On Vertex Partitions and some
Minor-Monotone Graph Parameters

D. Gonçalves

LIRMM UMR 5506, CNRS, Université Montpellier 2,
161 rue Ada, 34092 Montpellier Cedex 5, France

Abstract

We study vertex partitions of graphs according to some minor-monotone graph parameters. Ding et al. (DOS00) proved that for some of these parameters, we denote by \( P(G) \), any graph \( G \) with \( P(G) \geq k_P + 1 \) (\( k_P \) being a constant depending on \( P \)) admits a vertex partition into two graphs with parameter \( P \) at most \( P(G) - 1 \). Here we prove for some of these parameters \( P \), that any graph \( G \) with \( P(G) \geq k_P + 2 \) admits a vertex partition into three graphs with parameter \( P \) at most \( P(G) - 2 \).

Key words: minor-monotone parameters, vertex partition

1 Introduction

A graph parameter \( \rho \) is minor-monotone if for any minor \( H \) of any graph \( G \) we have \( \mu(H) \leq \mu(G) \). Let us define some minor-monotone parameters. The Hadwiger number \( \eta(G) \) of a graph \( G \) is the smallest integer \( t \) such that \( G \) is \( K_{t+1} \) minor-free. The well-known Hadwiger's Conjecture states that any graph \( G \) is \( \eta(G) \)-colorable. Since a \( k \)-coloring is a \( k \)-partition of the vertex set \( V(G) = V_1 \cup \ldots \cup V_k \) into stable subsets, this conjecture is an example of relation between vertex partitions and a graph parameter.

Let \( \pi \) denote another similarly defined minor-monotone parameter. Given a graph \( G \), \( \pi(G) \) be the smallest integer \( t \) such that \( G \) is \( K_{t+1} \) and \( K_{\lfloor \frac{22}{5} \rfloor}, \lfloor \frac{42}{5} \rfloor \) minor-free. Note that the graphs with \( \pi \) at most 2, 3 or 4 are respectively the forests, the outerplanar graphs and the planar graphs.

Email address: goncalves@lirmm.fr (D. Gonçalves).

Preprint submitted to Elsevier Science    February 27, 2008
In 1990, Y. Colin de Verdière (CdV90; CdV93) introduced an interesting minor-monotone parameter, \( \mu(G) \) (see (HLS99) for a survey on \( \mu \)). The parameter was motivated by the study of the maximum multiplicity of the second eigenvalue of certain Schrödinger operators. The parameter \( \mu(G) \) is described in terms of properties of matrices related to \( G \). Given a graph \( G \) with vertex set \( V(G) = \{1, \ldots, n\} \), \( \mu(G) \) is the largest corank of any real symmetric \( n \times n \) matrix \( M = (M_{i,j}) \) such that:

- for all \( i, j \) with \( i \neq j \): \( M_{i,j} < 0 \) if \( i \) and \( j \) are adjacent, and \( M_{i,j} = 0 \) if \( i \) and \( j \) are nonadjacent (there is no condition on the diagonal entries \( M_{i,i} \));
- \( M \) has exactly one negative eigenvalue, of multiplicity 1;
- there is no nonzero real symmetric \( n \times n \) matrix \( X = (X_{i,j}) \) such that \( MX = 0 \) and such that \( X_{i,j} = 0 \) whenever \( i = j \) or \( M_{i,j} \neq 0 \).

This parameter gives a characterization of well-known minor closed families of graphs. Indeed, the graphs with \( \mu \) at most 1, 2, 3 or 4 are respectively the forests of paths, the outerplanar graphs, the planar graphs, and the linkless embeddable graphs. A graph \( G \) is linkless embeddable if it has an embedding in the 3-dimensional space in such a way that for any two disjoint cycles of \( G \) there is a topological 2-sphere separating them. Y. Colin de Verdière proposed the following conjecture.

**Conjecture 1 (Colin de Verdière)** For any graph \( G \), \( \chi(G) \leq \mu(G) + 1 \).

Since \( \mu(G) + 1 \leq \eta(G) \) for any \( G \), this conjecture is a weaker version of Hadwiger's Conjecture.

The parameter \( \lambda(G) \) (HLS95) is the largest \( d \) for which there exists a \( d \)-dimensional subspace \( X \) of \( \mathbb{R}^{V(G)} \) such that:

\[
(* \quad \text{for each nonzero} \; x \in X, \; G[\text{supp}_+(x)] \text{ is a nonempty connected graph,}
\]

where \( \text{supp}_+(x) \) (the positive support of \( x \)) is the set \( \{u \in V \mid x(u) > 0\} \). Several conjectures on vertex partitions are very similar.

**Conjecture 2** For any parameter \( \rho \in \{\eta, \pi, \mu, \lambda\} \), any graph \( G \), and any integer \( k \in \{1, \ldots, \eta(G)\} \), the graph \( G \) has a vertex \( k \)-partition \( V(G) = V_1 \cup \ldots \cup V_k \), into \( k \) graphs \( G[V_i] \) such that \( \rho(G[V_i]) \leq \rho(G) + 1 - k \). for every \( i \in \{1, \ldots, k\} \).

The case \( \rho = \eta \) of the conjecture was proposed by Ding et al. (DOS00). Note that when \( k = \eta(G) \) it corresponds to Hadwiger's conjecture. The case \( \rho = \pi \) of the conjecture was proposed by Woodall in his survey (W90). Actually it is a "minor" reformulation of the so-called \( (m, n) \)-conjecture (CGH71) (which have been disproved (J89; HT94)). We propose the case \( \rho = \mu \) of the conjecture.
because it holds for small values of $\mu(G)$. Indeed:

- every outerplanar graph has a vertex partition into 2 forests of paths (M83; BM85; AEG89),
- every planar graph has a vertex partition into 2 outerplanar graphs (CGH71), and
- every planar graph has a vertex partition into 3 forests of paths (G91; P90).

We also propose the case $\rho = \lambda$ because it holds for some cases, the cases when $k$ is small. These cases are the purpose of this article. It is clear that Conjectures 2 holds for $k = 1$. Ding et al. (DOS00) proved a result that implies the conjecture for $k = 2$. Since this result uses other terminology and for completeness we prove a similar result in Section 3. In this section we also provide a result that implies the case $k = 3$. This implies for example that every linkless embeddable graph has a vertex partition into 3 outerplanar graphs. First let us focus on minor-monotone parameters.

2 Minor-monotone parameters

Let $G + v$ be the graph obtained from $G$ by adding a vertex $v$ adjacent to all the vertices of $G$. Let us define what is a convenient graph parameter.

**Definition 3** Given a graph parameter $\rho$ and an integer $k_\rho$, the couple $(\rho, k_\rho)$ is convenient if we have the following three properties:

1. Any minor $H$ of $G$ is such that $\rho(H) \leq \rho(G)$.
2. Any graph $G$ is such that $\rho(G) \leq \max\{\rho(G + v) - 1, k_\rho\}$.
3. For any pair of non-empty graphs $G_1$ and $G_2$, the disjoint union of $G_1$ and $G_2$, $G_1 \cup G_2$, is such that $\rho(G_1 \cup G_2) = \max\{\rho(G_1), \rho(G_2), k_\rho\}$.

Furthermore a convenient couple $(\rho, k_\rho)$ is minimum if $\rho(k_\rho - 1)$ is not convenient.

**Lemma 4** The couples $(\eta, 1)$, $(\pi, 1)$, $(\mu, 1)$ and $(\lambda, 1)$ are convenient and minimal.

**Proof.** By definition the graph parameters $\pi$ and $\eta$ are minor-monotone. Also by definition of $\pi$ and $\eta$, it is clear that $(\pi, 1)$ and $(\eta, 1)$ satisfy property (3). Finally, if $K_{t+1}$ is not a minor of $G$, then $K_{t+2}$ cannot be a minor of $G + v$. So, $(\eta, 1)$ is convenient. Similarly, if none of $K_{t+1}$ and $K_{t+2}$ is a minor of $G$, then none of $K_{t+2}$ and $K_{t+2}$ can be a minor of $G + v$. So, $(\pi, 1)$ is convenient. Here $k_{\pi} = k_{\eta} = 1$ because it is the
minimum possible value of $\pi(G)$ or $\eta(G)$. It is shown in (CdV90) that the couple $(\mu, 1)$ is convenient. Here $k_\mu = 1$ and not 0, because in property (2) we can have $\mu(G) = \mu(G + v) = 1$ for $G = K_2$. In (HLS95)(cf. Theorem 1 and 4) it is shown that $(\lambda, 1)$ satisfies property (1) and (3). For proving property (2), let $X \subseteq \mathbb{R}^V$ be a maximal subspace (of dimension $d$) that fulfills ($\ast$), and let $x_1, \ldots, x_d$ be $d$ vectors generating $X$. Now let the $d + 1$ dimensional subspace $X'$ of $\mathbb{R}^{V \cup \{v\}}$ be the subspace generated by $x_1, \ldots, x_d, x_{d+1}$ where

$$x_i(u) \quad \text{for} \quad 1 \leq i \leq d \quad \text{and} \quad u \in V$$

$$x'_i(u) = \begin{cases} 
0 & \text{for} \ 1 \leq i \leq d \quad \text{and} \quad u = v \\
 x_d(u) & \text{for} \ i = d + 1 \quad \text{and} \quad u \in V \\
1 & \text{for} \ i = d + 1 \quad \text{and} \quad u = v 
\end{cases}$$

The $(d + 1)$-dimensional subspace $X \times \mathbb{R}^{\{v\}}$ of $\mathbb{R}^{V \cup \{v\}}$ fulfills ($\ast$) for $G + v$. Indeed, consider any point $x' \in X \times \mathbb{R}^{\{v\}}$. If $x'(v) > 0$, since $v$ is adjacent to all the vertices of $G$, the graph $G[\text{supp}_+(x')]$ is connected. If $x'(v) \leq 0$, since the projection of $x'$ in $\mathbb{R}^V$ is a point $x \in X$, we have $\text{supp}_+(x') = \text{supp}_+(x)$, and so the graph $G[\text{supp}_+(x')]$ is nonempty and connected. So property (2) holds and $(\lambda, 1)$ is convenient.

Here $k_\lambda = 1$ and not 0, because in property (3) we have $\lambda(G_1 \cup G_2) = 1 > 0 = \max\{\lambda(G_1), \lambda(G_2)\}$ when $G_1$ and $G_2$ have only one vertex.

We could also mention that the treewidth is a convenient graph parameter but it is not relevant for the next section. Indeed, Ding et al. proved in (DOS98) that a graph of treewidth $k_1 + k_2 + 1$ admits a vertex partition into two graphs of treewidth at most $k_1$ and $k_2$. It is also possible that Conjecture 2 is true but not tight for some parameter $\rho$. We have been unable to construct, for any $k$ and $n$ such that $k \leq n$, a graph $G_{k,n}$ with $\rho(G_{k,n}) = n$ and such that in any vertex partition, one of the induced subgraphs has parameter at least $n + 1 - k$.

3 Vertex partitions

The following theorem is similar to the Theorem 4.2 in (DOS00) but uses other terminology.

**Theorem 5** Consider a convenient couple $(\rho, k_\rho)$. For any integer $k \geq k_\rho$, any graph $G$ with $\rho(G) \leq k + 1$, and any vertex $v_1 \in V(G)$, there is a vertex partition of $G$, $V(G) = V_1 \cup V_2$, such that:
(a) \( \rho(G[V_i]) \leq k, \) for all \( i \in \{1, 2\} \)
(b) \( v_1 \in V_1 \) and \( \deg_{G[V_1]}(v_1) = 0 \)

PROOF. Let \( G \) be a counter-example minimizing \( |V(G)| \). It is clear that \( G \) is a connected graph with at least two vertices. Let \( G' \) be the graph obtained by contracting all the edges incident to \( v_1 \) in \( G \). Denote \( v_2 \) the vertex of \( G' \) obtained from \( v_1 \) and its neighbors. Since \( G' \) is a minor of \( G \), by property (1), we have \( \rho(G') \leq \rho(G) \leq k + 1 \). Since \( |V(G')| < |V(G)| \), by minimality of \( |V(G)| \), there is a vertex partition of \( G' \), \( V(G') = V_1' \cup V_2' \), such that:

(a') \( \rho(G[V'_1]) \leq k \), for all \( i \in \{1, 2\} \).
(b') \( v_2 \in V_2' \) and \( \deg_{G[V'_2]}(v_2) = 0 \).

We extend this partition of \( G' \) to \( G \). Let the vertices of \( G' \setminus v_2 \) remain in the same subset of the partition \( (V'_1 \setminus v_2 \subseteq V_i) \). Put the vertex \( v_1 \) in \( V_1 \) and all its neighbors, \( N_G(v_1) \), in \( V_2 \). Point (b) clearly holds so focus on point (a).

Since \( G[V'_1] = G'[V'_1] \cup v_1 \) it is clear, by point (a') and property (3) of \( \rho \), that \( \rho(G[V'_1]) \leq k \). The graph induced by \( v_1 \) and \( N_G(v_1) \) is a minor of \( G \), so \( \rho(G[\{v_1\} \cup N_G(v_1)]) \leq k + 1 \), and by property (2) we have that \( \rho(G[N_G(v_1)]) \leq k \). Point (b') implies that there is no vertex in \( V_2 \) adjacent to a vertex of \( N_G(v_1) \). By property (3) we have \( \rho(G[V'_2]) \leq k \) and point (a) holds. So there is no counter-example \( G \) and the theorem holds.

Theorem 6 Consider a convenient couple \((\rho, k_\rho)\). For any integer \( k \geq k_\rho \), any graph \( G \) with \( \rho(G) \leq k + 2 \), and any edge \( v_1v_2 \in E(G) \), there is a vertex partition of \( G \), \( V(G) = V_1 \cup V_2 \cup V_3 \), such that:

(a) \( \rho(G[V_i]) \leq k - 2 \), for all \( i \in \{1, 2, 3\} \)
(b) \( v_1 \in V_1 \) and \( \deg_{G[V_1]}(v_1) = 0 \)
(c) \( v_2 \in V_2 \) and \( \deg_{G[V_2]}(v_2) = 0 \)

PROOF. Let \( G \) be a counter-example minimizing \( |V(G)| \).

Claim 7 The graph \( G \) is a 2-connected graph with at least three vertices.

If \( G \) is not 2-connected, let \( v \) be a separating vertex and let \( G_1 \) and \( G_2 \) be two non-empty graphs such that \( G = G_1 \cup G_2 \), \( V(G_1) \cap V(G_2) = \{v\} \) and \( v_1v_2 \in E(G_1) \). These graphs are minors of \( G \), so \( \rho(G_1) \) and \( \rho(G_2) \leq \rho(G) \leq k + 2 \). By minimality of \( |V(G)| \) we can consider a vertex partition of \( G_1 \) that fulfills points (a), (b) and (c). WLOG, we consider that \( v \in V_1 \). We apply now the induction hypothesis to \( G_2 \) with respect to any edge incident to \( v \). Since \( \deg_{G[V_1]}(v_1) = 0 \), it is clear that the union of these two 3-partitions is a 3-partition of \( V(G) \) that fulfills points (a), (b) and (c). So the counter-example \( G \) is 2-connected.
Let $u_1, \ldots, u_l$ and $v_1$ be the neighbors of $v_2$. Contract any edge incident to $v_1$ that is not $v_1v_2$ or an edge $v_1u_i$. Repeat this process until having only edges $v_1v_2$ or $v_1u_i$ incident to $v_1$. The graph obtained, $G'$, is a minor of $G$ and so $\rho(G') \leq \rho(G) \leq k + 2$. Consider that $u_1, \ldots, u_d$ (resp. $u_{d+1}, \ldots, u_l$) are the neighbors of $v_2$ that are (resp. are not) adjacent to $v_1$ in $G'$.

**Claim 8** $\rho(G'\{u_1, \ldots, u_d\}) \leq k$

Indeed, the induced graph $G'\{v_1, v_2, u_1, \ldots, u_d\}$ is a minor of $G$ and so $\rho(G'\{v_1, v_2, u_1, \ldots, u_d\}) \leq \rho(G) \leq k + 2$. Then, since $G'\{v_1, v_2, u_1, \ldots, u_d\} = (G'\{u_1, \ldots, u_d\} + v_1) + v_2$, the claim is implied by property (2).

Let $G_{2,3}$ be the graph obtained from $G'$ by contracting all the edges incident to $v_1$. Denote $v_3$ the vertex of $G_{2,3}$ obtained from $v_1$ and its neighbors. The graph $G_{2,3}$ is a minor of $G$, so we have $\rho(G_{2,3}) \leq \rho(G) \leq k + 2$. By minimality of $|V(G)|$, there is a vertex 3-partition of $G_{2,3}$ such that:

- $(a_{2,3})$ $\rho(G_{2,3}[V_i]) \leq k - 2$, for all $i \in \{1, 2, 3\}$
- $(b_{2,3})$ $v_3 \in V_3$ and $\deg_{G_{2,3}[V_3]}(v_3) = 0$
- $(c_{2,3})$ $v_2 \in V_2$ and $\deg_{G_{2,3}[V_2]}(v_2) = 0$

Let $G_{1,3}$ be the graph obtained from $G$ by contracting all the edges in $G' \setminus v_1$. Denote $v_3$ the vertex of $G_{1,3}$ obtained from $v_2$ and the other vertices of $G' \setminus v_1$. The graph $G_{1,3}$ is a minor of $G$, so we have $\rho(G_{1,3}) \leq \rho(G) \leq k + 2$. By minimality of $|V(G)|$, there is a vertex 3-partition of $G_{1,3}$ such that:

- $(a_{1,3})$ $\rho(G_{1,3}[V_i]) \leq k - 2$, for all $i \in \{1, 2, 3\}$
- $(b_{1,3})$ $v_1 \in V_1$ and $\deg_{G_{1,3}[V_1]}(v_1) = 0$
- $(c_{1,3})$ $v_3 \in V_3$ and $\deg_{G_{1,3}[V_3]}(v_3) = 0$

We consider the vertex 3-partition of $G$ induced by the vertex 3-partitions of $G_{2,3}$ and $G_{1,3}$. In this partition the vertices $u_1, \ldots, u_d$ are in $V_3$. It is clear that $(b_{1,3})$ (resp. $(c_{2,3})$) implies $(b)$ (resp. $(c)$). It is also clear that, since $\{u_1, \ldots, u_d\} \subseteq V_3$, none of the vertices in $G_{1,3}[V_1]$ (resp. $G_{1,3}[V_2]$) is adjacent to a vertex in $G_{2,3}[V_1]$ (resp. $G_{2,3}[V_2]$). So point (a) holds for $i = 1$ or 2. For $i = 3$, points $(b_{2,3})$ and $(c_{1,3})$ imply that $G[\{u_1, \ldots, u_d\}]$ is a connected component of $G[V_3]$. Finally Claim 8 points $(a_{1,3})$, $(a_{2,3})$ and property (3) imply point (a) for $i = 3$. So there is no counter-example $G$ and the theorem holds.

**4 Conclusion**

The proofs of these theorems is similar to the proofs of the facts that forests of paths and outerplanar graphs are respectively 2-colorable and 3-colorable.
Unfortunately, it seems difficult to use the proof of the 4 Color Theorem to find a proof for the next step, \( k = 4 \). To prove this case \( (k = 4) \) for Conjecture 9 we could use a different technique. Given two graphs \( H \) and \( G \), their lexicographic product, \( H \times_{\text{lex}} G \), is the graph with vertex set \( V(H) \times V(G) \) and such that \((u, v)(u', v')\) is an edge of \( H \times_{\text{lex}} G \) iff \( uu' \in E(H) \) or if \( u = u' \) and \( vv' \in E(G) \).

**Conjecture 9** For any graph \( X \) and any integer \( k \in [0 \ldots \mu(X)] \) there exist two graphs \( H \) and \( G \), with \( \mu(H) \leq k \) and \( \mu(G) \leq \mu(X) - k \), such that the graph \( X \) is a subgraph of \( H \times_{\text{lex}} G \).

Using the proofs in the previous section we see that Conjecture 9 holds for \( k < 3 \). Note that if for a given \( k \) Conjecture 1 holds for \( \mu(G) = k \) and if Conjecture 9 holds, then Conjecture 9 holds for \( k \). Since Conjecture 1 holds for \( \mu(G) \leq 4 \), we could prove Conjecture 9 for \( k = 3 \) (resp. \( k = 4 \)) by showing that Conjecture 9 holds for \( k = 3 \) (resp. \( k = 4 \)). Note that this scheme of proof would also work for Conjectures 9, 8, and 7.

**References**


