3-colorable planar graphs have an intersection segment representation using 3 slopes*

Daniel Gonçalves[a]

[a] *LIRMM, Université de Montpellier & CNRS, daniel.goncalves@lirmm.fr*

Abstract

In his PhD Thesis E.R. Scheinerman conjectured that planar graphs are intersection graphs of segments in the plane. This conjecture was proved with two different approaches by J. Chalopin and the author, and by the author, L. Isenmann, and C. Pennarun. In the case of 3-colorable planar graphs E.R. Scheinerman conjectured that it is possible to restrict the set of slopes used by the segments to only 3 slopes. Here we prove this conjecture by using an approach introduced by S. Felsner to deal with contact representations of planar graphs with homothetic triangles.

1 Introduction

In this paper, we consider intersection representations for planar graphs. A *segment representation* of a graph G maps every vertex $v \in V(G)$ to a segment $\mathbf{v}$ of the plane so that two segments $\mathbf{u}$ and $\mathbf{v}$ intersect if and only if $uv \in E(G)$. Although this graph family is simply defined, it is not easy to manipulate. Actually, even if this class of graphs is small (there are less than $2^{O(n \log n)}$ such graphs with n vertices [16]) a segment representation may be long to encode (in the representations of some of these graphs the endpoints of the segments need at least $2^{\sqrt{n}}$ bits to be coded [14]). There are also interesting open problems concerning this class of graphs. For example, we know that deciding whether a graph G admits a segment representation is NP-hard, actually it is even $\exists\mathbb{R}$-complete [13], but it is still open whether this problem belongs to NP or not. Here we focus on segment representations for planar graphs.

In his PhD Thesis, E.R. Scheinerman [17] conjectured that every planar graph has a segment representation. This conjecture attracted a lot of attention. H. de Fraysseix and P. Ossona de Mendez [6] proved it for a large family of planar graphs, the planar graphs having a 4-coloring in which every induced cycle of length 4 uses at most 3 colors. In particular, this implies the conjecture for 3-colorable planar graphs. Then J. Chalopin and the author finally proved this conjecture [2]. Recently, a much simpler

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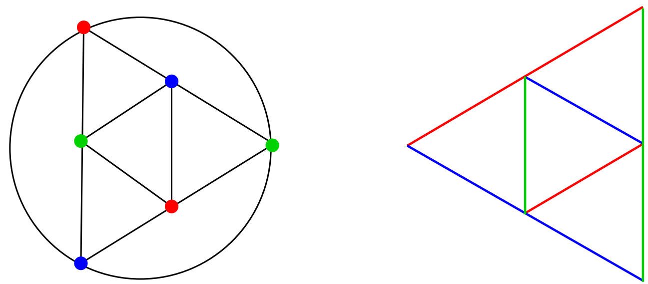

Figure 1: The octahedron and a 3-slopes contact representation. It is unique, up to vertex automorphism, up to scaling, and once the slopes are set.

proof was provided by the author, L. Isenmann, and C. Pennarun [10]. Here we focus on segment representations of planar graphs with further restrictions.

In his PhD Thesis, E.R. Scheinerman [17] proved that every outerplanar graph has a segment representation where only 3 slopes are used, and where parallel segments do not intersect. Let us call such a representation a *3-slopes segment representation*. This result led E.R. Scheinerman conjecture [18] (see also [6]) that such representation exists for every 3-colorable planar graph. Later, several groups proved a related result on bipartite planar graphs [3, 7, 11]. They proved that every bipartite planar graph has a 2-slopes segment representation, with the extra property that segments do not cross each other. Let us call such a representation a *2-slopes contact segment representation*. More recently de Castro *et al.* [1] considered a particular class of 3-colorable planar graphs. They proved that every triangle-free planar graph has a 3-slopes contact segment representation. Such a contact segment representation cannot be asked for any 3-colorable planar graph. Indeed, up to isomorphism, the octahedron has only one 3-slopes contact segment representation depicted in Figure 1, and one can easily check that this representation does not extend to the (3-colorable) graph obtained after gluing a copy of an octahedron in each of its faces. However, we will use 3-slopes contact segment representations in the proof of our main result.

Theorem 1 *Every 3-colored planar graph has a 3-slopes segment representation such that parallel segments correspond to the color classes.*

A 3-slopes contact representation of a graph naturally induces such a representation for its induced subgraphs. As every 3-colored planar graph is the induced subgraph of some 3-colored triangulation we only consider the case of triangulations in the following. In Section 2 we review some basic definitions. Section 3 is devoted to the so-called *triangular contact schemes*. It is shown that every 3-colorable triangulation admits such a scheme. Then, those schemes are used in Section 4 to build 3-slopes segment representations. Finally, we conclude with some remarks on 4-slopes segment representations.

2 Terminology

A *triangulation* is a plane graph where every face has size three. A triangulation is *simple* if it has no loops nor multiple edges. Throughout the paper the considered triangulations are not necessarily simple, unless stated otherwise. A triangulation T is *Eulerian* if every vertex has even degree. It is folklore that these triangulations are the 3-colorable triangulations. Actually these triangulations are uniquely 3-colorable (up to color permutation). Hence their vertex set $V(T)$ is canonically partitioned into three independent sets A, B and C. In the following we will denote the vertices of these sets respectively a_i with $0 \leq i < |A|$, b_j with $0 \leq j < |B|$, and c_k with $0 \leq k < |C|$. In such a triangulation T any face is incident to one vertex a_i, one vertex b_j, and one vertex c_k, and these vertices appear in this order either clockwisely or counterclockwisely. In the following, the vertices of the outerface are always denoted a_0, b_0 and c_0, and they appear clockwisely in this order around T. As the orders of two adjacent faces are opposite, the dual graph of T is bipartite. Given an Eulerian triangulation T with face set $F(T)$, let us denote by $F_1(T)$ and $F_2(T)$ (or simply F_1 and F_2 if it is clear from the context) the face sets partitioning $F(T)$, such that no two adjacent faces belong to the same set, and such that $F_2(T)$ contains the outer face. Note that by construction for any face $f \in F_1(T)$ (resp. $f \in F_2(T)$) its vertices a_i, b_j and c_k appear in clockwise (resp. counterclockwise) order around f. Note that the vertices a_0, b_0 and c_0 appear in clockwise order around T, but in counterclockwise order w.r.t. the outer face. Let $n = |V(T)|$. As T is a triangulation, by Euler's formula it has $2n-4$ faces. Hence, as T's dual is bipartite and 3-regular, $|F_1(T)| = |F_2(T)| = n-2$.

In the following we build 3-slopes segment representations. The 3 slopes used are expected to be distinct, but apart from that the exact 3 slopes considered do not matter. Indeed, for any two triples of slopes, (s_1, s_2, s_3) and (s'_1, s'_2, s'_3), there exists an affine map of the plane turning any 3-slopes segment representation using slopes (s_1, s_2, s_3) into a 3-slopes segment representation using slopes (s'_1, s'_2, s'_3). We denote $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ the vectors corresponding to slopes of the sets A, B , and C respectively. The magnitude of these vectors is chosen such that $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = \overrightarrow{0}$.

3 TC-representations and TC-schemes

We begin with the definition of particular 3-slopes contact representations illustrated in Figure 2.

Definition 2 *A* Triangular 3-slopes Contact segment representation *(*TC-representation *for short) is a 3-slopes contact segment representation using the same slopes as $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$, and where:*

- *Three segments $\mathbf{a_0}$, $\mathbf{b_0}$, and $\mathbf{c_0}$, form a triangle which contains all the other segments.*
- *Every inner region is a triangle, whose each side is contained in a segment of the representation.*

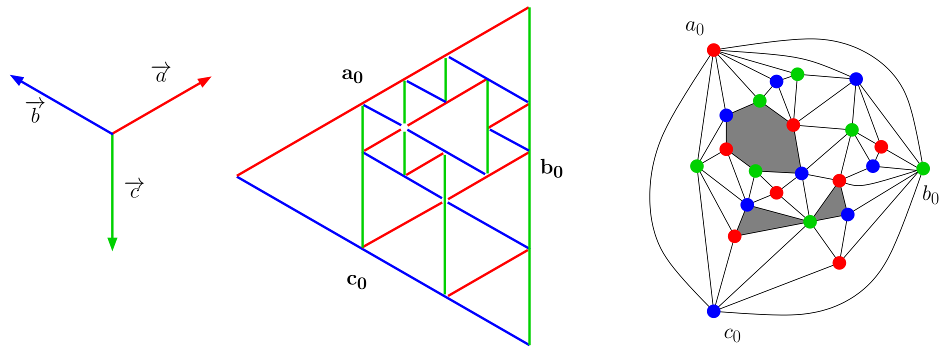

Figure 2: (left) Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ (middle) A TC-representation with various types of intersection points. (right) Its induced graph, where gray faces are particular degenerate faces. One has size six, and there are two faces of size three that correspond to the same intersection point.

- *Two parallel segments intersect on at most one point, their endpoint.*

Remark 3 *In such a representation, an intersection point* $\mathbf{p}$ *is of one of the following four types (see Figure 2).*

- *The intersection point of 2 outer segments;*
- *the intersection point of 3 segments (with 3 distinct slopes) such that exactly two of them end at* $\mathbf{p}$*;*
- *the intersection point of 5 segments such that exactly four of them end at* $\mathbf{p}$ *(such a point will be generally considered as the merge of two intersection points of 3 segments); or*
- *the intersection point of 6 segments that have an end at* $\mathbf{p}$.

Definition 4 *Let the plane graph* $M(\mathcal{R})$ induced *by a TC-representation* $\mathcal{R}$ *be the graph whose vertices correspond to the segments of the representation, and where two vertices are adjacent if and only if the corresponding segments form a corner of one of the inner triangles. The orders of the neighbors around a vertex* v *correspond to the order of the segments around the segment* $\mathbf{v}$.

Note that the plane graph induced by a TC-representation has several properties. For example, two parallel segments correspond to non-adjacent vertices. The slopes hence define a 3-coloring of the graph. Note also that the dual graph of $M(\mathcal{R})$ is bipartite. Indeed such a plane graph has two types of faces, one set contains the (triangular) faces corresponding to the inner regions of the TC-representation, and the other set contains the outerface and the faces corresponding to intersection points. Let us denote the latter faces *degenerate faces*, and note that those faces have size three or six. A size

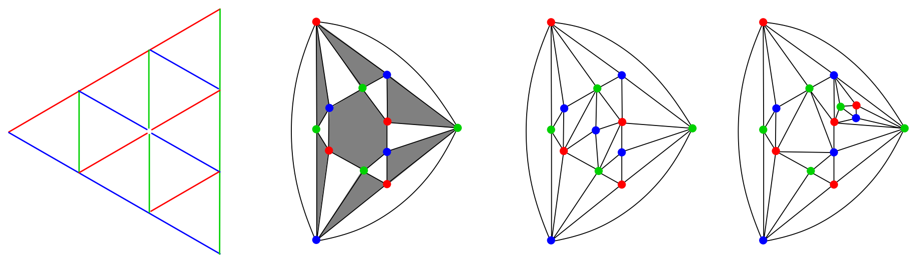

Figure 3: From left to right. A TC-representation $\mathcal{R}$; its induced plane graph $M(\mathcal{R})$, where gray faces are the degenerate faces; and two triangulations having $\mathcal{R}$ as TC-scheme.

six face $(a_i, b_j, c_k, a_{i'}, b_{j'}, c_{k'})$ comes from the intersection point of six segments, and as those six segments go in distinct directions they do not intersect elsewhere, so this cycle has no chord in $M(\mathcal{R})$. Finally note that going clockwise in any inner region one successively follows $\alpha\overrightarrow{a}$, $\alpha\overrightarrow{b}$, and then $\alpha\overrightarrow{c}$, for some not necessarily positive value α.

Definition 5 *A TC-representation $\mathcal{R}$ is a* TC-scheme *of an Eulerian triangulation T if $M(\mathcal{R})$ is a subgraph of T with the same outer face as T, and such that the vertices and edges of $V(T) \setminus V(M(\mathcal{R}))$ lie inside degenerate faces of $M(\mathcal{R})$ (see Figure 3).*

Actually as in $M(\mathcal{R})$, the inner faces around any vertex alternate among degenerate and non-degenerate. This implies that every edge of $M(\mathcal{R})$ bounds a non-degenerate face, and a face that is degenerate or that is the outerface. We thus have the following.

Remark 6 *A TC-representation $\mathcal{R}$ is a TC-scheme of T if and only if the non-degenerate faces of $M(\mathcal{R})$ and its outerface are faces of T.*

The main ingredient in the proof of Theorem 1 is the following.

Theorem 7 *Every Eulerian triangulation T has a TC-scheme, and this scheme is unique.*

To prove this theorem we proceed by the following steps. We first model TC-schemes of T by means of a system of linear equations in Section 3.1. We then show in Section 3.2 that such a linear system always has a solution, and that this solution is unique (c.f. Lemma 8). Finally we show in Section 3.3 that the solution of this linear system defines a TC-scheme of T (c.f. Lemma 12).

3.1 The linear system model

In a TC-representation all the triangles are homothetic. Let us define the *size* of a triangle as its relative size with respect to the outer triangle. We may require that the outer triangle has size 1, the triangles with a corner on the left have positive sizes, while

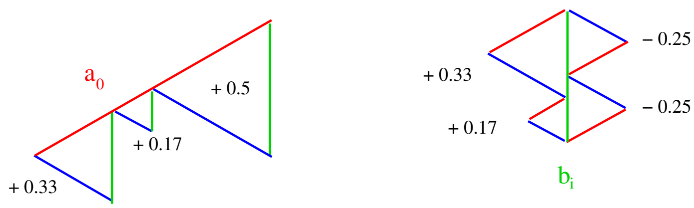

Figure 4: (left) The size of the triangles around a_0. (right) The size of the triangles around some inner vertex b_i.

the triangles with a corner on the right have negative sizes. The variables of our linear system correspond to the sizes of the triangular regions. So for each face $f \in F_1$ we have a variable x_f. Informally, the value of x_f will prescribe the size and shape of the corresponding triangle in a TC-representation. If $x_f < 0$, $x_f = 0$, or if $x_f > 0$ the corresponding triangle has a corner on the right, is missing, or has a corner on the left, respectively.

Let us denote by $F_1(v)$ the subset of faces of F_1 incident to v. As the outer triangle has size 1 and contains the other triangles, the faces in $F_1(a_0)$ should have non-negative sizes, and they should sum up to 1 (see Figure 4, left). We hence consider the following constraint.

$$\sum_{f \in F_1(a_0)} x_f = 1 \tag{a_0}$$

We add no constraint about the sign of these sizes. Note that similar constraints hold for b_0 and c_0.

$$\sum_{f \in F_1(b_0)} x_f = 1 \tag{b_0}$$

$$\sum_{f \in F_1(c_0)} x_f = 1 \tag{c_0}$$

Similarly, around an inner segment of a TC-representation all the triangles on one side have same size sign, which is opposite to the other side. Furthermore, by summing all these sizes one should obtain 0 (see Figure 4, right). Hence, for any inner vertex u we consider the following constraint.

$$\sum_{f \in F_1(u)} x_f = 0 \tag{u}$$

In the following, Equation (a_j) will refer to Equation (u) where vertex u is replaced by a_j. Note that every face $f \in F_1$ is incident to exactly one vertex of A, one vertex of B, and one vertex of C. Hence by summing Equations (a_0), (a_1),…,$(a_{|A|})$, one obtains that $\sum_{f \in F_1} x_f = 1$. The same holds with Equations (b_0), (b_1),…,$(b_{|B|})$, or with Equations (c_0), (c_1),…,$(c_{|C|})$. Equations (b_0) and (c_0) are hence implied by the others and thus we do not need to consider them anymore. Let us denote by $\mathcal{L}$ the obtained system of $n-2$ linear equations on $|F_1| = n-2$ variables.

3.2 $\mathcal{L}$ has a unique solution

Let us define the set $V' = V \setminus \{b_0, c_0\}$ of size $n-2$. Finding a solution to $\mathcal{L}$ is equivalent to finding a vector $S \in \mathbb{R}^{F_1}$ (that is a vector indexed by elements of F_1) such that $MS = I$, where $M \in \mathbb{R}^{V' \times F_1}$ (a square matrix indexed by elements of $V' \times F_1$) and $I \in \mathbb{R}^{V'}$ are defined by

$$M(x_i, f) = \begin{cases} 1 & \text{if } f \in F_1(x_i) \\ 0 & \text{otherwise.} \end{cases} \qquad I(x_i) = \begin{cases} 1 & \text{if } x_i = a_0 \\ 0 & \text{otherwise.} \end{cases}$$

Given some bijective mappings $g_{V'} : [1, \ldots, n-2] \longrightarrow V'$ and $g_{F_1} : [1, \ldots, n-2] \longrightarrow F_1$, one can index the elements of M by pairs $(i,j) \in [1, \ldots, n-2] \times [1, \ldots, n-2]$, and thus define the determinant of M. By the following lemma, $\mathcal{L}$ has a solution vector S, and this solution is unique.

Lemma 8 *The matrix M is non-degenerate, i.e.* $\det(M) \neq 0$.

The proof of this lemma is inspired by the work of S. Felsner [4] on contact representations with homothetic triangles. See also [5] for another proof using the same approach.

Proof. Let T_M be the bipartite graph with independent sets V' and F_1 such that $v \in V'$ and $f \in F_1$ are adjacent if and only if v and f are incident in T. Note that M is the biadjacency matrix of T_M. From the embedding of T one can easily embed T_M in such a way that all the inner faces have size 6, and such that a_0 is on the outerboundary.

Note that every perfect matching of T_M (if any) corresponds to a permutation σ on $[1, \ldots, n-2]$ defined by $\sigma(g_{F_1}^{-1}(f)) = g_{V'}^{-1}(v)$, for any edge vf of the perfect matching. If the obtained permutation is even we call such perfect matching positive, otherwise it is negative. From the Leibniz formula for the determinant,

$$\det(M) = \sum_{\sigma \in S_{n-2}} sgn(\sigma) \prod_{i \in [1, \ldots, n-2]} M(g_{V'}(\sigma(i)), g_{F_1}(i))$$

one can see that $\det(M)$ counts the number of positive perfect matchings of T_M minus its number of negative perfect matchings.

Claim 9 *The graph T_M admits at least one perfect matching.*

Proof. As T_M is bipartite, and as $|V'| = |F_1|$, it suffices to show that T_M has an F_1-saturating matching. This follows from Hall's mariage theorem, and the fact that for any set $X \subseteq F_1$ the set $Y \subset V'$ of vertices incident to a face in X is such that $|Y| \geq |X|$. Let us show this below for any $X \subseteq F_1$.

Consider the (planar) subgraph of T with all the edges and all the vertices incident to a face of X. Then, triangulate this graph and denote T_X the obtained triangulation. Note that as any two faces of X are not adjacent in T_X, this triangulation has at least $2|X|$ faces. Indeed, around each vertex there are at least twice as many faces as faces of X, and summing over every vertex one obtains the inequality. Together with the fact that T_X has $2|V(T_X)| - 4$ faces,

$$2|V(T_X)| - 4 \geq 2|X|$$

and that $V(T_X) \subseteq Y \cup \{b_0, c_0\}$,

$$|Y| + 2 \geq |V(T_X)|$$

one obtains that

$$2|V(T_X)| - 4 + 2|Y| + 4 \geq 2|X| + 2|V(T_X)|$$

$$|Y| \geq |X|$$

. □

Given a graph G and a perfect matching M of G, an *alternating cycle C* is a cycle of G with edges alternating between M and $E(G) \setminus M$. Note that replacing in M the edges of $M \cap C$ by the edges of $C \setminus M$ yields another perfect matching. We call such operation a *cycle exchange*. It is folklore that the perfect matchings of a graph are linked by cycle exchanges. Indeed, given any perfect matching M_1 of G one can reach any perfect matching M_2, by a succession of cycle exchanges. Actually, for T_M any such cycle has length congruent to $2 \pmod 4$.

Claim 10 *For any perfect matchings of T_M and any of its alternating cycles C, we have that the length $\ell(C)$ of C is congruent to* $2 \pmod 4$.

Proof. The subgraph G of T_M induced by the vertices and edges on or inside C is such that all the inner faces have length 6, and it is routine from Euler's formula to show that C has length congruent to $2 + 2|V_G| \pmod 4$, where V_G is the vertex set of G. Indeed,

$$\ell(C) - 6 + 6|F_G| = \sum_{f \in F_G} \ell(f) = 2|E_G| = 2|V_G| + 2|F_G| - 4$$

$$\ell(C) \equiv 2|V_G| + 2 \pmod 4$$

Finally, as the vertices of G are paired by the perfect matching we have that $|V_G|$ is even. □

The previous claim implies that all the perfect matchings of T_M induce permutations of the same sign. Indeed, performing a $(4k+2)$-cycle exchange does not change the sign of the permutation as it corresponds to performing $2k$ transpositions in the permutation. Hence, all the terms of $\det(M)$ have same sign, and this sum has at least one non-zero term (by Claim 9). Thus $\det(M) \neq 0$. □

3.3 A solution of $\mathcal{L}$ defines a TC-scheme

A TC-scheme $\mathcal{R}$ *corresponds* to a solution of $\mathcal{L}$, the linear system defined for an Eulerian triangulation T, if $\mathcal{R}$ is a TC-scheme of T such that each face $f \in F_1$ corresponds to an inner triangle of $\mathcal{R}$ of size x_f, the solution of $\mathcal{L}$, unless $x_f = 0$. In other words, the inner regions of $\mathcal{R}$ correspond to non-zero faces of F_1. By the embedding of $M(\mathcal{R})$, note that the non-zero faces of F_1 incident to a vertex v appear in the same order around **v** as in T. This implies the following.

Remark 11 *If a vertex v has two neighbors u and w, that are consecutive in $M(\mathcal{R})$ but not in T, then the vertices of T between u and w around v lie in a degenerate face of $M(\mathcal{R})$ with at least u, v and w on its border. Furthermore this degenerate face corresponds to the intersection point of* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$.

Let us now proceed to the main result of this section.

Lemma 12 *Every Eulerian triangulation T admits a TC-scheme $\mathcal{R}$ that corresponds to the solution of its linear system $\mathcal{L}$.*

Proof. Let us proceed by induction on the number of faces $f \in F_1$ such that $x_f = 0$. We start with the case where every face $f \in F_1$ is such that $x_f \neq 0$.

If every face $f \in F_1$ is such that $x_f \neq 0$, we construct a TC-scheme $\mathcal{R}$ corresponding to the solution of $\mathcal{L}$ as follows. First let Δ be a triangle formed by three vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ in this order. The sides of Δ correspond to $\mathbf{a_0}$, $\mathbf{b_0}$, and $\mathbf{c_0}$, respectively. For each face $f \in F_1$, let Δ_f be an homothetic copy of Δ with ratio x_f. The triangle Δ_f is thus obtained by following the vectors $x_f\overrightarrow{a}$, $x_f\overrightarrow{b}$, and $x_f\overrightarrow{c}$ in this order. We are going to show that all these triangles Δ_f can be arranged as a tiling of Δ, forming a TC-representation of T (i.e. such that $T = M(\mathcal{R})$).

Note that a necessary condition for this to work is that (1) every face of $f \in F_1$ around a_0, b_0, or c_0 is positive (i.e. $x_f > 0$), and that (2) for any inner vertex v of T its positive (resp. negative) incident faces in F_1 are consecutive around v. Otherwise this would result in overlapping triangles Δ_f (see Figure 4). We first show that (1) and (2) are fulfilled, and then we show that this suffices to ensure the construction of $\mathcal{R}$.

Consider the incidence graph I, between vertices of $V(T)$ and faces of F_2. First note that this plane graph has only size six faces and that they are in bijection with the faces in F_1. Let us orient the edges of I as follows. An edge vf of I, with $v \in V(T)$ and $f \in F_2(T)$, is oriented from v to f if and only if the incident faces (which correspond to faces in F_1) have different signs. Note that for an inner vertex of T, $d^+(v) \geq 2k$ for some $k \geq 1$ (as v is incident to positive and to negative faces in T), and that $d^+(f) = 1$ or 3 for a face $f \in F_2$. The graph I has $2n-2$ vertices (3 outer vertices of T, $n-3$ inner vertices of T, and $n-2$ faces of F_2) and $3n-6$ edges. The outerface of T, f^o, has outdegree 3 in I. Otherwise, among the three faces of F_1 incident to a_0b_0, a_0c_0, or b_0c_0 there would be positive ones and negative ones. This would imply that two of a_0, b_0, and c_0 have outdegree at least 2 in I. This would be impossible as $2+2+2(n-3)+(n-2) > 3n-6$. We thus have that $d^+(f^o) = 3$, and a counting argument gives us that the other faces f of F_2 have outdegree one, that the outer vertices have outdegree zero, and that inner vertices have outdegree two. Thus (1) and (2) are verified.

To construct the TC-representation of T, we define a plane graph G^Δ from I by replacing f^o with three vertices (Step 1), and for each vertex $v \in V(T)$, by turning its neighborhood in I from a star into a path (Step 2).

(Step 1) The vertex f^o is replaced by three new vertices $f^o_{\overrightarrow{a}}$, $f^o_{\overrightarrow{b}}$, and $f^o_{\overrightarrow{c}}$ in such a way that $f^o_{\overrightarrow{a}}$ is adjacent to b_0 and c_0 (see Figure 5). The six new edges are oriented

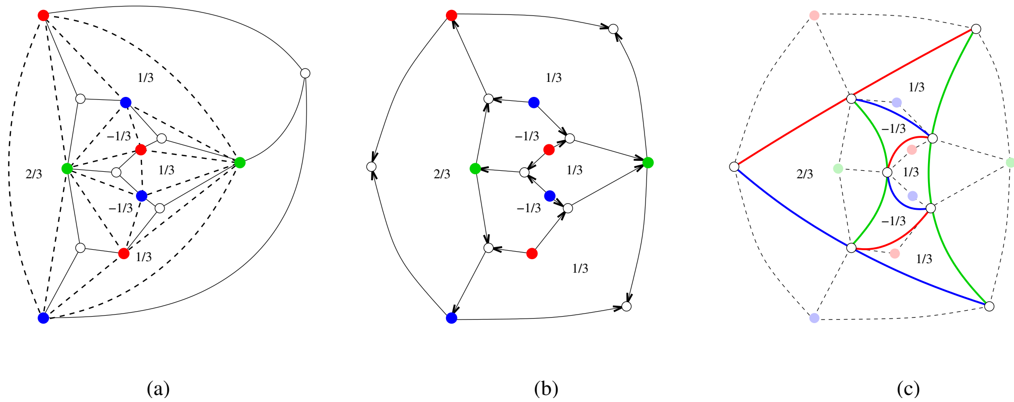

Figure 5: (a) Example of an Eulerian triangulation T (dashed lines), with incidence graph I. The numbers correspond to the solution of $\mathcal{L}$. (b) The graph I' obtained after (Step 1). (C) The graph G^{Δ}.

towards the newly created vertices. Let us denote I' this new oriented graph. Note that now every vertex $v \in V(T)$ has outdegree two, and that by assigning size -1 to the outerface, all faces incident to v sum up to zero.

(Step 2) For each vertex $v \in V(T)$, its neighborhood in I' is turned into a path P_v whose ends are the out-neighbors of v. The in-neighbors are ordered as follows in P_v. We first denote f^+ (resp. f^-) the out-neighbors of v such that the face following f^+ (resp. f^-), around v in clockwise order, has positive (resp. negative) size (*i.e.* solution in $\mathcal{L}$). Two in-neighbors f, f' of v are ordered along P_v in such a way that f is closer to f^+ than f', if and only if the sum of the face sizes going around v from f^+ to f is lower than the sum from f^+ to f'. If the two sums are equal, then f and f' are merged into a single vertex (see Figure 6). As all the faces around v have non-zero sizes, and as positive sizes are consecutive, a vertex f is merged at most once. The obtained plane graph is denoted G^{Δ}. Note that the inner faces of G^{Δ} correspond to a faces of F_1, and we assign them the corresponding sizes. Note also that a face of G^{Δ} corresponding to the face $a_i b_j c_k \in F_1$, is bordered by three subpaths of paths P_{a_i}, P_{b_j}, and P_{c_k}. We now assign positive length to the edges of G^{Δ} so that the length of these subpath corresponds to the size of the face, forgetting the sign. For an edge ff' of a path P_v we assign the absolute value of the sum of the face sizes between f and f' around v in I'.

By construction G^{Δ} has three types of vertices:

- The vertices $f^o_{\overline{a}}$, $f^o_{\overline{b}}$, and $f^o_{\overline{c}}$, which have degree two. Indeed, *e.g.* the vertex $f^o_{\overline{a}}$ is at the end of P_{b_0} and P_{c_0}.
- The vertices originating from a single vertex $f \in V(I) \setminus (V(T) \cup \{f^o\})$. As such f has in-degree two and out-degree one in I it is at the end of two paths and in the middle of a third one.

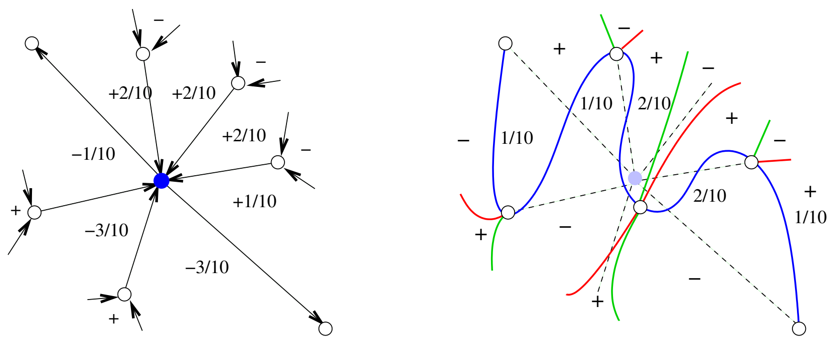

Figure 6: An example of (Step 2) with a merge of f and f'.

- The vertices originating from two vertices $f, f' \in V(I) \setminus (V(T) \cup \{f^o\})$. By construction, such vertex is in the middle of a path, and has two path ending on each side (corresponding to in-neighbors in I).

From the orientation of I', note that the sign of the faces alternate around any of these vertices (see Figure 6). We now want to draw G^Δ planarly, in such a way that its inner faces are all homothetic to the triangle formed by following the three vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$. More precisely, for a face f of size α that is bordered by subpaths $P_a^f \subseteq P_{a_i}$, $P_b^f \subseteq P_{b_j}$, and $P_c^f \subseteq P_{c_k}$, the subpath P_a^f, P_b^f and P_c^f should be mapped to vectors $\alpha\overrightarrow{a}$, $\alpha\overrightarrow{b}$, and $\alpha\overrightarrow{c}$, respectively, in such a way that the edge length along these paths are followed. Note that there is no local obstruction to the existence of such embedding.

- Each and edge ff' of G^Δ is consistently embedded. Indeed, the length of ff' is set in G^Δ, and whatever the incident face considered (as these faces have different signs) the vector $\overrightarrow{ff'}$ has the same direction.

- For the outer vertices $f^o_{\overline{a}}$, $f^o_{\overline{b}}$, and $f^o_{\overline{c}}$, their incident inner faces form an angle smaller than π (*e.g.* for $f^o_{\overline{a}}$ the angle is the one from $\overrightarrow{c}$ to $-\overrightarrow{b}$). For any other outer vertex f, which necessarily corresponds to a single vertex of I', its (three) incident inner faces form an angle of size exactly π. For example, if f is in the middle of the path P_{a_0} and at the end of paths P_{b_j} and P_{c_k}, we know by (1) that the inner faces incident to P_{a_0} are positive, while the third one is negative because the edge fb_j and fc_k are oriented towards f in I. Thus, the angles around f go from $\overrightarrow{a}$, to $-\overrightarrow{c}$, to $\overrightarrow{b}$, and to $-\overrightarrow{a}$.

- For any inner vertex f corresponding to a single vertex of I', its (four) incident faces form an angle of size exactly 2π. For example, if f is in the middle of a path P_{a_i} and at the end of paths P_{b_j} and P_{c_k}, as the four faces signs alternate, the angles around f go from $-x\overrightarrow{a}$ to $x\overrightarrow{a}$, to $-x\overrightarrow{c}$, to $x\overrightarrow{b}$, and back to $-x\overrightarrow{a}$, for $x \in \{-1, +1\}$.

- Similarly, for an inner vertex f originating from two vertices of I' the sum of the 6 angles is again 2π.

From these observations a simple variant of Lemma 6 of [5] ensures the existence of such embedding. Alternatively, one could triangulate G^{Δ} to use directly this lemma.

Note that this embedding is such that for each vertex $a_i \in V(T)$ (resp. $b_j \in V(T)$ and $c_k \in V(T)$) the corresponding path P_{a_i} (resp. P_{b_j} and P_{c_k}) forms a segment parallel to $\overrightarrow{a}$ (resp. $\overrightarrow{b}$ and $\overrightarrow{c}$). As in G^{Δ} a vertex f is in the middle of at most one path P_v, these segments do not cross. For any inner edge of T, say a_ib_j incident to a face $f \in F_2(T)$, the paths P_{a_i} and P_{b_j} touch at the vertex f of G^{Δ}. For the outer edges the contact points are $f^o_{\overline{a}}$, $f^o_{\overline{b}}$, and $f^o_{\overline{c}}$. We thus have a TC-scheme of T.

If some faces $f \in F_1$ are such that $x_f = 0$, consider a face $a_ib_jc_k \in F_1$ such that $x_{a_ib_jc_k} = 0$. Let $a_\ell \in A$ be the vertex such that $a_\ell c_k b_j$ is a face (of F_2). Let T' be the (non-necessarily simple) Eulerian triangulation obtained from T by deleting the edges b_jc_k, $a_\ell b_j$ and $a_\ell c_k$, and by merging a_i and a_ℓ. The resulting vertex of T' is also denoted a_i. Let $\mathcal{L}'$ be the linear system defined for T'. Note that a solution of $\mathcal{L}$ clearly induces a solution of $\mathcal{L}'$. Indeed, every vertex $v \in V(T') \setminus \{a_i, b_j, c_k\}$ is incident to the same faces as in T', so they sum up to 0 (or to 1 for outer vertices). For b_j, or c_k these vertices are incident to one less face of F_1, the face $a_ib_jc_k$, and as $x_{a_ib_jc_k} = 0$, their incident faces still sum up to 0 (or to 1) in T'. Similarly, as the faces of F_1 incident to a_i in T' are the faces of F_1 incident to a_i or to a_ℓ in T, except $a_ib_jc_k$, they sum up to 0. As the solution of $\mathcal{L}'$ has one less 0 entry we can apply the induction, and consider a TC-scheme $\mathcal{R}'$ of T' corresponding to this solution of $\mathcal{L}'$. We consider different cases according to whether a_i and a_ℓ have non-zero incident faces in T.

If a_i and a_ℓ only have zero incident faces in T, then a_i only has zero incident faces in T' and it lies inside a degenerate face of $M(\mathcal{R}')$. The vertices b_j and c_k thus lie inside or on the border of the same degenerate face. Thus to go from T' to T, it suffices to change the interior of a degenerate face of $M(\mathcal{R}')$. The TC-representation $\mathcal{R}'$ is thus a TC-scheme of T, which clearly follows $\mathcal{L}$.

If a_i has non-zero incident faces, while a_ℓ only has zero incident faces in T, let $f, f' \in F_1$ be the non-zero faces incident to a_i that are closer to the face $a_ib_jc_k$ around a_i. Let the faces f and f' appear respectively before and after $a_ib_jc_k$, while going clockwise around a_i, and let us denote c_r and b_s, the C-vertex of f and the B-vertex of f', respectively. Let us also denote $\mathbf{p}$ the intersection point of $\mathbf{a_i}$, $\mathbf{c_r}$, and $\mathbf{b_s}$ in $\mathcal{R}'$. By Remark 11 the neighbors of a_ℓ in T, that are neighbors of a_i in T', lie inside or are on the border of the degenerate face of $M(\mathcal{R}')$ corresponding to $\mathbf{p}$ (with at least b_s, a_i and c_r on its border). Thus to go from T' to T, it suffices to change the interior of this degenerate face of $M(\mathcal{R}')$. The TC-representation $\mathcal{R}'$ is thus a TC-scheme of T, which clearly follows $\mathcal{L}$.

The case where a_ℓ has non-zero incident faces, while a_i only has zero incident faces in T is similar.

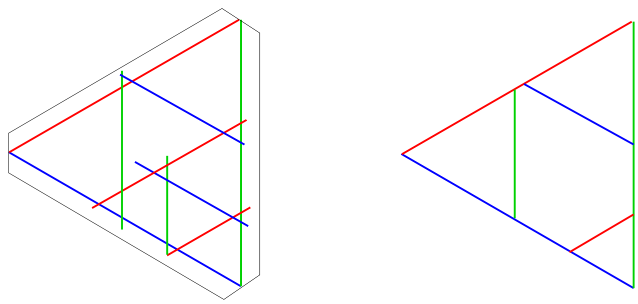

Figure 7: (left) A 3-slopes segment representation inside a hexagon. (right) A scheme representing its shape.

If both a_i and a_ℓ have non-zero incident faces in T, let us divide the segment $\mathbf{a'_i}$ of $\mathcal{R}'$ into two parts, one for each of a_i and a_ℓ. Note that the faces of $F_1 \setminus \{a_i b_j c_k\}$ incident to a_i (resp. a_ℓ) in T correspond to consecutive triangles arounds $\mathbf{a'_i}$. Furthermore as their sizes sum up to 0 there is a point $\mathbf{p} \in \mathbf{a'_i}$ that divides $\mathbf{a'_i}$ into two parts, $\mathbf{a_i}$ and $\mathbf{a_\ell}$, such that the faces of $F_1 \setminus \{a_i b_j c_k\}$ incident to a_i (resp. a_ℓ) in T correspond to triangles with a side contained inside $\mathbf{a_i}$ (resp. $\mathbf{a_\ell}$). Let us denote $\mathcal{R}$ the obtained TC-representation. As every non-degenerate face f of $\mathcal{R}$ corresponds to a face of $F_1(T)$ whose size is x_f, by Remark 6 we have that $\mathcal{R}$ is a TC-scheme of T following the solution of $\mathcal{L}$. This concludes the induction step of the proof. □

4 3-slopes segment representations

In this section we use Theorem 7 to prove the main theorem of the article, Theorem 1. As already mentioned, it is sufficent to prove it for Eulerian triangulations. Theorem 1 follows from the following technical proposition.

Proposition 13 *For every $0 < \epsilon < 1$, every simple Eulerian triangulation T admits a 3-slopes segment representations $\mathcal{R}$ such that:*

- *The segments $\mathbf{a_0}$, $\mathbf{b_0}$, and $\mathbf{c_0}$ form a triangle Δ of size 1 (its sides are obtained by following $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$).*
- *Every segment is contained in the hexagon centered on Δ, obtained by successively following $(1-\epsilon)\overrightarrow{a}$, $-2\epsilon\overrightarrow{c}$, $(1-\epsilon)\overrightarrow{b}$, $-2\epsilon\overrightarrow{a}$, $(1-\epsilon)\overrightarrow{c}$, and $-2\epsilon\overrightarrow{b}$ (see Figure 7).*
- *No three segments intersect at the same point.*

Given such representation $\mathcal{R}$ of a triangulation with some inner vertices, we define the *shape* of $\mathcal{R}$ as the triplet (s_a, s_b, s_c) of sizes in $\mathcal{R}$ of the triangles corresponding to $a_1b_0c_0$, $a_0b_1c_0$, $a_0b_0c_1$, respectively, where a_1, b_1 and c_1 are the vertices forming

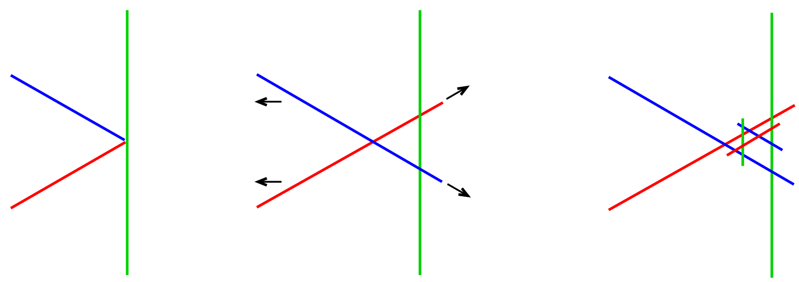

Figure 8: (left) A 3-degenerate point on $\mathcal{V}$ (middle) Small perturbation of $\mathcal{R}$ (right) The addition of a representation inside the new triangle.

an inner face with vertices b_0 and c_0, with a_0 and c_0, and with a_0 and b_0, respectively. Note that as the segments $\mathbf{a_1}$, $\mathbf{b_1}$, and $\mathbf{c_1}$ are contained in the hexagon, we have that $s_a > 0$, $s_b > 0$, and $s_c > 0$.

Proof. We proceed by induction as we assume that the proposition holds for any simple Eulerian triangulation with less vertices. The initial case of this induction, when $|V(T)| = 3$ clearly holds.

Given an Eulerian triangulation T with more vertices, we consider a TC-scheme $\mathcal{R}$ of T (given by Theorem 7), and by successively *resolving* degenerate points (i.e. intersection points of at least three segments) from left to right, we eventually reach the sought representation. Here resolving means that the segments of a 3-degenerate point (resp. a 6-degenerate point) are moved to form a triangle (resp. a polygon) inside which we are going to draw a 3-slopes representation of the graph corresponding to this degenerate face of $M(\mathcal{R})$, this is possible by using the induction on this smaller graph. The degenerate points of $\mathcal{R}$ are resolved from left to right. This means that at a given stage of this process there is a vertical line (parallel with $\overrightarrow{b}$) $\mathcal{V}$ such that on its left there is no intersection point of three or more segments. This implies that on the left of $\mathcal{V}$ the representation handles some small perturbations: one can slightly move the segments without changing the intersections.

Let $\mathcal{V}$ be the leftmost vertical line containing degenerate points. We resolve those degenerate points by slightly moving segments on the left of or on $\mathcal{V}$, while maintaining the right side of the representation unchanged. We consider different cases according to the degenerate points on $\mathcal{V}$.

If $\mathcal{V}$ contains a 3-degenerate point p in the interior of a (vertical) segment $\mathrm{b_j}$ and at the end of two segments $\mathrm{a_i}$ and $\mathrm{c_k}$ lying on the left of $\mathcal{V}$, the situation is rather simple. Move these segments a little to the left and slightly prolong them to intersect $\mathbf{b_j}$ (see Figure 8). As there is no degenerate point on the left of $\mathcal{V}$ these moves can be done while maintaining the existing intersections and avoiding new intersections. If $a_ib_jc_k$ is not a face of T, consider the triangulation T' induced by the vertices in the cycle $a_ib_jc_k$ of T. By induction T' has a representation that can be drawn inside the

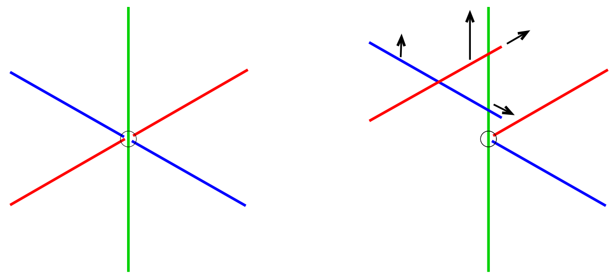

Figure 9: (left) A double 3-degenerate point on $\mathcal{V}$ (right) Small perturbation of $\mathcal{R}$.

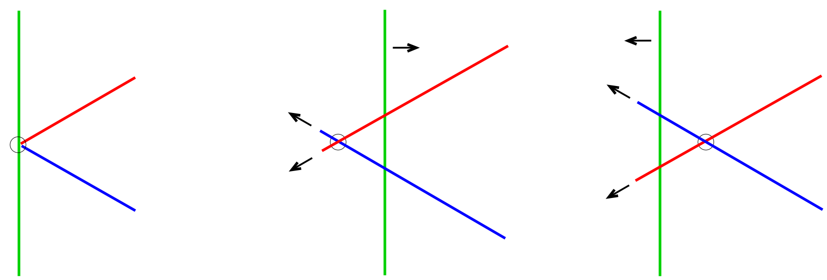

Figure 10: (left) A 3-degenerate point on $\mathcal{V}$ (middle) Slightly moving $\mathbf{b_j}$ to the right (right) Slightly moving $\mathbf{b_j}$ to the left.

newly formed triangle bordered by the segments $\mathbf{a_i}$, $\mathbf{b_j}$ and $\mathbf{c_k}$.

If $\mathcal{V}$ contains a double 3-degenerate point p in the interior of a (vertical) segment $\mathbf{b_j}$, the situation is similar to the previous one. Move the segments on the left of $\mathcal{V}$ as depicted in Figure 9. If the new triangle is not a face of T, we add a representation inside. We are now left with a simple 3-degenerate point at $\mathbf{p}$. This corresponds to the following case.

If $\mathcal{V}$ contains a 3-degenerate point p in the interior of a (vertical) segment $\mathrm{b_j}$ and at the end of two segments, $\mathrm{a_i}$ and $\mathrm{c_k}$, lying on the right of $\mathcal{V}$, one can move $\mathbf{b_j}$ slightly to the right or slightly to the left and resolve these points without changing the right part of the representation. The choice of moving $\mathbf{b_j}$ to the right or to the left is explained in the next paragraph, but we can assume this move to be arbitrarily small. Whatever the direction $\mathbf{b_j}$ is moved, one has to prolong $\mathbf{a_i}$ and $\mathbf{c_k}$ to have all the intersections, between these segments or with $\mathbf{b_j}$ (see Figure 10). Note that in order to preserve the representation on the right of $\mathcal{V}$ the segments $\mathbf{a_i}$ and $\mathbf{c_k}$ are not moved, they are only prolonged around $\mathbf{p}$. Again, if $a_i b_j c_k$ is not a face of T, we draw a representation inside the newly formed triangle. Note that if $\mathbf{b_j}$ moves to the right, the triangle bordered by $\mathbf{a_i}$, $\mathbf{b_j}$ and $\mathbf{c_k}$ has negative size, but it suffices to apply a homothety

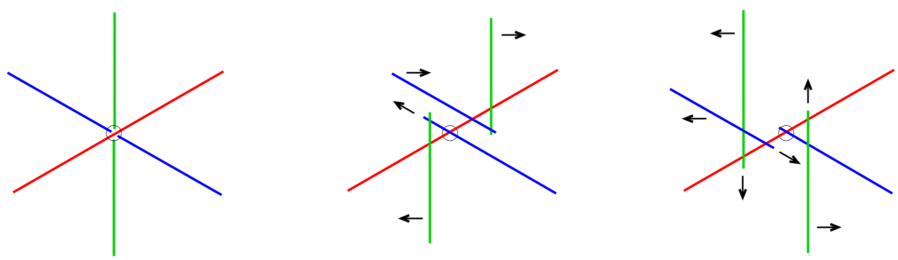

Figure 11: (left) A double 3-degenerate point on $\mathcal{V}$ (middle) & (right) Small moves that resolve this point.

with negative ratio to obtain a representation that can be drawn inside.

Consider now the degenerate points at the end of a (vertical) segment $\mathbf{b_j}$ of $\mathcal{V}$. Let $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_t}$ be a maximal sequence of segments on $\mathcal{V}$ such that $\mathbf{b_j}$ and $\mathbf{b_{j+1}}$ intersect on a point. We are going to move these segments alternatively to the right and to the left, for example the segments with even index are moved to the left while the ones with odd index are moved to the right. The exact magnitude of these moves will be set later, but first note that the 3-degenerate points in the interior of the segments $\mathbf{b_j}$ with $1 \leq j \leq t$ can be dealt if the move of $\mathbf{b_j}$ is sufficiently small (see previous cases). Consider the intersection point $\mathbf{p}$ between $\mathbf{b_j}$ and $\mathbf{b_{j+1}}$. The case of $\mathbf{b_1}$ and $\mathbf{b_t}$'s end is similar and it is not detailed here.

If there is a segment $\mathbf{a_i}$ going through $\mathbf{p}$. It is shown in Figure 11 how to resolve these two overlapped 3-degenerate points, in order to create two triangles, where one can add a small representation if needed. The case where there is a segment $\mathbf{c_k}$ going through $\mathbf{p}$ is similar.

Assume now that six segments intersect at $\mathbf{p}$. Let $\mathbf{b_j}$ be the one below $\mathbf{p}$, and let $\mathbf{a}$, $\mathbf{c}$, $\mathbf{b_{j+1}}$, $\mathbf{a'}$, and $\mathbf{c'}$ be the other ones around $\mathbf{p}$ clockwisely. Let us assume w.l.o.g. that $\mathbf{b_j}$ has to move to the left, while $\mathbf{b_{j+1}}$ has to move to the right. The degenerate face corresponding to $\mathbf{p}$ is bounded by these six vertices and there are several cases according to whether there are chords among them in T (see Figure 12).

If there is no chord inside the cycle $b_j a c b_{j+1} a' c'$ we consider the subgraph of T induced by the vertices on and inside this cycle, add we add the edges ab_{j+1}, $b_{j+1}c'$, and ac' outside the cycle, and we denote by T' the obtained simple Eulerian triangulation. By the induction we know that T' admits a 3-slope segment representation $\mathcal{R}'$, and let (s_a, s_b, s_c) be the shape of $\mathcal{R}'$. We resolve the point by moving the segments as depicted in Figure 12, and the magnitude of each of these moves is prescribed by the shape (s_a, s_b, s_c) in order to allow us to copy $\mathcal{R}'$ inside the triangle formed by $\mathbf{a}$, $\mathbf{b_{j+1}}$, and $\mathbf{c'}$. Then we shorten $\mathbf{a}$, $\mathbf{b_{j+1}}$, and $\mathbf{c'}$ to avoid intersections among them. Actually, the case where none of ab_{j+1}, $b_{j+1}c'$, or ac' is a chord is identical.

If none of $b_j c$, ca', or $a'b_j$ is a chord of $b_j a c b_{j+1} a' c'$ we proceed similarly. The only difference is that we add the edge $b_j c$, ca', or $a'b_j$ outside $b_j a c b_{j+1} a' c'$ to obtain T', and that we have to perform a homothety with negative ratio to include $\mathcal{R}'$.

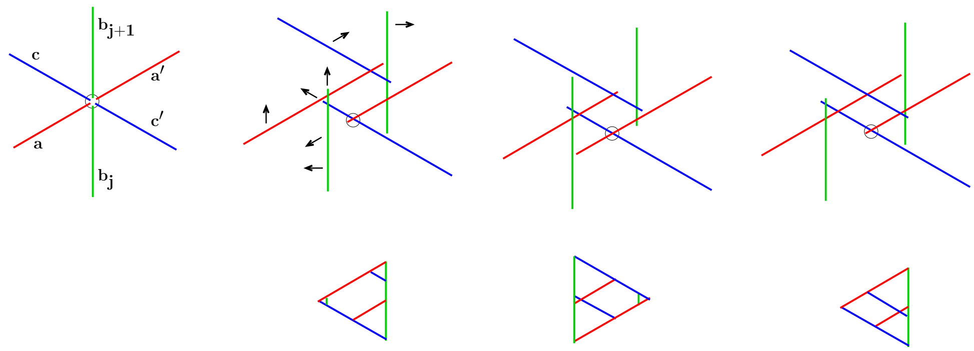

Figure 12: From left to right : A 6-degenerate point on $\mathcal{V}$. Resolution if there is no chord in $b_j a c b_{j+1} a' c'$ with the shape of $\mathcal{R}'$. Resolution if none of $b_j c$, ca', or $a' b_j$ is a chord, with the shape of $\mathcal{R}'$. Resolution if ac' and ca' are chords, with the shape of $\mathcal{R}_2$.

Finally, if there are two opposite chords on $b_j a c b_{j+1} a' c'$, say ac' and ca', we consider two triangulations. Let T_1 be the one inside the cycle $c' b_j a$ and let T_2 be the one obtained from the interior of the 5-cycle $a c b_{j+1} a' c'$ by adding the edges $a b_{j+1}$ and $b_{j+1} c'$. By the induction we know that T_1 and T_2 admit 3-slopes segment representations $\mathcal{R}_1$, and $\mathcal{R}_2$, and let (s_a, s_b, s_c) be the shape of $\mathcal{R}_2$. We resolve the point by moving the segments as depicted in Figure 12, and the magnitude of each of these moves, except for $\mathbf{b_j}$, is prescribed by the shape (s_a, s_b, s_c) in order to allow us to copy $\mathcal{R}_2$ inside the triangle formed by $\mathbf{a}$, $\mathbf{b_{j+1}}$, and $\mathbf{c}'$. Then we shorten $\mathbf{a}$, and $\mathbf{b_{j+1}}$ to avoid the intersections corresponding to $a b_{j+1}$ and $b_{j+1} c'$. The segment $\mathbf{b_j}$ is moved sufficiently to the left to avoid the interior of the triangle containing $\mathcal{R}_2$. Then $\mathcal{R}_1$ is drawn inside the triangle bordered by $\mathbf{b_j}$, $\mathbf{a}$ and $\mathbf{c}'$. This is possible because $\mathcal{R}_2$ does not intersect this triangle.

Finally note that the moves of $\mathbf{b_j}$ and $\mathbf{b_{j+1}}$ are opposite but of proportional magnitudes (up to some constant depending on the shapes (s_a, s_b, s_c) of $\mathcal{R}'$ or $\mathcal{R}_2$). So it is clear that we can simultaneously move all the segments $\mathbf{b_j}$ on $\mathcal{V}$. This concludes the proof of the lemma. □

5 Conclusion

Our result implies that for $k \leq 3$, planar graphs that are k-colorable admit a k-slopes segment representation, where parallel segment induce an independent set. These graphs have a so-called *PURE-k-DIR* representation. Unfortunately this does not extend to the final case $k = 4$ as conjectured by D. West [19]. Recently, the author [9] built a conter-example based on a construction of Kardoš and Narboni [12]. Their construction is an example of a signed planar graph that is not 4-colorable, in the sense of signed graphs, and it thus contradicts a conjecture of E. Máčajová, A. Raspaud, and M.

Škoviera [15].

However, it remains open to know whether (4-colorable) planar graphs admit a PURE-k-DIR representation (resp. a non-necessarily pure one) for some $k > 4$ (resp. $k > 2$).

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