Covering planar graphs with degree bounded forests

D. Gonçalves

LaBRI, U.M.R. 5800, Université Bordeaux 1,
351 cours de la Libération 33405 Talence Cedex, France.

Abstract

We prove that every planar graphs has an edge partition into three forests, one having maximum degree 4. This answers a conjecture of Balogh *et al.* (J. Combin. Theory B. 94 (2005) 147-158).We also prove that every planar graphs with girth $g \geq 6$ (resp. $g \geq 7$) has an edge partition into two forests, one having maximum degree 4 (resp. 2).

Key words: planar graphs, edge partition, forests, trees

1 Introduction

A graph G is *covered* by subgraphs $G_1, \ldots, G_k$ of G if every edge of G belongs to one of these subgraphs. A graph G is (t, D)*-coverable* if it can be covered by t forests and a graph H of maximum degree $\Delta(H) \leq D$. A graph is $F(d_1, \ldots, d_k)$*-coverable* if it can be covered by k forests $F_1, \ldots, F_k$ such that $\Delta(F_i) \leq d_i$ for all $1 \leq i \leq k$. If $d_i = \infty$ the maximum degree of F_i is unbounded. By a result of Nash-Williams [8], we know that planar graphs are $(3, 0)$-coverable (*i.e.* $F(\infty, \infty, \infty)$-coverable) and that planar graphs of girth $g \geq 4$ are $(2, 0)$-coverable (*i.e.* $F(\infty, \infty)$-coverable). In [6], the authors proved that planar graphs are $(2, 8)$-coverable. The authors also asked what could be the minimal d such that every planar graph is $(2, d)$-coverable. In [2], the authors proved that planar graphs are more than $(2, 8)$-coverable, they are $F(\infty, \infty, 8)$-coverable. They also proved that there exist non-$(2, 3)$-coverable planar graphs and they conjectured that planar graphs are $(2, 4)$-coverable. Our main result is slightly stronger than this conjecture.

Email address: goncalve@labri.fr (D. Gonçalves).

Preprint submitted to Journal of Combinatorial Theory, Series B

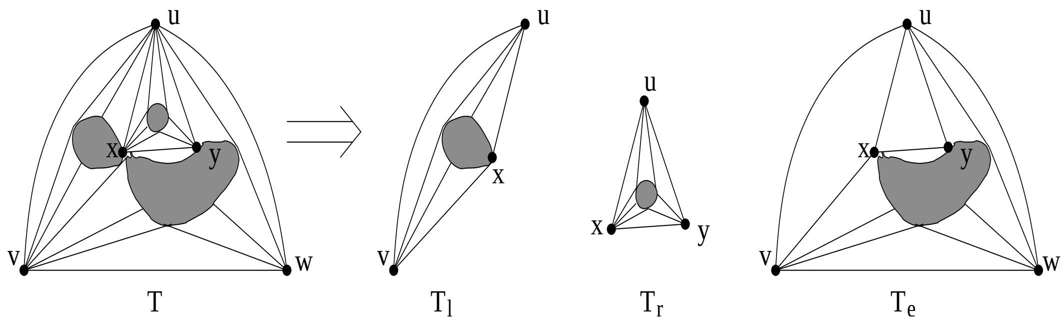

Fig. 1. From T to T_e, T_l and T_r.

Theorem 1 *Planar graphs are $F(\infty,\infty,4)$-coverable.*

The case of bounded girth planar graphs has also been studied. It is proven in [6] that planar graphs with girth at least 5 (resp. 7) are $(1,4)$-coverable (resp.$(1,2)$-coverable). In [1], the authors proved that planar graphs with girth at least 10 are $(1,1)$-coverable (*i.e.* $F(\infty,1)$-coverable). Here we have some results on forest coverings of planar graphs of girth at least 6 or 7.

Theorem 2 *Planar graphs of girth $g \geq 6$ are $F(\infty,4)$-coverable.*

Theorem 3 *Planar graphs of girth $g \geq 7$ are $F(\infty,2)$-coverable.*

2 Planar graphs

A *triangulation* is a planar graph in which every face is triangular. In [4] the author proved that planar graphs are coverable by four forests of caterpillars. His proof works by induction using a decomposition of triangulations into three smaller triangulations. We prove Theorem 1 using the same decomposition tool.

Consider an embedded triangulation T with at least four vertices and such that going counter-clockwise on the external face we successively meet the vertices u, v and w. For a couple (u,v) of these vertices we define its *partner couple* (x,y) of vertices. In a triangulation with at least four vertices, any edge ab is such that its ends, a and b, have at least two common neighbors. We consider the sequence of u's neighbors going in the clockwise sense from w to v. Let x be the second of these vertices being a neighbor of v (the first one being w, $x \neq w$). Note that every common neighbor of u and v other than x or w is inside the cycle (u,v,x). Then, let y be the first vertex of the sequence that is a neighbor of x. Since u and x have at least two common neighbors, one of these vertices appears before v in the sequence, so $y \neq v$. On the other hand note that the vertex y may be equal to w. Note that partner couples are defined for every triangulation $T \neq K_3$.

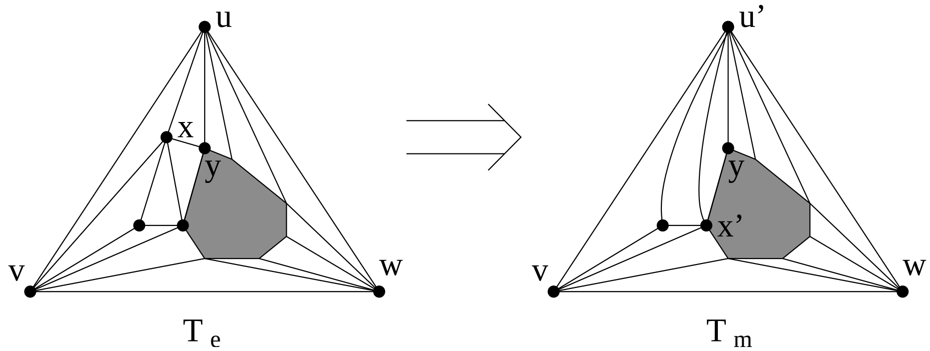

Fig. 2. From T_e to T_m.

Let T_l (resp. T_r) be the triangulation induced by the vertices on and inside the cycle (u, v, x) (resp. (u, x, y)). Then let T_e be the triangulation induced by the vertices on and outside the cycle (u, v, x, y) (see Figure 1). Since $v \notin V(T_r)$ and $w \notin V(T_l)$, T_r and T_l have less vertices than T. This is not the case for T_e if (u, v, x) and (u, x, y) both delimit an inner-face of T. In T_e, the vertices u and v (resp. u and x) have only two common neighbors, x and w (resp. u and y). So in T_e, the partner couple of (u, v) is still (x, y).

We construct T_m from T_e by deleting three edges, vx, ux and yx, and then merging u and v in a single vertex u' (see Figure 2). Since u and x have only two common neighbors v and y in T_e, T_m is a well defined triangulation, without loop or multiple edges. Since we merged two vertices, T_m has less vertices than T_e. If $T_m \neq K_3$, let (x', y') be the partner couple of (u', v) in T_m. Note that since u and v have exactly two neighbors in T_e, x and w, the vertex x' is adjacent to x and not adjacent to u in T_e. Using this decomposition, we prove the following theorem illustrated in Figure 3.

Theorem 4 *Given any triangulation $T = (V, E)$ and any triplet (u, v, w) of vertices on the external face, let (x, y) be the partner couple of (u, v). The graph $T' = T \backslash \{uv, uw, vw\}$ has an $F(\infty, \infty, 4)$-covering by F_1, F_2 and F_3. If x is defined, this is if $T \neq K_3$ these forests are such that :*

- *the edges of T' incident to v are in F_1,*
- *the edges of T' incident to w are in F_2,*
- *the edge ux is in F_3,*
- *the edges of T' incident to u strictly between ux and uw are in F_1,*
- *the edges of T' incident to u strictly between uv and ux are in F_2, and*
- *the vertices u, v and w are in distinct connected components of F_i, for $1 \leq i \leq 3$. Furthermore, the connected component of F_2 containing the vertex u only contains u and some vertices inside the cycle (u, v, x).*

Note that each of these forests has exactly 3 connected components. Indeed, an acyclic graph on n vertices with c connected components has $n - c$ edges and the graph T' has $3n - 9$ edges. For example the forest F_2 has 2 connected

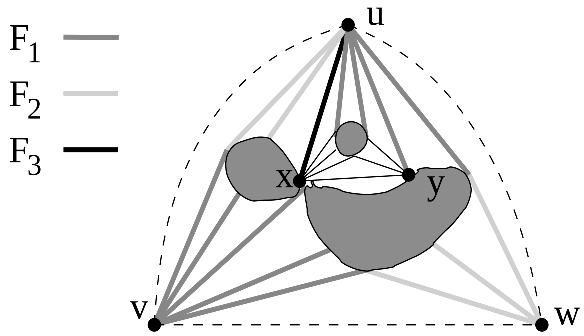

Fig. 3. The edge partition of Theorem 1.

components of one vertex each, respectively u and v, and a third connected component containing all the remaining vertices. We can extend this edge-partition of T' to T by putting for example the edges uv and uw in F_1 and the edge vw in F_2. This partition clearly implies Theorem 1.

PROOF. This proof works by induction on $|V(T)|$. The theorem clearly holds for K_3, so we consider the induction step of the proof. Given a triangulation T with $|V(T)| \geq 4$, consider the three triangulations T_m, T_l and T_r obtained by the decomposition of T described before. Since T_m, T_l and T_r have less vertices than T, we can use the induction hypothesis. Let F_1', F_2' and F_3' be the three forests given by the theorem for the triangulation T_m and the triplet (u', v, w). These forests cover $T_m' = T_m \setminus \{u'v, u'w, vw\}$ and we use them to define the graphs F_1, F_2 and F_3 that cover $T_e' = T_e \setminus \{uv, uw, vw\}$.

- $F_i' \setminus \{u, x\} \subset F_i$
- $ua \in F_i$ if $a \neq x$ and $u'a \in F_i'$
- $xa \in F_i$ if $a \neq u$, v or y, and $u'a \in F_i'$
- $vx \in F_1$
- $yx \in F_2$
- $ux \in F_3$

These forests verify the conditions of the theorem for the triangulation T_e and the triplet (u, v, w).

- The edges incident to u ,v or w are clearly well partitionned.
- The graph F_1 is a forest. If F_1 had a cycle, this cycle should either pass through vx or not. In the first case, this would imply that there is a path from u' to v in $F_1' \subset T_m'$. In the second case, this would imply that there is a cycle in $F_1' \subset T_m'$. Both cases are impossible since the partition of T_m' verifies the theorem.

  Similarly the vertices u ,v and w are in distinct connected components of F_1. If there was a path in F_1 linking two of these vertices this path should either pass through vx or not. In the first case, this would imply that there is either a path from u' to w or a cycle (passing through u') in $F_1' \subset T_m'$. In

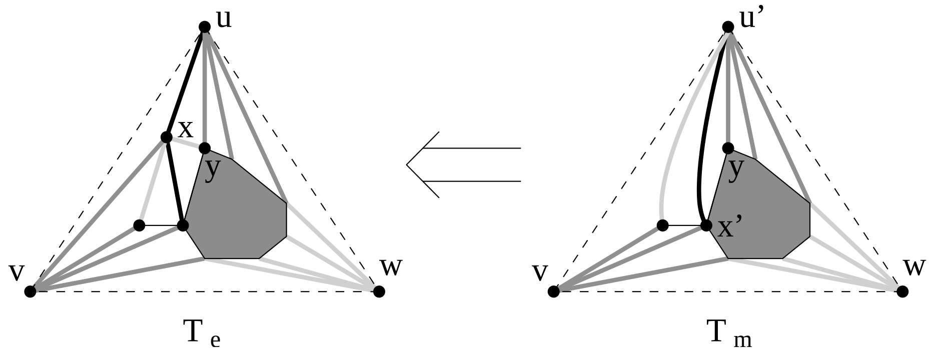

Fig. 4. From T_m to T_e.

the second case, this would imply that there is a path in $F'_1 \subset T'_m$ linking two of the vertices u', v or w. Both cases are impossible since the partition of T'_m verifies the theorem.

- The graph F_2 is a forest. If F_2 had a cycle, this cycle should either pass through yx or not. In the first case, this would imply that there is a path from u' to y in $F'_2 \subset T'_m$. In the second case, this would imply that there is a cycle in $F'_2 \subset T'_m$. Both cases are impossible since the partition of T'_m verifies the theorem.
  Similarly the vertices u ,v and w are in distinct connected components of F_2. If there was a path in F_2 linking two of these vertices this path should either pass through yx or not. In the first case, this would imply that there is either a path from u' to v or w, or a cycle (passing through u') in $F'_2 \subset T'_m$. In the second case, this would imply that there is a path in $F'_2 \subset T'_m$ linking two of the vertices u', v or w. Both cases are impossible since the partition of T'_m verifies the theorem.
  Furthermore, since there is no vertex inside (u, v, x) and no edge incident to u in F_2, the connected component of F_2 containing u is as expected.
- The graph F_3 is a forest. If F_3 had a cycle, this cycle should either pass through ux or not. In the first case, this would imply that there is a cycle (passing through u') in $F'_3 \subset T'_m$. In the second case, this would imply that there is a cycle in $F'_3 \subset T'_m$. Both cases are impossible since the partition of T'_m verifies the theorem.
  Similarly the vertices u ,v and w are in distinct connected components of F_3. If there was a path in F_3 linking two of these vertices this path should either pass through ux or not. In the first case, this would imply that there is a path from u' to v or w in $F'_3 \subset T'_m$. In the second case, this would imply that there is a path in $F'_3 \subset T'_m$ linking two of the vertices u', v or w. Both cases are impossible since the partition of T'_m verifies the theorem.
  Furthermore note that every vertex $a \in V(T_e) \setminus \{u, x\}$ has as many incident edges in F_3 as in F'_3. Since u and x have respectively one and two incident edges in F_3, F_3 has maximum degree at most four.

For the rest of the proof it is important to remember that the theorem holds

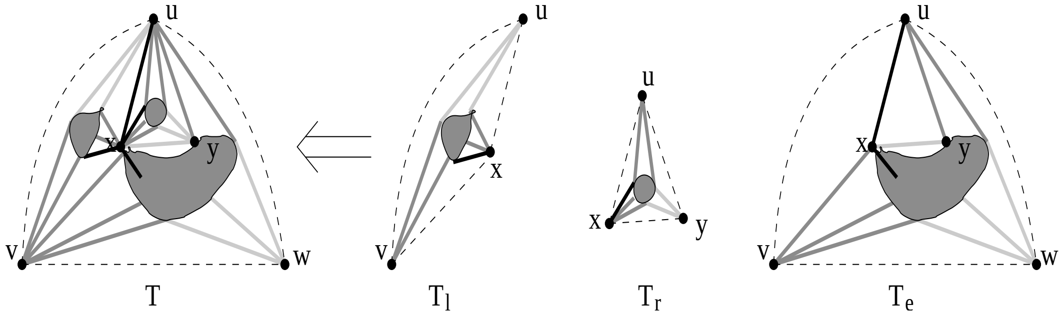

Fig. 5. From T_e, T_l and T_r to T.

for T_e in such way that x has degree two in F_3.

The graph T' is the disjoint union of T'_e, $T'_l = T_l \backslash \{uv, ux, vx\}$, and $T'_r = T_r \backslash \{ux, uy, xy\}$. We construct an edge-partition of T' into three forest F_1, F_2, and F_3, by partitionning each of T'_e, T'_l, and T'_r into three forests. To do this, we apply the induction hypothesis to T_l according to the triplet (x, v, u). This means that the edges incident to v (resp. u) in T'_l belongs to F_1 (resp. F_2). Similarly, we apply the induction hypothesis to T_r according to the triplet (x, u, y). This means that the edges incident to u (resp. y) in T'_l belongs to F_1 (resp. F_2). We have seen that the induction hypothesis holds for T_e according to the triplet (u, v, w), and we consider such partition in which the vertex x has degree two in F_3. This yelds to a partition of T' into the three forests described in the theorem.

- The edges incident to u ,v or w are clearly well partitionned.
- The graph F_i, for any $1 \leq i \leq 3$, is a forest. If F_i would contain a cycle, since there is no such cycle in T'_e, T'_l, or T'_r, this cycle should pass through T_l or T_r. This would imply that there is a path in $F_i \cap T'_l$ or $F_i \cap T'_r$ linking two of the vertices u, v, x, and y, which is impossible according to the partitions of T'_l or T'_r.
- The graph F_i, for any $1 \leq i \leq 3$, does not contain any path linking two of the vertices u, v, and w. If F_i would contain such path, since there is no such path in T'_e, this path should pass through T'_l or T'_r from u to v, x or y, or from v to x, which is impossible according to the partitions of T'_l or T'_r.
- Since the vertex u has no incident edges in $F_2 \cap T'_e$ and $F_2 \cap T'_r$, and since there is no path from u to v or x in $F_2 \cap T'_l$, the connected component of F_2 containing u only contains u ans some vertices inside (u, v, x).
- The graph F_3 is such that, $\Delta(F_3) \leq 4$. Indeed, x has at most 2, 1, and 1 incident edges in $F_2 \cap T'_e$, $F_2 \cap T'_l$, and $F_2 \cap T'_r$; and the other vertices have as many incident edges in F_2 as in $F_2 \cap T'_e$, $F_2 \cap T'_l$, or $F_2 \cap T'_r$.

This complete the proof of the theorem.

3 Planar graphs of bounded girth

The results in [1, 2, 5, 6] are all proved using discharging methods. We use this method for proving Theorem 2 and Theorem 3. This method consists roughly in showing that a counter-example H minimizing $|V(H)|$ would be too "dense" (*i.e.* has too many edges per vertex) to verify Euler's Formula. This formula says that any connected planar graph G with n vertices, m edges and f faces verifies $m = n + f - 2$. Let us define a *k-vertex* (resp. *$^{\leq}k$-vertex* and *$^{\geq}k$-vertex*) as a vertex of degree k (resp. at most k and at least k).

3.1 Planar graphs with girth $g \geq 6$

Let H be a counter-example of Theorem 2 minimizing $|V(H)|$.

Lemma 5 *The counter-example H:*

(1) is connected,
(2) has minimum degree $\delta(H) \geq 2$, and
(3) does not contain any edge uv such that $\deg(u) = 2$ *and* $\deg(v) \leq 5$.

PROOF. (1) If H was disconnected, one of its connected component would be a smaller counter-example. (2) If H had a 1-vertex u, the graph $H \setminus \{u\}$ would have girth $g \geq 6$ and would have an $F(\infty, 4)$-covering by F_1 and F_2. Adding the incident edge of u in F_1 we would obtain an $F(\infty, 4)$-covering of H, which is impossible. (3) Consider that H had an edge uv such that $\deg(u) = 2$ and $\deg(v) \leq 5$. Since H is minimal, the graph $H \backslash \{u\}$ has an $F(\infty, 4)$-covering by F_1 and F_2. We extend those forests to obtain an $F(\infty, 4)$-covering of H. Let w be the second neighbor of u. If all the edges incident to v in $H \backslash \{u\}$ are in F_2 then let $F'_1 = F_1 \cup \{wu, uv\}$ and $F'_2 = F_2$. Else, v has degree at most 3 in F_2 and let $F'_1 = F_1 \cup \{wu\}$ and $F'_2 = F_2 \cup \{uv\}$. In both cases the forests F'_1 and F'_2 cover H, and $\Delta(F'_2) \leq 4$. Since H is not $F(\infty, 4)$-coverable we have a contradiction and H does not contain such edge uv.

We now use a discharging procedure on the vertices of H in order to estimate $2|E(H)|/|V(H)|$. Let the intitial charge of the vertices be equal to their degree, $ch(v) = \deg(v)$ for all $v \in V(H)$. Then, every $^{\geq}6$-vertex gives charge $\frac{1}{2}$ to its neigbhors of degree 2. After this procedure the total charge of the graph is preserved and all the vertices have a final charge $ch^*(v) \geq 3$. Indeed :

- If $\deg(v) = 2$, then v receives $\frac{1}{2}$ from each of its neighbors (Lemma 5.(3)) and $ch^*(v) = 2 + 2\frac{1}{2} = 3$.

- If $3 \leq \deg(v) \leq 5$, then v does not give any charge, so $ch^*(v) \geq 3$.
- If $6 \leq \deg(v)$, then v gives at most $\frac{1}{2}$ to each of its neigbors, so $ch^*(v) \geq 6 - 6\frac{1}{2} = 3$.

So we have that $2|E(H)| = \sum_{v \in V(H)} \deg(v) = \sum_{v \in V(H)} ch^*(v) \geq 3|V(H)|$. Let n, m and f denote respectively the number of vertices, edges and faces in H. We know that $2m \geq 3n$ and since H has girth at least 6, each face is bounded by at least 6 edges and $2m \geq 6f$. Combining these two equations we obtain that $m \geq n + f$ contradicting Euler's Formula. So H does not exist and Theorem 2 holds.

3.2 Planar graphs with girth $g \geq 7$

Let H be a counter-example of Theorem 3 minimizing $|V(H)|$.

Lemma 6 *The counter-example H:*

(1) is connected,
(2) has minimum degree $\delta(H) \geq 2$,
(3) does not contain any edge uv such that $\deg(u) = 2$ and $\deg(v) \leq 3$, and
(4) does not contain any 3-vertex u adjacent to three 3-vertices.

PROOF. (1) If H was disconnected, one of its connected component would be a smaller counter-example. (2) If H had a 1-vertex u, the graph $H \setminus \{u\}$ would have girth $g \geq 6$ and would have an $F(\infty, 4)$-covering by F_1 and F_2. Adding the incident edge of u in F_1 we would obtain an $F(\infty, 4)$-covering of H, which is impossible.

For the cases (3) and (4) we consider the graph $H \backslash \{u\}$. By minimality of $|V(H)|$, the graph $H \backslash \{u\}$ has an $F(\infty, 4)$-covering by F_1 and F_2. We consider a pair (F_1, F_2) maximizing the number of edges in F_1. This implies that every 2-vertex in $H \backslash \{u\}$ has at most one incident edge in F_2. In case (3), let w be the second neighbor of u. Since v has degree at most one in F_2, the forests $F_1 \cup \{uw\}$ and $F_2 \cup \{uv\}$ would be an $F(\infty, 4)$-covering of H, which is impossible. In case (4), let v_1, v_2 and v_3 be the neighbors of u. Since v_1, v_2 and v_3 have degree at most two in $H \backslash \{u\}$, they have degree at most one in F_2. Since each connected component of F_2 contains at most two $^{\leq}$1-vertices, two of the vertices v_1, v_2 and v_3 are in distinct connected components of F_2, say v_1 and v_2. In this case, the forests $F_1 \cup \{uv_3\}$ and $F_2 \cup \{uv_1, uv_2\}$ would be an $F(\infty, 4)$-covering of H, which is impossible.

Since $\delta(H) \geq 2$, we distinguish 6 types of edges in H :

(a) For every 2-vertex v, let one of its incident edges be an *a-edge* and the other one be an *$\overline{a}$-edge*.

Let us distinguish 2 types of 3-vertices. An *isolated* 3-vertex has no 3-vertex in its neighborhood. The rest of the 3-vertices are *linked* 3-vertices, this means adjacent to at least one 3-vertex.

(b) For every isolated 3-vertex v, let one of its incident edges be a *b-edge* and the two remaining ones be *$\overline{b}$-edges*.
(c) We consider the subgraph K of H induced by the linked 3-vertices. This subgraph K is such that $\Delta(K) \leq 2$ (by Lemma 6.(4)) and $\delta(K) \geq 1$ (by definition of linked 3-vertices). Let $C \subseteq E(K)$ be the smallest set of edges in K such that all the linked 3-vertices have an incident edge in C. The minimality of $|C|$ implies that in each connected component of K (a cycle or a path), there is at most one vertex with two incident edges in C. The edges of C are the *c-edges* and all the edges of H (not just K) adjacent to a c-edge are $\overline{c}$-edges.

It is clear given Lemma 6 that the sets of *a*-, *b*-, *c*-, $\overline{a}$-, $\overline{b}$- and $\overline{c}$-edges, respectively A, B, C, $\overline{A}$, $\overline{B}$ and $\overline{C}$, are pairwise disjoint. Now we transform H into another graph H' by contracting the *a*-, *b*- and *c*-edges. Since every 2-vertex (resp. 3-vertex) is adjacent to a ${}^{\geq}4$-vertex (resp. ${}^{\geq}3$-vertex) by an *a*-edge (resp. *b*- or *c*-edge), and since it has at most one (resp. two) incident a-edge (resp. b- or c-edges), there is no more vertices of degree less than 4 in H'.

Lemma 7 *The graph H' is connected and after the transformation every cycle $\mathcal{C}$ in H becomes a cycle $\mathcal{C}'$ in H'.*

(1) If $\mathcal{C}$ has length 7 and if all its vertices are 3-vertices, then the cycle $\mathcal{C}'$ has length 3 and contains a 5-vertex.
(2) Else, the cycle $\mathcal{C}'$ has length $l(\mathcal{C}') \geq 4$.

PROOF. It is clear that, by contracting edges, a graph remains connected. For the cycles we distinguish the two cases. In case (1), since $\mathcal{C}$ has 4 *c*-edges and 3 $\overline{c}$-edges, the cycle $\mathcal{C}'$ has length 3 and the two consecutive *c*-edges produce a 5-vertex. In case (2), the cycle $\mathcal{C}$ contains at least one ${}^{\geq}4$-vertex (case (2.1)) or contains only 3-vertices and has length $l \geq 8$ (case(2.2)).

In case (2.1), consider any path $\mathcal{P} = (v_0, v_1, \ldots, v_k) \subseteq \mathcal{C}$ linking two ${}^{\geq}4$-vertices, v_0 and v_k, and going through ${}^{\leq}3$-vertices. Actually this path may be a cycle if $v_0 = v_k$.

Claim 8 *There is at least as many $\overline{a}$-edges (resp. $\overline{b}$-edges and $\overline{c}$-edges) in $\mathcal{P}$ than a-edges (resp. b-edges and c-edges).*

Indeed :

(-) If $\mathcal{P}$ is just an edge linking two $^{\geq}4$-vertices, then this edge is not an a-, b- or c-edge.
(a) If $\mathcal{P}$ goes through a 2-vertex, then $\mathcal{P}$ has length 2 and contains exactly one a-edge and one $\overline{a}$-edge.
(b) If $\mathcal{P}$ goes through an isolated 3-vertex, then $\mathcal{P}$ has length 2 and contains at most one b-edge and at least one $\overline{b}$-edge.
(c) If $\mathcal{P}$ goes through $(k-1)$ 3-vertices, then $\mathcal{P}$ contains $\left\lceil \frac{k-1}{2} \right\rceil$ c-edges and the remaining $k - \left\lceil \frac{k-1}{2} \right\rceil$ edges are $\overline{c}$-edges.

This claim implies that at most half of the edges in $\mathcal{C}$ are contracted. Since $l(\mathcal{C}) \geq 7$ this implies that $\mathcal{C}'$ has length $l' \geq 4$.

In case (2.2), the cycle $\mathcal{C}$ has length $l \geq 8$ and contains $\left\lceil \frac{l}{2} \right\rceil$ c-edges and the remaining $\left\lfloor \frac{l}{2} \right\rfloor$ edges are $\overline{c}$-edges Since $\left\lfloor \frac{l}{2} \right\rfloor \geq 4$ when $l \geq 8$, we have $l(\mathcal{C}') \geq 4$ and the lemma holds.

Let n_4 and $n_{\geq 5}$ be the number of 4-vertices and $^{\geq}5$-vertices in H'. Let c_3 be the number of cycles of length 3 in H'. Note that all the cycles of length 3 in H' contain a 5-vertex. Since these cycles of length 3 in H' come from cycles of 3-vertices in H, Lemma 6.(4) implies that these cycles of length 3 are vertex disjoint. This implies that $n_{\geq 5} \geq c_3$. Let f_3 and $f_{\geq 4}$ be the number of faces of length respectively $l = 3$ and $l \geq 4$ in H'. Since $c_3 \geq f_3$, we have $n_{\geq 5} \geq f_3$. Now, let n, m and f be the number of vertices, edges and faces in H'. It is clear that $n = n_4 + n_{\geq 5}$ and $f = f_3 + f_{\geq 4}$. Since the edges have two end points and are incident to at most two faces, we have :

$$2m \geq 4n_4 + 5n_{\geq 5} = 4n + n_{\geq 5} \geq 4n + f_3$$

$$2m \geq 3f_3 + 4f_{\geq 4} = 4f - f_3$$

Suming these two equations we obtain that $m \geq n + f$ contradicting Euler's Formula. So H' and H do not exist and Theorem 3 holds.

4 Perspectives

In [3] Colin de Verdière introduced a graph parameter μ. For a graph G this parameter is defined by spectral properties of matrices associated to G. This parameter is such that :

- $\mu(G) \leq 1$ iff G is a forest of paths.
- $\mu(G) \leq 2$ iff G is an outerplanar graph.

- $\mu(G) \leq 3$ iff G is a planar graph.

Since forests of paths, outerplanar graphs [2], and planar graphs are respectively $F(2)$-, $F(\infty, 3)$-, and $F(\infty, \infty, 4)$-coverable we conjecture the following.

Conjecture 9 *Every graph G has an edge partition into $\mu(G)$ forests, one having maximum degree $\Delta \leq \mu(G)+1$.*

A weaker result would be that every graph G is $(\mu(G)-1, \mu(G)+1)$-coverable. This result would be sharp, indeed:

Theorem 10 *For any integer $k \geq 1$ there is a graph G with $\mu(G) = k$ that is not $(k-1, k)$-coverable.*

PROOF. It is know for $k \leq 3$, so consider that $k \geq 4$. For any pair of positive integers (k, l) with $k \geq 4$ and $l \geq 0$ we define the graph G_k^l. Let $G_k^0 = K_{k+1}$. For $l > 0$ we construct the graph G_k^l from G_k^{l-1} by adding, for each copy of K_k in G_k^{l-1} that contain a k-vertex a new vertex adjacent to the vertices in this copy of K_k. According to [7] we have $\mu(G_k^l) = k$ for any k and l.

Claim 11 *For any $l \geq 1$, the graph G_k^l has $(k+1)k^{l-1}$ k-vertices that form an independant set and $(k+1)\left(1 + \sum_{i=0}^{l-2} k^i\right)$ $^>k$-vertices. Furthermore, this graph has $\frac{k(k+1)}{2} + (k+1)\left(\sum_{i=1}^{l-1} k^i\right)$ edges linking two $^>k$-vertices and $(k+1)k^l$ edges linking a k-vertex and a $^>k$-vertex.*

Indeed, it is clear for $l = 1$ and for the induction we just note that each of the $(k+1)k^{l-2}$ k-vertices in G_k^{l-1} belongs to k copies of K_k. Since these vertices form an independant set there is $k \times (k+1)k^{l-2}$ copies of K_k in G_k^{l-1} that contain a vertex of degree k. So G_k^l has $(k+1)k^{l-1}$ new vertices of degree k and all the vertices that were in G_k^{l-1} have now degree more than k, and there are $(k+1)\left(1 + \sum_{i=0}^{l-2} k^i\right)$ such vertices. Furthermore, since every k-vertex of G_k^l is incident to $^>k$-vertices, these k-vertices clearly form an independant set. For the number of edges, it is clear that from G_k^{l-1} to G_k^l we add k new edges per new vertex (of degree k) and that every edge present in G_k^{l-1} link two $^>k$-vertices in G_k^l.

We consider now the following theorem of Balogh *et al.* [2].

Theorem 12 *For every (t, D)-coverable graph G and any two disjoint subsets A and B of $V(G)$,*

$$f_t(A) + e(A, B) \leq D|A| + t(|A| + |B| - 1)$$

where $e(X, Y)$ denotes the number of edges of G with one end in X and the other in Y, and where $f_t(A) = e(A, A)$ if $e(A, A) \leq t(|A| - 1)$, and $f_t(A) =$

$2e(A,A) - t(|A|-1)$ *otherwise.*

For any $k \geq 4$ consider the graph G_k^3, let A be the set of ${}^{>}k$-vertices and let B be the set of k-vertices. This theorem says that if G_k^3 was $(k-1,k)$-coverable we should have

$$f_{k-1}(A) + e(A,B) \leq k|A| + (k-1)(|A|+|B|-1)$$

Note that according to Claim 11 $|A| = (k+1)(k+2)$ and $e(A,A) = k(k+1)(k+3/2)$, so we have $e(A,A) > (k-1)(|A|-1)$. This implies that $f_{k-1}(A) = 2e(A,A) - (k-1)(|A|-1) = \frac{1}{2}(k^2+7k+2)$. Thus if G_k^3 was $(k-1,k)$-coverable we should have

$$\frac{1}{2}(k^2+7k+2) + (k+1)k^3 \leq k(k+1)(k+2) + (k-1)((k+1)(2+k+k^2)-1)$$

, which is equivalent to

$$2 + 6k - 6k^2 - k^3 + k^4 \leq 0$$

, and which does not hold for $k \geq 4$. Thus G_k^3 is not $(k-1,k)$-coverable for $k \geq 4$ and this complete the proof of the theorem.

Another interesting question concerns the consequences of Theorem 1. Since the forests of maximum degree four are coverable by two linear forests or by two star forests with maximum degree three we have the following corollary.

Corollary 13 *Planar graphs are coverable by :*

- *6 star forests, two of them having maximum degree at most three.*
- *2 forests and 2 linear forests.*

Planar graphs with girth $g \geq 6$ are coverable by :

- *4 star forests, two of them having maximum degree at most three.*
- *1 forest and 2 linear forests.*

We have seen that Theorem 1 is optimal, we wonder if it is also the case for Theorem 2, Theorem 3 and for this corollary.

References

[1] A. Bassa, J. Burns, J.Campbell, A. Deshpande, J. Farley, M. Halsey, S. Michalakis, P.-O.Persson, P. Pylyavskyy, L. Rademacher, A. Riehl, M. Rios, J. Samuel, B. Tenner, A. Vijayasaraty, L. Zhao, and D.J. Kleitman.

Partitioning a Planar Graph of Girth Ten into a Forest and a Matching *manuscript* (2004).

[2] J. Balogh, M. Kochol, A. Pluhár and X. Yu. Covering planar graphs with forests, *J. Combin. Theory B.* **94** (2005), 147-158.

[3] Y. Colin de Verdière, Sur un nouvel invariant des graphes et un critère de planarité *J. Combin. Theory Ser. B* **50** (1990) 1, 11–21.

[4] D. Gonçalves. Caterpillar arboricity of planar graphs, *Discrete Math.* to appear.

[5] D.J. Guan, X. Zhu, Game chromatic number of outerplanar graphs *J. Graph Theory* **30** (1999) 1, 67–70.

[6] W. He, X. Hou, K-W Lih, J. Shao, W. Wang and X. Zhu, Edge-partitions of planar graphs and their game coloring numbers *J. Graph Theory* **41** (2002) 4, 307–317.

[7] H. van der Holst, L. Lovász and A. Schrijver On the invariance of Colin de Verdière's graph parameter under clique sums *Linear Algebra Appl.* **226/228** (1995), 509–517.

[8] C.St.J.A. Nash-Williams. Decompositions of finite graphs into forests, *J. London Math. Soc.* **39** (1964), 12.