On Vertex Partitions and some Minor-Monotone Graph Parameters

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Abstract

We study vertex partitions of graphs according to some minor-monotone graph parameters. Ding et al. (DOS00) proved that for some of these parameters, we denote by $\mathcal{P}(G)$, any graph G with $\mathcal{P}(G) \geq k_{\mathcal{P}} + 1$ ($k_{\mathcal{P}}$ being a constant depending on $\mathcal{P}$) admits a vertex partition into two graphs with parameter $\mathcal{P}$ at most $\mathcal{P}(G) - 1$. Here we prove for some of these parameters $\mathcal{P}$, that any graph G with $\mathcal{P}(G) \geq k_{\mathcal{P}} + 2$ admits a vertex partition into three graphs with parameter $\mathcal{P}$ at most $\mathcal{P}(G) - 2$.

Key words: minor-monotone parameters, vertex partition

1 Introduction

A graph parameter ρ is *minor-monotone* if for any minor H of any graph G we have $\mu(H) \leq \mu(G)$). Let us define some minor-monotone parameters. The Hadwiger number $\eta(G)$ of a graph G is the smallest integer t such that G is K_{t+1} minor-free. The well-known Hadwiger's Conjecture states that any graph G is $\eta(G)$-colorable. Since a k-coloring is a k-partition of tha vertex set $V(G) = V_1 \cup \ldots \cup V_k$ into stable subsets, this conjecture is an example of relation between vertex partitions and a graph parameter.

Let π denote another similarly defined minor-monotone parameter. Given a graph G, $\pi(G)$ be the smallest integer t such that G is K_{t+1} and $K_{\lceil \frac{t+2}{2} \rceil, \lfloor \frac{t+2}{2} \rfloor}$ minor-free. Note that the graphs with π at most 2, 3 or 4 are respectively the forests, the outerplanar graphs and the planar graphs.

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In 1990, Y. Colin de Verdière (CdV90; CdV93) introduced an interesting minor-monotone parameter, $\mu(G)$ (see (HLS99) for a survey on μ). The parameter was motivated by the study of the maximum multiplicity of the second eigenvalue of certain Schrödinger operators. The parameter $\mu(G)$ is described in terms of properties of matrices related to G. Given a graph G with vertex set $V(G) = \{1, \ldots, n\}$, $\mu(G)$ is the largest corank of any real symmetric $n \times n$ matrix $M = (M_{i,j})$ such that :

- for all i, j with $i \neq j$: $M_{i,j} < 0$ if i and j are adjacent, and $M_{i,j} = 0$ if i and j are nonadjacent (there is no condition on the diagonal entries $M_{i,i}$);
- M has exactely one negative eigenvalue, of multiplicity 1;
- there is no nonzero real symmetric $n \times n$ matrix $X = (X_{i,j})$ such that $MX = 0$ and such that $X_{i,j} = 0$ whenever $i = j$ or $M_{i,j} \neq 0$.

This parameter gives a characterization of well-known minor closed families of graphs. Indeed, the graphs with μ at most 1, 2, 3 or 4 are respectively the forests of paths, the outerplanar graphs, the planar graphs, and the linkless embeddable graphs. A graph G is linkless embeddable if it has an embedding in the 3-dimensional space in such a way that for any two disjoint cycles of G there is a topological 2-sphere separating them. Y. Colin de Verdière proposed the following conjecture.

Conjecture 1 (Colin de Verdière) *For any graph G, $\chi(G) \leq \mu(G) + 1$.*

Since $\mu(G) + 1 \leq \eta(G)$ for any G, this conjecture is a weaker version of Hadwiger's Conjecture.

The parameter $\lambda(G)$ (HLS95) is the largest d for which there exists a d-dimensional subspace X of $\mathbb{R}^{V(G)}$ such that:

(*) for each nonzero $x \in X$, $G[\text{supp}_+(x)]$ is a nonempty connected graph,

where $\text{supp}_+(x)$ (the positive support of x) is the set $\{u \in V \mid x(u) > 0\}$. Several conjectures on vertex partitions are very similar.

Conjecture 2 *For any parameter $\rho \in \{\eta, \pi, \mu, \lambda\}$, any graph G, and any integer $k \in \{1, \ldots, \eta(G)\}$, the graph G has a vertex k-partition $V(G) = V_1 \cup \ldots \cup V_k$, into k graphs $G[V_i]$ such that $\rho(G[V_i]) \leq \rho(G) + 1 - k$. for every $i \in \{1, \ldots, k\}$.*

The case $\rho = \eta$ of the conjecture was proposed by Ding et al. (DOS00). Note that when $k = \eta(G)$ it corresponds to Hadwiger's conjecture. The case $\rho = \pi$ of the conjecture was proposed by Woodall in his survey (W90). Actually it is a "minor" reformulation of the so-called (m, n)-conjecture (CGH71) (which have been disproved (J89; HT94)). We propose the case $\rho = \mu$ of the conjecture because it holds for small values of $\mu(G)$. Indeed:

- every outerplanar graph has a vertex partition into 2 forests of paths (M83; BM85; AEG89),
- every planar graph has a vertex partition into 2 outerplanar graphs (CGH71), and
- every planar graph has a vertex partition into 3 forests of paths (G91; P90).

We also propose the case $\rho = \lambda$ because it holds for some cases, the cases when k is small. These cases are the purpose of this article. It is clear that Conjectures 2 holds for $k = 1$. Ding et al. (DOS00) proved a result that implies the conjecture for $k = 2$. Since this result uses other terminology and for completeness we prove a similar result in Section 3. In this section we also provide a result that implies the case $k = 3$. This implies for example that every linkless embeddable graph has a vertex partition into 3 outerplanar graphs. First let us focus on minor-monotone parameters.

2 Minor-monotone parameters

Let $G + v$ be the graph obtained from G by adding a vertex v adjacent to all the vertices of G. Let us define what is a convenient graph parameter.

Definition 3 *Given a graph parameter ρ and an integer k_ρ, the couple (ρ, k_ρ) is* convenient *if we have the following three properties :*

(1) Any minor H of G is such that $\rho(H) \leq \rho(G)$.
(2) Any graph G is such that $\rho(G) \leq \max\{\rho(G + v) - 1, k_\rho\}$.
(3) For any paire of non-empty graphs G_1 and G_2, the disjoint union of G_1 and G_2, $G_1 \cup G_2$, is such that $\rho(G_1 \cup G_2) = \max\{\rho(G_1), \rho(G_2), k_\rho\}$.

Furthermore a convenient couple (ρ, k_ρ) is minimum *if $(\rho, k_\rho - 1)$ is not convenient.*

Lemma 4 *The couples $(\eta, 1)$, $(\pi, 1)$, $(\mu, 1)$ and $(\lambda, 1)$ are convenient and minimal.*

PROOF. By definition the graph parameters π and η are minor-monotone. Also by definition of π and η, it is clear that $(\pi, 1)$ and $(\eta, 1)$ satisfy property (3). Finally, if K_{t+1} is not a minor of G, then K_{t+2} cannot be a minor of $G+v$. So, $(\eta, 1)$ is convenient. Similarly, if none of K_{t+1} and $K_{\lceil \frac{t+2}{2} \rceil, \lfloor \frac{t+2}{2} \rfloor}$ is a minor of G, then none of K_{t+2} and $K_{\lceil \frac{t+2}{2} \rceil, \lfloor \frac{t+2}{2} \rfloor + 1} = K_{\lfloor \frac{(t+1)+2}{2} \rfloor, \lceil \frac{(t+1)+2}{2} \rceil}$ can be a minor of $G + v$. So, $(\pi, 1)$ is convenient. Here $k_\pi = k_\eta = 1$ because it is the minimum possible value of $\pi(G)$ or $\eta(G)$. It is shown in (CdV90) that the couple $(\mu, 1)$ is convenient. Here $k_\mu = 1$ and not 0, because in property (2) we can have $\mu(G) = \mu(G+v) = 1$ for $G = \overline{K_2}$. In (HLS95)(c.f. Theorem 1 and 4) it is shown that $(\lambda, 1)$ satisfies property (1) and (3). For proving property (2), let $X \subseteq \mathbb{R}^V$ be a maximal subspace (of dimension d) that fulfills $(*)$, and let $x_1, \ldots, x_d$ be d vectors generating X. Now let the $d+1$ dimensional subspace X' of $\mathbb{R}^{V \cup \{v\}}$ be the subspace generated by $x'_1, \ldots, x'_d, x'_{d+1}$ where

$$x'_i(u) = \{ \begin{array}{ll} x_i(u) & \text{for } 1 \leq i \leq d \text{ and } u \in V \\ 0 & \text{for } 1 \leq i \leq d \text{ and } u = v \\ x_d(u) & \text{for } i = d+1 \text{ and } u \in V \\ 1 & \text{for } i = d+1 \text{ and } u = v \end{array}$$

The $(d+1)$-dimensional subspace $X \times \mathbb{R}^{\{v\}}$ of $\mathbb{R}^{V \cup \{v\}}$ fulfills $(*)$ for $G + v$. Indeed, consider any point $x' \in X \times \mathbb{R}^{\{v\}}$. If $x'(v) > 0$, since v is adjacent to all the vertices of G, the graph $G[\text{supp}_+(x')]$ is connected. If $x'(v) \leq 0$, since the projection of x' in $\mathbb{R}^V$ is a point $x \in X$, we have $\text{supp}_+(x') = \text{supp}_+(x)$, and so the graph $G[\text{supp}_+(x')]$ is nonempty and connected. So property (2) holds and $(\lambda, 1)$ is convenient.

Here $k_\lambda = 1$ and not 0, because in property (3) we have $\lambda(G_1 \cup G_2) = 1 > 0 = \max\{\lambda(G_1), \lambda(G_2)\}$ when G_1 and G_2 have only one vertex.

We could also mention that the treewidth is a convenient graph parameter but it is not relevant for the next section. Indeed, Ding et al. proved in (DOS98) that a graph of treewidth $k_1 + k_2 + 1$ admits a vertex partition into two graphs of treewidth at most k_1 and k_2. It is also possible that Conjecture 2 is true but not tight for some parameter ρ. We have been unable to construct, for any k and n such that $k \leq n$, a graph $G_{k,n}$ with $\rho(G_{k,n}) = n$ and such that in any vertex partition, one of the induced subgraphs has parameter at least $n + 1 - k$.

3 Vertex partitions

The following theorem is similar to the Theorem 4.2 in (DOS00) but uses other terminology.

Theorem 5 *Consider a convenient couple (ρ, k_ρ). For any integer $k \geq k_\rho$, any graph G with $\rho(G) \leq k+1$, and any vertex $v_1 \in V(G)$, there is a vertex partition of G, $V(G) = V_1 \cup V_2$, such that:*

(a) $\rho(G[V_i]) \leq k$, *for all* $i \in \{1, 2\}$
(b) $v_1 \in V_1$ *and* $\deg_{G[V_1]}(v_1) = 0$

PROOF. Let G be a counter-example minimizing $|V(G)|$. It is clear that G is a connected graph with at least two vertices. Let G' be the graph obtained by contracting all the edges incident to v_1 in G. Denote v_2 the vertex of G' obtained from v_1 and its neighbors. Since G' is a minor of G, by property (1), we have $\rho(G') \leq \rho(G) \leq k + 1$. Since $|V(G')| < |V(G)|$, by minimality of $|V(G)|$, there is a vertex partition of G', $V(G') = V'_1 \cup V'_2$, such that :

(a') $\rho(G[V'_i]) \leq k$, for all $i \in \{1, 2\}$.
(b') $v_2 \in V'_2$ and $\deg_{G[V'_2]}(v_2) = 0$.

We extend this partition of G' to G. Let the vertices of $G' \setminus v_2$ remain in the same subset of the partition ($V'_i \setminus v_2 \subseteq V_i$). Put the vertex v_1 in V_1 and all its neighbors, $N_G(v_1)$, in V_2. Point (b) clearly holds so focus on point (a). Since $G[V_1] = G'[V'_1] \cup v_1$ it is clear, by point (a') and property (3) of ρ, that $\rho(G[V_1]) \leq k$. The graph induced by v_1 and $N_G(v_1)$ is a minor of G, so $\rho(G[\{v_1\} \cup N_G(v_1)]) \leq k+1$, and by property (2) we have that $\rho(G[N_G(v_1)]) \leq k$. Point (b') implies that there is no vertex in V'_2 adjacent to a vertex of $N_G(v_1)$. By property (3) we have $\rho(G[V_2]) \leq k$ and point (a) holds. So there is no counter-example G and the theorem holds.

Theorem 6 *Consider a convenient couple* (ρ, k_ρ)*. For any integer* $k \geq k_\rho$*, any graph* G *with* $\rho(G) \leq k + 2$*, and any edge* $v_1v_2 \in E(G)$*, there is a vertex partition of* G*,* $V(G) = V_1 \cup V_2 \cup V_3$*, such that:*

(a) $\rho(G[V_i]) \leq k - 2$, *for all* $i \in \{1, 2, 3\}$
(b) $v_1 \in V_1$ *and* $\deg_{G[V_1]}(v_1) = 0$
(c) $v_2 \in V_2$ *and* $\deg_{G[V_2]}(v_2) = 0$

PROOF. Let G be a counter-example minimizing $|V(G)|$.

Claim 7 *The graph* G *is a 2-connected graph with at least three vertices.*

If G is not 2-connected, let v be a separating vertex and let G_1 and G_2 be two non-empty graphs such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{v\}$ and $v_1v_2 \in E(G_1)$. These graphs are minors of G, so $\rho(G_1)$ and $\rho(G_2) \leq \rho(G) \leq k + 2$. By minimality of $|V(G)|$ we can consider a vertex partition of G_1 that fulfills points (a), (b) and (c). W.l.o.g. we consider that $v \in V_1$. We apply now the induction hypothesis to G_2 with respect to any edge incident to v. Since $\deg_{G_2[V_1]}(v_1) = 0$, it is clear that the union of these two 3-partitions is a 3-partition of $V(G)$ that fulfills points (a), (b) and (c). So the counter-example G is 2-connected.

Let $u_1, \ldots, u_t$ and v_1 be the neighbors of v_2. Contract any edge incident to v_1 that is not v_1v_2 or an edge v_1u_i. Repeat this process until having only edges v_1v_2 or v_1u_i incident to v_1. The graph obtained, G', is a minor of G and so $\rho(G') \leq \rho(G) \leq k+2$. Consider that $u_1, \ldots, u_d$ (resp. $u_{d+1}, \ldots, u_t$) are the neighbors of v_2 that are (resp. are not) adjacent to v_1 in G'.

Claim 8 $\rho(G'[\{u_1, \ldots, u_d\}]) \leq k$

Indeed, the induced graph $G'[\{v_1, v_2, u_1, \ldots, u_d\}]$ is a minor of G and so $\rho(G'[\{v_1, v_2, u_1, \ldots, u_d\}]) \leq \rho(G) \leq k+2$. Then, since $G'[\{v_1, v_2, u_1, \ldots, u_d\}] = ((G'[\{u_1, \ldots, u_d\}] + v_1) + v_2)$, the claim is implied by property (2).

Let $G_{2,3}$ be the graph obtained from G' by contracting all the edges incident to v_1. Denote v_3 the vertex of $G_{2,3}$ obtained from v_1 and its neighbors. The graph $G_{2,3}$ is a minor of G, so we have $\rho(G_{2,3}) \leq \rho(G) \leq k+2$. By minimality of $|V(G)|$, there is a vertex 3-partition of $G_{2,3}$ such that:

($a_{2,3}$) $\rho(G_{2,3}[V_i]) \leq k-2$, for all $i \in \{1, 2, 3\}$
($b_{2,3}$) $v_3 \in V_3$ and $\deg_{G_{2,3}[V_3]}(v_3) = 0$
($c_{2,3}$) $v_2 \in V_2$ and $\deg_{G_{2,3}[V_2]}(v_2) = 0$

Let $G_{1,3}$ be the graph obtained from G by contracting all the edges in $G' \setminus v_1$. Denote v_3 the vertex of $G_{1,3}$ obtained from v_2 and the other vertices of $G' \setminus v_1$. The graph $G_{1,3}$ is a minor of G, so we have $\rho(G_{1,3}) \leq \rho(G) \leq k+2$. By minimality of $|V(G)|$, there is a vertex 3-partition of $G_{1,3}$ such that:

($a_{1,3}$) $\rho(G_{1,3}[V_i]) \leq k-2$, for all $i \in \{1, 2, 3\}$
($b_{1,3}$) $v_1 \in V_1$ and $\deg_{G_{1,3}[V_1]}(v_1) = 0$
($c_{1,3}$) $v_3 \in V_3$ and $\deg_{G_{1,3}[V_3]}(v_3) = 0$

We consider the vertex 3-partition of G induced by the vertex 3-partitions of $G_{2,3}$ and $G_{1,3}$. In this partition the vertices $u_1, \ldots, u_d$ are in V_3. It is clear that ($b_{1,3}$) (resp. ($c_{2,3}$)) implies (b) (resp. (c)). It is also clear that, since $\{u_1, \ldots, u_d\} \subseteq V_3$, none of the vertices in $G_{1,3}[V_1]$ (resp. $G_{1,3}[V_2]$) is adjacent to a vertex in $G_{2,3}[V_1]$ (resp. $G_{2,3}[V_2]$). So point (a) holds for $i = 1$ or 2. For $i = 3$, points ($b_{2,3}$) and ($c_{1,3}$) imply that $G[\{u_1, \ldots, u_d\}]$ is a connected component of $G[V_3]$. Finally Claim 8, points ($a_{1,3}$), ($a_{2,3}$) and property (3) imply point (a) for $i = 3$. So there is no counter-example G and the theorem holds.

4 Conclusion

The proofs of these theorems is similar to the proofs of the facts that forests of paths and outerplanar graphs are respectively 2-colorable and 3-colorable.

Unfortunately, it seems difficult to use the proof of the 4 Color Theorem to find a proof for the next step, $k = 4$. To prove this case ($k = 4$) for Conjecture **??** we could use a different technique. Given two graphs H and G, their lexicographic product, $H \times_{lex} G$, is the graph with vertex set $V(H) \times V(G)$ and such that $(u, v)(u', v')$ is an edge of $H \times_{lex} G$ iff $uu' \in E(H)$ or if $u = u'$ and $vv' \in E(G)$.

Conjecture 9 *For any graph X and any integer $k \in [0 \dots \mu(X)]$ there exist two graphs H and G, with $\mu(H) \leq k$ and $\mu(G) \leq \mu(X) - k$, such that the graph X is a subgraph of $H \times_{lex} G$.*

Using the proofs in the previous section we see that Conjecture 9 holds for $k < 3$. Note that if for a given k Conjecture 1 holds for $\mu(G) = k$ and if Conjecture 9 holds, then Conjecture **??** holds for k. Since Conjecture 1 holds for $\mu(G) \leq 4$, we could prove Conjecture **??** for $k = 3$ (resp. $k = 4$) by showing that Conjecture 9 holds for $k = 3$ (resp. $k = 4$). Note that this scheme of proof would also work for Conjectures **??**, **??** and **??**.

References

[AEG89] J. Akiyama, H. Era, S.V. Gervacio and M. Watanabe, Path chromatic numbers of graphs, *J. Graph Theory* **13** (1989), 569–575.

[BM85] I. Broere, C.M. Mynhardt, Generalized colorings of outerplanar and planar graphs, *In: Y. Alavi, et al. Graph Theory with Applications to Algorithms and Computer Science* New York: Wiley Intersci. Publ., John Wiley & Sons, Inc. (1985), 151–161.

[CGH71] G. Chartrand, D. Geller and S. Hedetniemi, Graphs with forbidden subgraphs, *J. Combinatorial Theory Ser. B* **10** (1971), 12–41.

[CdV90] Y. Colin de Verdière, Sur un nouvel invariant des graphes et un critère de planarité, *J. Combin. Theory Ser. B* **50** (1990) 1, 11–21.

[CdV93] Y. Colin de Verdière, On a new graph invariant and a criterion for planarity, *Graph structure theory (Seattle, WA, 1991) Contemp. Math.* **147** (1993), 137–147.

[DOS98] G. Ding, B. Oporowski, D.P. Sanders and D. Vertigan, Partitioning graphs of bounded tree-width, *Combinatorica*, **18** (1998) 1, 1–12.

[DOS00] G. Ding, B. Oporowski, D.P. Sanders and D. Vertigan, Surfaces, tree-width, clique-minors, and partitions, *J. Combin. Theory Ser. B*, **79** (2000) 2, 221–246.

[G91] W. Goddard, Acyclic colorings of planar graphs, *Discrete Math.* **91** (1991), 91–94.

[HT94] D. Hanson and B. Toft, The (m, n)-conjecture is false, *Bull. Inst. Combin. Appl.* **11** (1994), 59–66.

[HLS95] H. van der Holst, M. Laurent and A. Schrijver, On a Minor-Monotone Graph Invariant, *J. Combin. Theory Ser. B* **65** (1995) 2, 291–304.

[HLS99] H. van der Holst, L. Lovász and A. Schrijver, The Colin de Verdière graph parameter, *Graph theory and combinatorial biology (Balatonlelle, 1996) Bolyai Soc. Math. Stud.* **7** (1999), 29–85.

[J89] L. Jørgensen, Some probabilistic and extremal results on subdivisions and odd subdivisions of graphs, *J. Graph Theory* **13** (1989) 1, 75–85.

[M83] P. Mihók, On vertex partition numbers of graphs, *In: M. Fiedler: Graphs and Other Combinatorial Topics* Leipzig: Teubner (1983), 183–188.

[P90] K. S. Poh, On the linear vertex-arboricity of a planar graph, *J. Graph Theory* **14** (1990) 1, 73–75.

[W90] D. Woodall Improper colourings of graphs *Graph colourings (Milton Keynes, 1988) Pitman Res. Notes Math. Ser.* **218** (1990), 45–63.