ and that each inner-face of $T$ has a corresponding face-region in $R$.

### 2.3 Special Triangulations

In a near-triangulation $T$, a separating 3-cycle $C$ is a cycle of length 3 such that some vertices of $T$ lie inside $C$ whereas other vertices lie outside. It is well known that a triangulation is 4-connected if and only if it contains no separating 3-cycle. In [12], Whitney considered a special family of near-triangulations, it is why we call them W-triangulations.

**Definition 3** A W-triangulation is a 2-connected near-triangulation containing no separating 3-cycle.

In particular, any 4-connected triangulation is a W-triangulation. Note that since a W-triangulation has no cut vertex, its outer-edges induce a cycle. The following lemma gives a sufficient condition for a subgraph of a W-triangulation $T$ to be a W-triangulation.

**Lemma 2** Let $T$ be a W-triangulation and consider a cycle $C$ of $T$. The subgraph induced by the vertices lying on and inside $C$ is a W-triangulation.

**Proof.** Consider the near-triangulation $T'$ inside some cycle $C$ of $T$. By definition, $T$ has no separating 3-cycle and consequently $T'$ does not have any separating 3-cycle. Since $T'$ is clearly connected and has more than two vertices, we prove that it is 2-connected by showing that it does not contain any cut vertex.

Since the cycle $C$ delimits the outer-face of $T'$, any vertex $v \in V(T')$ appears at most once on the outer face. Since the outerface appears at most once around $v$ and since all its other incident faces are triangles, $T'$ contains a path linking all the neighbors of $v$. This implies that $T' \setminus v$ is connected and thus $T'$ has no cut vertex.

**Definition 4** A W-triangulation $T$ is 3-boundary if the outer-boundary of $T$ is the union of three paths $(a_1, \ldots, a_p)$, $(b_1, \ldots, b_q)$, and $(c_1, \ldots, c_r)$ that satisfy the following conditions (see Figure 2):

- $a_1 = c_r$, $b_1 = a_p$ and $c_1 = b_q$.
- the paths are non-trivial, i.e., $p \geq 2$, $q \geq 2$ and $r \geq 2$.
- there exists no chord $a_i a_j$, $b_i b_j$ or $c_i c_j$.

Such a 3-boundary of $T$ will be denoted by $(a_1, \ldots, a_p)$-$\cdot$$(b_1, \ldots, b_q)$-$\cdot$$(c_1, \ldots, c_r)$.

In the following, we will use the order on the three paths and their directions, i.e., $(a_1, \ldots, a_p)$-$\cdot$$(b_1, \ldots, b_q)$-$\cdot$$(c_1, \ldots, c_r)$ will be different from $(b_1, \ldots, b_q)$-$\cdot$$(c_1, \ldots, c_r)$-$\cdot$$(a_1, \ldots, a_p)$ and $(a_p, \ldots, a_1)$-$\cdot$$(c_r, \ldots, c_1)$-$\cdot$$(b_q, \ldots, b_1)$.
3 Proof for 4-connected triangulations.

The following property describes the shape of a PS-representation of a 3-bounded W-triangulation.

**Property 1** Consider a 3-bounded W-triangulation $T$ with a 3-boundary $(a_1, \ldots, a_p)-(b_1, \ldots, b_q)-(c_1, \ldots, c_r)$. The W-triangulation $T$ has Property 1 if $T$ has a PS-representation $(\Sigma, R, F)$ contained inside a region $\tau$ of the plane homeomorphic to the disk that satisfies the following properties (see Figure 3):

(a) $F = E_o(T) \setminus \{a_1a_2\}$ (i.e., the missing crossings correspond to the outer edges, except $a_1a_2$),

(b) on the boundary of $\tau$ we successively have the ends of $a_2, a_3, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_r$.

If going clockwise (resp. anticlockwise) around the boundary of $\tau$, we cross the strings in the order described in (b), we say that the PS-representation is clockwise (resp. anticlockwise). Note that by an axial symmetry, one can obtain a clockwise PS-representation from an anticlockwise PS-representation, and vice versa. Observe that since $a_p = b_1$, $b_q = c_1$, and $c_r = a_1$, both ends of $b_1$ and $c_1$ lie on the boundary of $\tau$, but it is not the case for $a_1$ or any other string (i.e., all the strings appearing on the boundary of $\tau$ have an end inside $\tau$ except $b_1$ and $c_1$).

![Figure 3: Property 1](image)

Before proving that each 3-bounded W-triangulation has Property 1, we give some definitions and we present Property 2. Consider a 3-bounded W-triangulation $T \neq K_3$ whose boundary is $(a_1, \ldots, a_p)-(b_1, \ldots, b_q)-(c_1, \ldots, c_r)$ and such that $T$ does not contain any chord $a_ib_j$ or $a_ic_j$. Let $D \subseteq V_i(T)$ be the set of inner-vertices of $T$ that are adjacent to some vertex $a_i$ with $i > 1$ (the black vertices on the left of Figure 4). Since $T$ has at least 4 vertices, no
separating 3-cycle, and no chord \( a_i a_j, a_i b_j, \) or \( a_i c_j, \) then \( a_1 \) and \( a_2 \) (resp. \( b_1 \) and \( b_2 \)) have exactly one common neighbor in \( V_i(T) \) that will be denoted \( a \) (resp. \( d_1 \)).

Since there is no chord \( a_i a_j, a_i b_j, \) or \( a_i c_j, \) for each vertex \( a_i \) with \( i \in [2, p - 1], \) all the neighbors of \( a_i \) (resp. \( a_p \)) except \( a_{i-1} \) and \( a_{i+1} \) (resp. \( a_{p-1} \) and \( b_2 \)) are in \( D. \) Since for each \( i \in [2, p], \) there is a path linking the neighbors of \( a_i \) in \( D_i \) and since the vertices \( a_i \) and \( a_{i+1} \) have a common neighbor in \( D, \) then the set \( D \) induces a connected graph. Since \( a \) is in \( D, \) the set \( D \cup \{a_1\} \) also induces a connected graph.

**Definition 5** The adjacent path of \( T \) with respect to the 3-boundary \((a_1, \ldots, a_p)-(b_1, \ldots, b_q)-(c_1, \ldots, c_r)\) is the shortest path linking \( d_1 \) and \( a_1 \) in \( T[D \cup \{a_1\}] \) (the graph induced by \( D \cup \{a_1\} \)). This path will be denoted \( (d_1, d_2, \ldots, d_s, a_1). \)

**Observation 1** There exists neither an edge \( d_i d_j \) with \( 2 \leq i + 1 < j \leq s, \) nor an edge \( a_1 d_s \) with \( 1 \leq i < s. \) Otherwise, \( (d_1, d_2, \ldots, d_s, a_1) \) would not be the shortest path between \( d_1 \) and \( a_1. \)

![Diagram](https://via.placeholder.com/150)

**Figure 4:** the adjacent path of \( T \) and the graph \( T_{d_2 a_5}. \)

**Definition 6** For each edge \( d_x a_y \in E(T) \) with \( x \in [1, s] \) and \( y \in [2, p], \) the graph \( T_{d_x a_y} \) is the graph lying inside the cycle \( C = (a_1, d_x, \ldots, d_x, a_y, \ldots, a_p, b_2, \ldots, b_q, c_1, \ldots, c_r) \) (see Figure 4).

Note that since \( D \subseteq V_i(T), \) \( C \) is a cycle and by Lemma 2, \( T_{d_x a_y} \) is a W-triangulation. The following property describes the shape of a PS-representation of \( T_{d_x a_y}. \)

**Property 2** Consider a 3-bounded W-triangulation \( T \) with a 3-boundary \((a_1, \ldots, a_p)-(b_1, \ldots, b_q)-(c_1, \ldots, c_r)\) that does not have any chord \( a_i b_j, \) or \( a_i c_j, \) and let \( (d_1, d_2, \ldots, d_s, a_1) \) be its adjacent path. Consider an edge \( d_x a_y \in E(T) \) with \( y > 1. \) The W-triangulation \( T_{d_x a_y} \) has Property 2 if \( T_{d_x a_y} \) has a PS-representation \((\Sigma, R, F)\) satisfying the following properties (see Figure 5):

(a) \( F = E_o(G) \setminus \{d_x a_y\}, \)

(b) Every string \( v \in \Sigma \setminus \{d_x, a_y\} \) is contained in a region \( \tau \) of the plane homeomorphic to the disk. Furthermore \( d_x \) and \( a_y \) have their ends in \( \tau \) (or on the boundary of \( \tau \)) but they cross each other outside \( \tau. \)
(c) each face-region of $R$ is contained inside $\tau$,

(d) on the boundary of $\tau$ we successively have the ends of $a_y, a_p, b_1, \ldots, b_q, c_1, \ldots, c_r, a_1, d_s, \ldots, d_{x+1}$, and then we successively have internal points of $d_x, a_y, d_x$, and $a_y$.

Figure 5: Property 2.

Here again, if going clockwise (resp. anticlockwise) around the boundary of $\tau$, we cross the strings in the order described in (d), we say that the PS-representation is clockwise (resp. anticlockwise). In the proof of Theorem 2, we only use Property 1. However, in order to prove Property 1, we use Property 2. We prove these two properties by doing a “crossed” induction.

**Proof of Property 1 and Property 2**

We prove, by induction on $m \geq 3$, that the following two statements hold:

- Property 1 holds if $T$ has at most $m$ edges.

- Property 2 holds if $T_{d_xa_y}$ has at most $m$ edges.

The initial case, $m = 3$, is easy to prove since there is only one W-triangulation having at most 3 edges, $K_3$. For Property 1, we have to consider all the possible 3-boundaries of $K_3$. All these 3-boundaries are equivalent, so let $V(K_3) = \{a, b, c\}$ and consider the 3-boundary $(a, b)-(b, c)-(c, a)$. In Figure 6 there is a PS-representation $(\Sigma, R, F)$ of $K_3$ with $F = \{bc, ac\}$ that fulfills Property 1. For Property 2, since a W-triangulation $T_{d_xa_y}$ has at least 4 vertices, $a_1, b_1, c_1$, and $d_1$, we have $T_{d_xa_y} \neq K_3$ and there is no W-triangulation $T_{d_xa_y}$ with at most 3 edges. So by vacuity, Property 2 holds for $T_{d_xa_y}$ with at most 3 edges.

The induction step applies to both Property 1 and Property 2. This means that we prove Property 1 (resp. Property 2) for the W-triangulations $T$ (resp. $T_{d_xa_y}$) with $m$ edges using both Property 1 and Property 2 on W-triangulations with less than $m$ edges. We first prove the induction for Property 1.
Case 1: Proof of Property 1 for a W-triangulation $T$ with $m$ edges. Let $(a_1, \ldots, a_p)$-
$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ be the 3-boundary of $T$ considered. We distinguish different cases
according to the existence of a chord $a_ib_j$ or $a_ic_j$ in $T$. We successively consider the case
where there is a chord $a_ib_i$ with $1 < i < q$, the case where there is a chord $a_ib_j$, with
$1 < i < p$ and $1 < j < q$, and the case where there is a chord $a_ic_j$, with $1 < i < p$ and
$1 < j < r$. We then finish with the case where there is no chord $a_ib_j$, with $1 < i < p$ and
$1 < j < q$ (by definition of 3-boundary, $T$ has no chord $a_ib_q$, $a_ib_1$, or $a_ib_2$), and no chord $a_ic_j$,
with $1 < i < p$ and $1 < j < r$ (by definition of 3-boundary, $T$ has no chord $a_p c_1$, $a_ic_r$, or $a_1 c_j$).

Figure 7: Case 1.1: Chord $a_ib_i$.

Case 1.1: There is a chord $a_ib_i$, with $1 < i < q$ (see Figure 7). Let $T_1$ (resp. $T_2$) be the
subgraph of $T$ that lies inside the cycle $(a_1, b_1, \ldots, b_q, c_2, \ldots, c_r, a_1)$ (resp. $(a_1, a_2, \ldots, a_p, b_2, \ldots, b_i,
\ldots, a_1)$). By Lemma 2, $T_1$ and $T_2$ are W-triangulations. Since $T$ has no chord $a_xa_y$, $b_xb_y$, or
c_xc_y, $(b_i, a_1)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_1)$ (resp. $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_i)$-$(b_1, a_1)$) is a 3-boundary of
$T_1$ (resp. $T_2$). Furthermore, since $a_1a_2 \notin E(T_1)$ and $c_1c_2 \notin E(T_2)$, $T_1$ and $T_2$ have less
edges than $T$ and Property 1 holds for $T_1$ and $T_2$ with the mentioned 3-boundaries. Let
$(\Sigma_1, R_1, F_1)$ (resp. $(\Sigma_2, R_2, F_2)$) be a clockwise (resp. anticlockwise) PS-representation con-
tained in the region $\tau_1$ (resp. $\tau_2$) obtained for $T_1$ (resp. $T_2$) with $F_1 = E_o(T_1) \setminus \{a_ib_i\}$ (resp.
$F_2 = E_o(T_2) \setminus \{a_1a_2\}$). In Figure 8 we show how to associate these two representations to obtain
$(\Sigma, R, F)$, an anticlockwise PS-representation of $T$ contained in $\tau$. Note that the two
strings $a_1$ (resp. $b_i$) from $\Sigma_1$ and $\Sigma_2$ have been linked.

We easily verify that $(\Sigma, R, F)$ satisfies Property 1:

- $\Sigma$ is a string representation of $T \setminus F$ with $F = E_o(T) \setminus \{a_1a_2\}$. Indeed, since $V(T_1) \cup
V(T_2) = V(T)$ and $V(T_1) \cap V(T_2) = \{a_1, b_i\}$, every vertex $v \in V(T)$ has exactly one
string in $\Sigma$. Furthermore, since $(E(T_1) \setminus F_1) \cup (E(T_2) \setminus F_2) = E(T) \setminus F$, $\Sigma$ is a string representation of $T \setminus F$.

- $\Sigma$ is a 1-string representation. The only edge that belongs to both $T_1$ and $T_2$ is $a_ib_i$. Since $a_1$ and $b_1$ cross each other in $\Sigma_1 (a_1b_i \notin F_1)$ but not in $\Sigma_2 (a_1b_i \in F_2)$, $a_1$ and $b_1$ cross exactly once in $\Sigma$.

- $(\Sigma, R)$ is ‘strong’: Each inner-face of $T$ is an inner-face in $T_1$ or $T_2$ and the regions $\tau_1$ and $\tau_2$ are disjoint (so the face-regions in $\tau_1$ are disjoint from the face-regions in $\tau_2$).

Finally we see in Figure 8 that point (b) of Property 1 is satisfied.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{Case 1.1: $\Sigma, R, F$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{Case 1.2: Chord $a_ib_j$.}
\end{figure}

**Case 1.2:** There is a chord $a_ib_j$, with $1 < i < p$ and $1 < j < q$ (see Figure 9). If there are several chords $a_ib_j$, we consider one that maximizes $j$, i.e., there is no chord $a_ib_k$ with $j < k \leq q$. Let $T_1$ (resp. $T_2$) be the subgraph of $T$ that lies inside the cycle $(a_1, a_2, \ldots, a_i, b_j, \ldots, b_q, c_2, \ldots, c_r)$ (resp. $(a_i, \ldots, a_p, b_2, \ldots, b_j, a_i)$). By Lemma 2, $T_1$ and $T_2$ are W-triangulations. Since $T$ has no chord $a_xa_y, b_xb_y, c_xc_y$, or $a_ib_k$ with $k > j$, $(a_1, \ldots, a_i)-(a_i, b_j, \ldots, b_q)-(c_1, \ldots, c_r)$ (resp. $(a_i, b_j)-(b_1, \ldots, b_j)-(a_1, \ldots, a_i)$) is a 3-boundary of $T_1$ (resp. $T_2$). Furthermore, since $b_1b_2 \notin E(T_1)$ and $a_1a_2 \notin E(T_2)$, $T_1$ and $T_2$ have less edges than $T$ and Property 1 holds for $T_1$ and $T_2$ with the mentioned 3-boundaries. Let $(\Sigma_1, R_1, F_1)$ (resp. $(\Sigma_2, R_2, F_2)$) be an anticlockwise (resp. clockwise) PS-representation contained in the region $\tau_1$ (resp. $\tau_2$) obtained for $T_1$ (resp. $T_2$), with $F_1 = E_o(T_1) \setminus \{a_1a_2\}$ (resp. $F_2 = E_o(T_2) \setminus \{a_1b_2\}$). In Figure 10 we show how to associate these two representations to obtain $(\Sigma, R, F)$, an anticlockwise PS-representation of $T$ contained in $\tau$. Note that in this construction the two strings $a_1$ (resp. $b_1$) from $\Sigma_1$ and $\Sigma_2$ have been linked.
As in Case 1.1, we easily verify that $(\Sigma, R, F)$ satisfies Property 1.

**Figure 10:** Case 1.2: $(\Sigma, R, F)$.

**Case 1.3:** There is a chord $a_ic_j$, with $1 < i \leq p$ and $1 < j < r$ (see Figure 11). If there are several chords $a_ic_j$, we consider one which maximizes $i$, i.e., there is no chord $a_kc_j$ with $i < k \leq p$. Let $T_1$ (resp. $T_2$) be the subgraph of $T$ that lies inside the cycle $(a_1, a_2, \ldots, a_i, c_j, \ldots, c_r)$ (resp. $(c_j, a_i, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_j)$). By Lemma 2, $T_1$ and $T_2$ are W-triangulations. Since $T$ has no chord $a_xa_y$, $b_xb_y$, $c_xc_y$ or $a_kc_j$ with $k > i$, $(a_1, \ldots, a_i)$-$(a_i, c_j)$ or $T_2$). Furthermore, since $b_1b_2 \notin E(T_1)$ and $a_1a_2 \notin E(T_2)$, $T_1$ and $T_2$ have less edges than $T$ and Property 1 holds for $T_1$ and $T_2$ with the mentioned 3-boundaries. Let $(\Sigma_1, R_1, F_1)$ (resp. $(\Sigma_2, R_2, F_2)$) be an anticlockwise PS-representation contained in the region $\tau_1$ (resp. $\tau_2$) obtained for $T_1$ (resp. $T_2$), with $F_1 = E_0(T_1) \setminus \{a_1a_2\}$ (resp. $F_2 = E_0(T_2) \setminus \{c_ja_i\}$).

In Figure 12 we show how to associate these two representations to obtain $(\Sigma, R, F)$, an anticlockwise PS-representation of $T$ contained in $\tau$. Note that in this construction the two strings $a_i$ (resp. $c_j$) from $\Sigma_1$ and $\Sigma_2$ have been linked.

As in Case 1.1, we easily verify that $(\Sigma, R, F)$ satisfies Property 1.

**Figure 11:** Case 1.3: Chord $a_ic_j$.

**Case 1.4:** There is no chord $a_ib_j$, with $1 \leq i \leq p$ and $1 \leq j \leq q$, and no chord $a_ic_j$, with $1 \leq i \leq p$ and $1 \leq j \leq r$ (see Figure 13). In this case we consider the adjacent path $(a_1, \ldots, a_p, a_1)$ (see Figure 4) of $T$ with respect to its 3-boundary, $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. Consider the edge $d_ay$, with $1 < y \leq p$ and which minimizes $y$. This edge exists since, by definition of the adjacent path, $d_ay$ is adjacent to some vertex $a_y$ with $y > 1$. The W-triangulation $T_{d_ay}$ having less edges than $T$ $(a_1a_2 \notin E(T_{d_ay}))$, Property 2 holds for

10
Figure 12: Case 1.3: \((\Sigma, R, F)\).

\(T_{d_s a_y}\). Let \((\Sigma', R', F')\) be an anticlockwise PS-representation almost contained in the region \(\tau'\) obtained for \(T_{d_s a_y}\), with \(F' = E_o(T_{d_s a_y}) \setminus \{d_s a_y\}\).

Figure 13: Case 1.4: No chord \(a_i b_j\) or \(a_i c_j\).

Now we distinguish two cases according to the position of \(a_y\): either \(y = 2\) (Case 1.4.1), or \(y > 2\) (Case 1.4.2).

**Case 1.4.1: \(y = 2\).** In Figure 14, starting from \((\Sigma', R', F')\), we show how to extend the string \(a_1 \in \Sigma'\) (in order to cross \(d_s\) and \(a_2\)) and how to draw the \((a_1, a_2, d_s)\)-region \(a_1 a_2 d_s\) to obtain \((\Sigma, R, F)\), an anticlockwise PS-representation of \(T\) contained in a region \(\tau\).

One can verify on Figure 14 that \((\Sigma, R, F)\) satisfies Property 1.

**Case 1.4.2: \(y > 2\).** Let us denote \(e_1, e_2, \ldots, e_t\) the neighbors of \(d_s\) strictly inside the cycle \((d_s, a_1, a_2, \ldots, a_y, d_s)\), going “from right to left” (see Figure 13). By minimality of \(y\) we have \(e_i \neq a_j\), for all \(1 \leq i \leq t\) and \(1 \leq j \leq y\).

Let \(T_1\) be the subgraph of \(T\) that lies inside the cycle \((a_1, \ldots, a_y, e_1, \ldots, e_t, a_1)\). By Lemma 2, \(T_1\) is a W-triangulation. Since the W-triangulation \(T\) has no separating 3-cycle \((d_s, a_1, e_1)\), \((d_s, a_y, e_1)\) or \((d_s, e_i, e_j)\), there exists no chord \(a_1 e_i, a_y e_i\) or \(e_i e_j\) in \(T_1\). So \((a_2, a_1)\)-\((a_1, e_t, \ldots, e_1, a_y)-(a_y, a_2)\) is a 3-boundary of \(T_1\). Finally, since \(T_1\) has less edges than \(T\) \((a_1 d_s \notin E(T_1))\), Property 1 holds for \(T_1\) with respect to the mentioned 3-boundary. Let \((\Sigma_1, R_1, F_1)\) be a clockwise PS-representation contained in the region \(\tau_1\) obtained for \(T_1\), with \(F_1 = E_o(T_1) \setminus \{a_2 a_1\}\).
In Figure 15, starting from \((\Sigma', R', F')\) and \((\Sigma_1, R_1, F_1)\), we show how to join the strings \(a_1\) (resp. \(a_y\)) of \(\Sigma'\) and \(\Sigma_1\), how to extend the strings \(e_i\), for \(1 \leq i \leq t\), and how to draw the face-regions \(a_y e_1 d_s, e_y a_1 d_s\), and \(e_y e_{i-1} d_s\), for \(2 \leq i \leq t\), in order to obtain \((\Sigma, R, F)\), an anticlockwise PS-representation of \(T\) contained in a region \(\tau\).

We verify that \((\Sigma, R, F)\) satisfies Property 1:

- \(\Sigma\) is a string representation of \(T \setminus F\) with \(F = E_\circ(T) \setminus \{a_1 a_2\}\). Indeed, since \(V(T_{d,a_y}) \cup \)
$V(T_1) = V(T)$ and $V(T_{d,a_y}) \cap V(T_1) = \{a_1, a_y\}$, every vertex $v \in V(T)$ has exactly one string in $\Sigma$. Furthermore, since $E(T) \setminus F = (E(T_{d,a_y}) \setminus F') \cup (E(T_1) \setminus F_1) \cup \{a_y e_1, e_i a_1, d_x a_1\} \cup \{e_i e_{i-1} | i \in [2, t]\} \cup \{d_x e_i | i \in [1, t]\}$, $\Sigma$ is a string representation of $T \setminus F$.

- $\Sigma$ is a 1-string representation. Indeed $T_{d,a_y}$ and $T_1$ do not have common edges, and the new crossings added correspond to edges missing in both $E(T_{d,a_y}) \setminus F'$ and $E(T_1) \setminus F_1$.

- $(\Sigma, R)$ is “strong”: The only inner-faces of $T$ not in $T_{d,a_y}$ nor in $T_1$ are the faces $d_y a_y e_1, d_x a_1 e_i$ and $d_x e_i e_{i+1}$, with $1 \leq i < t$. These faces correspond to the new face-regions.

Finally we see in Figure 15 that point (b) of Property 1 is satisfied.

So Property 1 holds for any W-triangulation $T$ with $m$ edges and this concludes the proof of Case 1.

**Case 2: Proof of Property 2 for a W-triangulation $T_{d,a_y}$ with $m$ edges.** Recall that the W-triangulation $T_{d,a_y}$ is a subgraph of a W-triangulation $T$ with 3-boundary $(a_1, \ldots, a_y)-(b_1, \ldots, b_y)-(c_1, \ldots, c_r)$. Moreover, $T$ has no chord $a_i b_j$ or $a_i c_j$ and its adjacent path is $(d_1, \ldots, d_s, a_1)$, with $s \geq 1$. We distinguish the case where $d_x a_y = d_1 a_p$ and the case where $d_x a_y \neq d_1 a_p$.

![Figure 16: Case 2.1: $T_{d,a_y} = T_{d_1 a_p}$](image)

**Case 2.1: $d_x a_y = d_1 a_p$ (see Figure 16).** Let $T_1$ be the subgraph of $T_{d_1 a_p}$ that lies inside the cycle $(a_1, d_1, d_2, \ldots, b_y, c_2, \ldots, c_r)$. By Lemma 2, $T_1$ is a W-triangulation. This W-triangulation has no chord $b_i b_j$, $c_i c_j$, $d_i d_j$, or $a_1 d_j$. We consider two cases according to the existence of an edge $d_1 b_i$ with $2 < i \leq q$.

- If $T_1$ has no chord $d_1 b_i$ then $(d_1, b_2, \ldots, b_y) - (c_1, \ldots, c_r) - (a_1, d_s, \ldots, d_1)$ is a 3-boundary of $T_1$.

- If $T_1$ has a chord $d_1 b_i$, with $2 < i \leq q$, note that $q > 2$ and that there cannot be a chord $b_2 a_1$ or $b_2 d_j$, with $1 < j \leq s$ (this would violate the planarity of $T_{d,a_y}$, see Figure 16) So in this case, $(b_2, d_1, d_2, a_1) - (c_r, \ldots, c_1) - (b_q, \ldots, b_2)$ is a 3-boundary of $T_1$.

Finally, since $T_1$ is a W-triangulation with less edges than $T_{d_1 a_p}$ ($b_1 b_2 \notin E(T_1)$), Property 1 holds for $T_1$ with respect to at least one of the two mentioned 3-boundaries. Whichever 3-boundary we consider, we obtain a PS-representation $(\Sigma_1, R_1, F_1)$ of $T_1$ contained in a region $\tau_1$, with the same following characteristics:
• $F_1 = E_o(T) \setminus \{d_1b_2\},$

• in the boundary of $\tau_1$ we successively meet the ends of $d_1, \ldots, d_s, a_1, c_r, \ldots, c_1, b_q, \ldots, b_2$ (clockwise or anticlockwise).

In Figure 17 we modify $(\Sigma_1, R_1, F_1)$, by extending the strings $d_1$ and $b_2$ and by adding a new string $a_p$ and a new face-region $d_1b_2a_p$. This leads to $(\Sigma, R, F)$, a PS-representation of $T_{d_1a_p}$ contained in a region $\tau$.

![Figure 17: Case 2.1: $(\Sigma, R, F)$.

We verify that $(\Sigma, R, F)$ satisfies Property 2:

• $\Sigma$ is a 1-string representation of $T_{d_1a_p} \setminus F$: Indeed, $E(T_{d_1a_p}) \setminus F$ is the disjoint union of $E(T_1) \setminus F_1$ and $\{a_p, d_1\}$.

• $(\Sigma, R)$ is “strong”: The only inner-face of $T_{d_1a_p}$ that is not an inner-face of $T_1$ is $d_1a_pb_2$, which corresponds to the new face-region $d_1a_pb_2$.

Finally we see in Figure 17 that the other points of Property 2 are satisfied.

Case 2.2: $T_{d_xa_y} \neq T_{d_1a_p}$. In this case we consider an edge $d_xa_w \in E(T_{d_xa_y})$ such that $d_xa_w \neq d_xa_y$. Among all the possible edges $d_xa_w$ we choose the one that first maximizes $z$ and then minimizes $w$. Such an edge necessarily exists and actually one can see that $d_z = d_x$ or $d_z = d_{x-1}$. Indeed, if $d_x = d_1$ there is at least one edge $d_1a_w$ with $w > y$, the edge $d_1a_p$. If $x > 1$, it is clear by definition of the adjacent path that the vertex $d_x$ is adjacent to at least one vertex $a_w$ with $w \geq y$.

By Lemma 2, $T_{d_xa_w}$ is a W-triangulation. Since $d_xa_y \notin E(T_{d_xa_w})$, the W-triangulation $T_{d_xa_w}$ has less edges than $T_{d_xa_y}$, and so Property 2 holds for $T_{d_xa_w}$. Let $(\Sigma', R', F')$ be an anticlockwise PS-representation almost contained in the region $\tau'$ obtained for $T_{d_xa_w}$, with $F' = E_o(T_{d_xa_w}) \setminus \{d_xa_w\}$. We distinguish 4 cases according to the edge $d_xa_w$. When $z = x$ we consider the case where $w = y + 1$ and the case where $w > y + 1$. When $z = x - 1$ we consider the case where $w = y$ and the case where $w > y$.
Case 2.2.1: $T_{d_xa_y} \neq T_{d_1a_p}$, $z = x$ and $w = y + 1$ (see Figure 18). In Figure 19 we modify $(\Sigma', R', F')$, by adding a new string $a_y$ and a new face-region $a_ya_w d_x$. This leads to $(\Sigma, R, F)$, an anticlockwise PS-representation of $T_{d_xa_y}$ almost contained in a region $\tau$.

We verify that $(\Sigma, R, F)$ satisfies Property 2:

- $\Sigma$ is a 1-string representation of $T_{d_xa_y} \setminus F$: Indeed, $E(T_{d_xa_y}) \setminus F$ is the disjoint union of $E(T_{d_1a_p}) \setminus F' \cup \{d_xa_y\}$.
- $(\Sigma, R)$ is “strong”: The only inner-face of $T_{d_xa_y}$ that is not an inner-face of $T_{d_1a_p}$ is $d_xa_ya_w$, which corresponds to the new face-region $d_xa_ya_w$.

Finally, we see in Figure 19 that the other points of Property 2 are satisfied.

Case 2.2.2: $z = x - 1$ and $w = y$ (see Figure 20). In Figure 21, we modify $(\Sigma', R', F')$ by extending the string $d_x$ and by adding a new face-region $d_xd_xa_y$. This leads to $(\Sigma, R, F)$, an anticlockwise PS-representation of $T_{d_xa_y}$ almost contained in a region $\tau$.

We verify that $(\Sigma, R, F)$ satisfies Property 2:

- $\Sigma$ is a 1-string representation of $T_{d_xa_y} \setminus F$: Indeed, $E(T_{d_xa_y}) \setminus F$ is the disjoint union of $E(T_{d_1a_p}) \setminus F' \cup \{d_xd_z, d_xa_y\}$. 

15
Figure 20: Case 2.2.2: \( T_{d_xa_w} \neq T_{d_1a_p} \), \( z = x - 1 \) and \( w = y \).

Figure 21: Case 2.2.2: \((\Sigma, R, F)\).

- \((\Sigma, R)\) is “strong”: The only inner-face of \( T_{d_xa_y} \) that is not an inner-face of \( T_{d_1a_w} \) is \( d_xd_za_y \), which corresponds to the new face-region \( d_xd_2a_y \).

Finally we see in Figure 21 that the other points of Property 2 are satisfied.

**Case 2.2.3:** \( z = x \) and \( w > y + 1 \) (see Figure 22). Let us denote \( e_1, e_2, \ldots, e_t \) the neighbors of \( d_x \) strictly inside the cycle \((d_x, a_y, \ldots, a_w, d_x)\), going “from right to left” (see Figure 22). Since there is no chord \( a_ia_j \) we have \( t \geq 1 \). Furthermore by minimality of \( w \) we have \( e_i \neq a_j \), for all \( 1 \leq i \leq t \) and \( y \leq j \leq w \). Let \( T_1 \) be the subgraph of \( T_{d_xa_y} \) that lies inside the cycle \((a_y, a_w, e_1, \ldots, e_t, a_y)\). By Lemma 2, \( T_1 \) is a W-triangulation. Since the W-triangulation \( T_{d_2a_y} \) has no separating 3-cycle \((d_x, a_w, e_i)\) or \((d_x, e_i, e_j)\), there exists no chord \( a_we_i \) or \( e_ie_j \) in \( T_1 \). With the fact that \( t \geq 1 \), we know that \((e_1, a_y)\), \((a_y, \ldots, a_w)\), \((a_w, e_1, \ldots, e_t)\) is a 3-boundary of \( T_1 \). Finally, since \( T_1 \) has less edges than \( T_{d_2a_y} \) \((d_2a_y \notin E(T_1))\), Property 1 holds for \( T_1 \) with respect to the mentioned 3-boundary. Let \((\Sigma_1, R_1, F_1)\) be an anticlockwise
PS-representation contained in the region \( r_1 \) obtained for \( T_1 \), with \( F_1 = E_0(T_1) \setminus \{ e_i a_y \} \).

In Figure 23, starting from \((\Sigma', R', F')\) and \((\Sigma_1, R_1, F_1)\), we show how to join the strings \( a_w \) of \( \Sigma' \) and \( \Sigma_1 \), how to extend the string \( a_y \) and the strings \( e_i \), for \( 1 \leq i \leq t \), and how to draw the face-regions \( a_y e_i d_x \), \( e_i a_w d_x \), and \( e_i e_{i-1} d_x \), for \( 1 < i \leq t \), in order to obtain \((\Sigma, R, F)\), an anticlockwise PS-representation of \( T_{d_x a_y} \) contained in a region \( r \).

We verify that \((\Sigma, R, F)\) satisfies Property 2:

- \( \Sigma \) is a 1-string representation of \( T_{d_x a_y} \setminus F \) with \( F = E_0(T_{d_x a_y}) \setminus \{ d_x a_y \} \); Indeed, \( E(T_{d_x a_y}) \setminus F \) is the disjoint union of \( E(T_{d_x a_y}) \setminus F', \ E(T_1) \setminus F_1 \), and \( \{ a_w e_1, d_x a_y \} \cup \{ e_i e_{i-1} \mid i \in [2, t] \} \cup \{ d_x e_i \mid i \in [1, t] \} \).

- \((\Sigma, R)\) is “strong”: The only inner-faces of \( T_{d_x a_y} \) that are not inner-faces in \( T_{d_x a_y} \) or \( T_1 \) are \( d_x a_y e_1 \), \( d_x a_w e_1 \), and the faces \( d_x e_i e_{i-1} \), for \( 2 \leq i \leq t \), which correspond to the new face-regions.

Finally we see in Figure 23 that the other points of Property 2 are satisfied.
**Case 2.2.4:** $z = x - 1$ and $w > y$ (see Figure 24). Let us denote $e_1, e_2, \ldots, e_t$ the neighbors of $d_z$ strictly inside the cycle $(d_z, d_x, a_y, \ldots, a_w, d_x)$, going “from right to left” (see Figure 24). By maximality of $z$, there is no edge $d_z a_w$, so $t \geq 1$. Let us denote $f_1, \ldots, f_u$ the neighbors of $d_z$ strictly inside the cycle $(d_z, a_y, \ldots, a_w, d_x)$, going “from right to left” (see Figure 24). Note that $f_1 = e_y$ and that by minimality of $w$, there is no edge $d_z a_y$, so $u \geq 1$.

By minimality of $w$ (resp. maximality of $z$) we have $e_i \neq a_j$ (resp. $f_i \neq a_j$), for all $1 \leq i \leq t$ (resp. $1 \leq i \leq u$) and $y \leq j \leq w$. Let $T_1$ be the subgraph of $T_{d_z a_y}$ that lies inside the cycle $(a_y, \ldots, a_w, e_1, \ldots, e_t, f_2, \ldots, f_u, a_y)$. By Lemma 2, $T_1$ is a W-triangulation. Since the W-triangulation $T_{d_z a_y}$ has no separating 3-cycle $(d_z, a_w, e_i)$, $(d_z, e_i, e_j)$, $(d_z, f_i, f_j)$, or $(d_x, f_i, a_y)$, there exists no chord $a_w e_i e_j f_i f_j$, or $f_i a_y$ in $T_1$. With the fact that $t \geq 1$ and $u \geq 1$, we know that $(f_1, f_2, \ldots, f_u, a_y) - (a_y, \ldots, a_w) - (a_w, e_1, \ldots, e_t)$ is a 3-boundary of $T_1$. Finally, since $T_1$ has less edges than $T_{d_z a_y}$ ($d_x a_y \notin E(T_1)$), Property 1 holds for $T_1$ with respect to the mentioned 3-boundary. Let $(\Sigma_1, R_1, F_1)$ be an anticlockwise PS-representation contained in the region $T_1$, with $F_1 = E_0(T_1) \setminus \{f_1 f_2\}$.

In Figure 25, starting from $(\Sigma', R', F')$ and $(\Sigma_1, R_1, F_1)$, we show how to join the strings $a_w$ of $\Sigma'$ and $\Sigma_1$, how to extend the string $d_x$, $a_y$, the strings $e_i$ for $1 \leq i \leq t$, and the strings $f_i$ for $2 \leq i \leq u$, and how to draw the face-regions $d_x d_w e_1$, $d_x f_1 f_2$, $d_x f_1 f_2$, $d_x a_y f_u$ in order to obtain $(\Sigma, R, F)$, an anticlockwise PS-representation of $T_{d_z a_y}$ almost contained in a region $\tau$.

We verify that $(\Sigma, R, F)$ satisfies Property 2:

- $\Sigma$ is a 1-string representation of $T_{d_z a_y} \setminus F$ with $F = E_0(T_{d_z a_y}) \setminus \{d_x a_y\}$: Indeed, $E(T_{d_z a_y}) \setminus F$ is the disjoint union of $E(T_{d_z a_w}) \setminus F'$, $E(T_1) \setminus F_1$, and $\{d_x a_y, d_x d_y, a_w e_1, a_y f_u\} \cup \{d_x e_i \mid i \in [1, t] \} \cup \{d_x f_i \mid i \in [1, u] \} \cup \{e_i e_{i-1} \mid i \in [2, t] \} \cup \{f_i f_{i-1} \mid i \in [2, u] \}$.

- $(\Sigma, R)$ is “strong”: The only inner-faces of $T_{d_z a_y}$ that are not inner-faces in $T_{d_z a_w}$ or $T_1$ are $d_x a_w e_1$, $d_x e_i e_{i-1}$ for $2 \leq i \leq t$, $d_x d_y e_t$, $d_x f_1 f_{i-1}$ for $2 \leq i \leq u$, and $d_x a_y f_u$, which correspond to the new face-regions.

Finally we see in Figure 25 that the other points of Property 2 are satisfied. So, Property 2 holds for any W-triangulation $T_{d_z a_y}$ with $m$ edges and this completes the proofs of Property 1 and Property 2. □
4 Proof in the general case

**Theorem 3** Every triangulation $T$ admits an $S$-representation $(\Sigma, R)$.

**Proof.** We prove this result by induction on the number of separating 3-cycles. Note that any triangulation $T$ is 3-connected, and that if $T$ has no separating 3-cycle, then $T$ is 4-connected and is a W-triangulation. Consequently, if $T$ is a 4-connected triangulation whose outer-vertices are $a$, $b$, and $c$, then $T$ is a W-triangulation 3-bounded by $(a, b)-(b, c)-(c, a)$. By Property 1, $T$ admits a PS-representation $(\Sigma, R, F)$, with $F = \{bc, ca\}$, that is contained in a region $\tau$. Furthermore, in the boundary of $\tau$ we successively meet the ends of $b, b, c, c, a$. To obtain an $S$-representation of $T$, it is sufficient to extend $a, b,$ and $c$ outside of $\tau$ so that $c$ crosses $a$ and $b$, as depicted in Figure 26.

![Figure 26: S-representation of $T$ from $(\Sigma, R, F)$](image)

Suppose now that $T$ is a triangulation that contains at least one separating 3-cycle. Consider a separating 3-cycle $(a, b, c)$ such that there is no other separating 3-cycle lying inside. This implies that the triangulation $T'$ induced by the vertices on and inside $(a, b, c)$ is 4-connected.

Let $T_1$ be the triangulation obtained by removing the vertices lying strictly inside $(a, b, c)$. Let $T_2$ be the subgraph of $T$ induced by the vertices lying strictly inside $(a, b, c)$ (i.e., $T_2 = \ldots$)
$T' \setminus \{a, b, c\}$. In $T_1$, the cycle $(a, b, c)$ is a face of the triangulation and is no more a separating 3-cycle. Thus $T_1$ has one separating cycle less than $T$, and so we have by induction hypothesis that $T_1$ admits an S-representation $(\Sigma_1, R_1)$. This S-representation contains a face-region $abc$ corresponding to the face $abc$. Without loss of generality, say that $abc$ is an $(a, b, c)$-region, as depicted in Figure 27.

![Figure 27](image)

Figure 27: In the S-representation $(\Sigma_1, R_1)$ of $T_1$, the $(a, b, c)$-region $abc$.

Since $T'$ is a triangulation with at least four vertices, the neighbors of any vertex $v \in V(T')$ induce a cycle. Suppose that the vertex $a$ (resp. $b$ and $c$) has exactly one neighbor $v$ that lies inside $(a, b, c)$. Then there exists a cycle $(b, v, c)$ (resp. $(a, v, c)$ and $(a, v, b)$) in $T'$ and consequently $v$ is a neighbor of $a$, $b$, and $c$ in $T'$.

Suppose that there exists another vertex $w$ in $T'$, then $w$ lies either inside the cycle $(a, v, b)$, inside $(a, v, c)$, or inside $(b, v, c)$ and then one of these cycles is a separating 3-cycle. This is impossible by definition of $(a, b, c)$. So we can distinguish two cases (see Figure 28), (A) the case where $T_2$ is a single vertex, and (B) the case where each of the vertices $a$, $b$, and $c$ has at least two neighbors inside $(a, b, c)$.

![Figure 28](image)

Figure 28: The cases (A) and (B).

**Case (A):** $T_2$ is a single vertex $v$. To obtain an S-representation $(\Sigma, R)$ of $T$ (see Figure 29), we add a string $v$ in $(\Sigma_1, R_1)$. Since $E(T) \setminus E(T_1) = \{va, vb, vc\}$ this string $v$ crosses $a, b, c$. Moreover, we also define three disjoint face-regions $acv, vbc, vab$ that correspond respectively to the faces $ace, vbc, vab$.

Since $(\Sigma_1, R_1)$ is an S-representation of $T_1$ and since $v, acv, vbc, vab$ are drawn inside $abc$, it is clear that $(\Sigma \cup \{v\}, (R \setminus \{abc\}) \cup \{acv, vbc, vab\})$ is an S-representation of $T$.

**Case (B):** Each of the vertices $a$, $b$, and $c$ has at least two neighbors inside $(a, b, c)$. There exists a cycle $(c, a_1, \ldots, a_p, b)$ (resp. $(a, b_1, \ldots, b_q, c)$ and $(b, c_1, \ldots, c_r, a)$) in $T'$ whose vertices are exactly the neighbors of $a$ (resp. $b$ and $c$). We already know that $p > 1, q > 1, r > 1$ and that $a_p = b_1, b_q = c_1$, and $c_r = a_1$. Moreover, since $b_1$ and $c$ (resp. $c_1$ and $a$, and $a_1$ and $b$) are the only two common neighbors of $a$ and $b$ (resp. $b$ and $c$, and $a$ and $c$) in $T'$ (otherwise
there would be a separating 3-cycle \( (a_1, \ldots , a_p = b_1, \ldots , b_q = c_1, \ldots , c_r = a_1) \) is a cycle. This implies from Lemma 2 that \( T_2 \) is a W-triangulation.

Suppose that there exists an edge \( a_i a_j \) (resp. \( b_i b_j, c_i c_j \)) with \( 1 < i+1 < j \leq p \) (resp. \( 1 < i+1 < j \leq q \)). Then, the cycle \( (a_i, a_j) \) (resp. \( (b_i, b_j), (c_i, c_j) \)) would be a separating 3-cycle of \( T' \). Consequently, \( T_2 \) is 3-bounded by \( (a_1, \ldots , a_p)-(b_1, \ldots , b_q)-(c_1, \ldots , c_r) \).

With respect to this 3-boundary, \( T_2 \) has an anticlockwise PS-representation \( (\Sigma_2, R_2, F_2) \), with \( F_2 = E_0 \setminus \{a_1 a_2\} \) (c.f. Property 1). Let \( T_2 \) be a region of \( abc \) containing this representation.

Since \( abc \) is an \((a,b,c)\)-region, on its boundary we successively cross \( a, a, b, b, c, a \) and \( c \) when going anticlockwise (by doing an axial symmetry if necessary).

In Figure 30, starting from \((\Sigma_1, R_1)\) and \((\Sigma_2, R_2)\) we obtain \((\Sigma, R)\). We extend the strings \( a_2, \ldots , a_p, b_1, \ldots , b_q, c_1, \ldots , c_r \) to obtain the crossings that correspond to the edges in the set \( E(T) \setminus (E(T_1) \cup (E(T_2) \setminus F_2)) = \{ a_{i+1} \mid i \in [1, p] \} \cup \{ b_{i+1} \mid i \in [1, q] \} \cup \{ c_{i+1} \mid i \in [1, r] \} \cup \{ a_i a_{i+1} \mid i \in [2, p-1] \} \cup \{ b_i b_{i+1} \mid i \in [1, q-1] \} \cup \{ c_i c_{i+1} \mid i \in [1, r-1] \} \). We also define face-regions for the faces in the set \( \{ a b_1 a_{i+1}, a c_1 c_{i+1} \} \cup \{ a_{i+1} a_i, b_{i+1} b_i, c_{i+1} c_i \} \). Since \((\Sigma_1, R_1)\) is an S-representation of \( T_1 \) and \((\Sigma_2, R_2, F_2)\) is a PS-representation of \( T_2 \), \((\Sigma, R, F)\) is an S-representation of \( T \).

- \( \Sigma \) is a 1-string representation of \( T \): Indeed, we added all the crossings corresponding to the edges in \( E(T) \setminus (E(T_1) \cup (E(T_2) \setminus F_2)) \).
- \((\Sigma, R)\) is “strong”: Indeed, we added all the face-regions corresponding to the inner-faces of \( T \) that are neither in \( T_1 \) nor in \( T_2 \).

Consequently, every triangulation admits an S-representation, which proves Theorem 3 and then Theorem 2.

\[ \square \]

5 Conclusion

The first and the second author recently improved the result presented in this article by proving Conjecture 1 [2]. For this they use the same decomposition of triangulation but their notion of face-region is quite different. One should also mention that their construction does not correspond to a stretching of the 1-string representation presented here.

Finally, an interesting question is whether the result presented here holds for other surfaces. For example, does any graph embedded on a surface \( S \) have a 1-string representation on \( S \)?
Figure 30: Case (B): Modifications inside abc.

References


