A planar linear hypergraph whose edges cannot be represented as straight line segments

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Abstract

H. de Fraysseix and P. Ossona de Mendez asked whether every planar linear hypergraph H admits a planar representation in which the edges are straight line segments. In this article we exhibit a planar linear hypergraph that does not admit such repesentation.

Key words: planar graph, hypergraph, segment

A *straight line representation* of a hypergraph H maps each vertex v to a point $p(v)$ and each edge e to a straight line segment $s(e)$, in such way that:

- for each vertex $v \in V(H)$ and each edge $e \in E(H)$ we have $p(v) \in s(e)$ if and only if $v \in e$, and
- for each each couple of distinct edges e_1 and e_2 we have $s(e_1) \cap s(e_2) = \{p(v) : v \in e_1 \cap e_2\}$

Since two segments have either 0, 1, or an infinite number of common points, a trivial condition for a finite hypergraph H for having a straight line representation is to be linear, this is that any two edges of H share at most one vertex. In (1) (see also (3)) H. de Fraysseix and P. Ossona de Mendez asked whether every planar linear hypergraph H has a straight line representation.

This clearly holds for graphs (*i.e.* hypergraphs whose edges have size two). Furthmermore the author proved (2) that it also holds for hypergraphs with maximum degree two and without any restriction on the size of the edges. However this does not hold in general. Here we prove that the hypergraph H depicted in Figure 1 is a counter-example. The vertex set of H is the set of subsets of $S = \{a, b, c, d\}$ of size two or three, $V(H) = \mathcal{P}_2(S) \cup \mathcal{P}_3(S)$.

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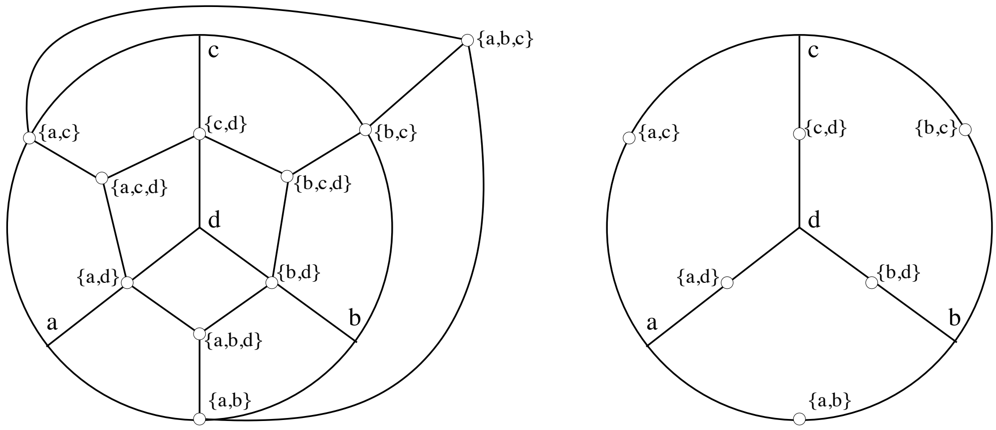

Figure 1. The hypergraphs H and K.

A straight line representation of a hypergraph H is *minimum* if for any edge $e \in E(H)$ the two ends of the segment $s(e)$ belong to $p(V(H))$. It is clear that any straight line representation can easely be turned into a minimum one. So now on, we only consider minimum straight line representations.

Let us consider the sub-hypergraph K of H with vertex set $V(K) = \mathcal{P}_2(S)$ and edge set $E(K) = S$. If H had a straight line representation, this would induce a straight line representation of K. In the following, we show that none of the possible straight line representations of K can be extended to a straight line representation of H.

The reader can easely verify that any straight line representation of K is equivalent to the one depicted in Figure 2, given that e_1, e_2, e_3, and e_4 is an alternative way to denote the elements of S (*i.e.* the edges of K).

Now observe that if we extend this representation to H we have some constraints for the location of $p(v)$ for $v \in \mathcal{P}_3(S)$, the vertices in $V(H) \setminus V(K)$. Indeed, for exemple the point $p(\{e_1, e_3, e_4\})$ is linked to $p(\{e_1, e_3\})$, $p(\{e_1, e_4\})$, and $p(\{e_3, e_4\})$ by the three segments $s(\{\{e_1, e_3\}, \{e_1, e_3, e_4\}\})$, $s(\{\{e_1, e_4\}, \{e_1, e_3, e_4\}\})$, and $s(\{\{e_3, e_4\}, \{e_1, e_3, e_4\}\})$. Furthermore, by minimality of the representation and since these three segments correspond to edges of size two, these three segments only intersect other segments in their end points. This implies that $p(\{e_1, e_3, e_4\})$ should either belong to Z_1 or Z_3 (the dashed regions in Figure 2). Similarly $p(\{e_1, e_2, e_4\})$ should either belongs to Z_2 or Z_3. It is now easy to see that whichever the regions that contain $p(\{e_1, e_3, e_4\})$ and $p(\{e_1, e_2, e_4\})$, the four cases lead to problematic segment intersections. Thus H has no straight line representation.

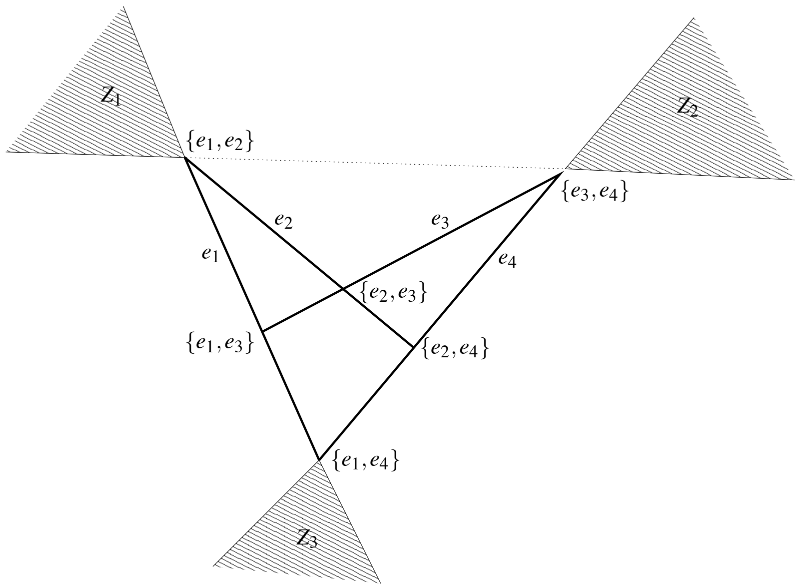

Figure 2. minimum straight line representation of K

References

[1] H. de Fraysseix and P. Ossona de Mendez, Representations by Contact and Intersection of Segments, *Algorithmica* **47** (2007) 453–463.

[2] J. Chalopin and D. Gonçalves, Every planar graph is the intersection graph of segments in the plane, *In preparation*.

[3] Open Problem Garden, retrieved Oct. 2007 at `http://garden.irmacs.sfu.ca`.