On Oriented Labelling Parameters

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Abstract

We introduce two notions, (i)oriented labelling and (ii) oriented $L(p, q)$ labelling, to explore frequency assignment problem under half- duplex setting and compute bounds of oriented labelling of special classes of graphs : trees, bipartite graphs and planar graphs. We do study an oriented version of clique cover problem as well.

The frequency assignment problem (FAP) arises in wireless communication systems. There are several models based on genetic algorithms, neural networks, constrained programming and combinatorial enumeration to explore and optimize different features of FAP such as available frequencies and limiting interference among radio signals. In 1970, Metzger [14] introduced graph coloring techniques as a tool to optimize the frequency spectrum used in FAP. Motivated by FAP, Hale [10] developed the concept of T-coloring and it lead to many indepth graph theoretical results. A common feature of all these graph theoretical models of FAP is that they assume communication is viable in both direction (duplex) between two radio trasmitters and hence model FAP as a non-oriented graph. This is far away from reality. In fact, Aardal et al emphasised the importance of "direction/orientation" of transmission in their recent survey (for details see [1] page 4) on FAP.

In this paper, we explore FAP under half-duplex setting (i.e., there are radio transimitters in a network in which at most one way transimission is effective between any two of them). Hence FAP

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can be modeled as an oriented multi graph. As a preliminary step, we focus on oriented simple graphs.

Oriented vertex partitioning problems

The partitioning of the vertex set of a non-oriented graph into minimum number of subsets in which each subset possesses a special property is a fundamental graph theory problem with many applications. For example : vertex coloring, star (vertex) coloring and clique cover problem. Note that T-coloring is a generlization of vertex coloring . We suggest a general frame work to incorporate "orientation" into various vertex partitioning problems. Two subsets A, B of the vertex set of an oriented graph is called *one way oriented* if all edges with one end vertex in A and other in B are oriented from A (respectively B) to B (respectively A).

Oriented vertex partitioning problem : Partition the vertex set of an oriented graph into minimum number of pairwise one way oriented subsets in which each set possesses a special property.

Oriented coloring is a well known oriented vertex partitioning problem [16, 17].

Notation and Terminology

In this paper, we consider only finite (oriented and non-oriented) simple graphs. As usual, $N^+(v)$ = $\{u : \ \ \overrightarrow{vu} \in E(\vec{G})\}$ and $N^-(v)$ = $\{u : \ \ \overrightarrow{uv} \in E(\vec{G})\}$. A proper k-coloring of the vertices of a non-oriented graph is called *acyclic* if the subgraph induced by the vertices with any two colors has no cycle. The acyclic chromatic number of a graph G is the smallest integer k such that G has an acyclic k-coloring. The length of the shortest path joining two vertices x, y in a non-oriented graph is called distance between x and y. A path with an orientation such that all internal vertices have both in-degree and out-degree one is called an *oriented path*. The length of a shortest oriented path joining two vertices x, y in an oriented graph is called *oriented distance* between x and y. The *girth* of a non-oriented graph G is the length of its smallest cycle. Let $\vec{G}(V, E)$ and $\vec{H}(U, F)$ be two oriented graphs. An oriented homomoriphism from $\vec{G}$ to $\vec{H}$ is a map $f : V \longrightarrow U$ which preserves adjacency, that is for any edge $\overrightarrow{xy} \in E$, the correspoding pair $\overrightarrow{f(x)f(y)} \in F$.

Oriented coloring [16, 17] Let $\vec{G}(V, E)$ be an oriented graph. A map $c : V \longrightarrow \{1, 2, ..., k\}$ is called a k-oriented coloring of $\vec{G}$ if it satisfies the following.

(i) For any edge $\overrightarrow{xy}$, $\quad c(x) \neq c(y)$.

(ii) There are no two edges $\overrightarrow{xy}$, $\overrightarrow{uv}$ such that $c(x) = c(v)$ and $c(y) = c(u)$.

The least integer k in which $\vec{G}$ has k-oriented coloring is called the oriented chromatic number $\vec{\chi}(\vec{G})$ of $\vec{G}$. The *oriented chromatic number* $\vec{\chi}(G)$ of a non-oriented graph G is defined as max $\{$ $\vec{\chi}(\vec{G})$: $\vec{G}$ is an orientation of G $\}$.

Note that the condition (i) of oriented coloring ensures that adjacent vertices do not belong to same color class. The condition (ii) guarantees that any two color classes preserve one way oriented property in an oriented graph. Moreover, $\chi(G) \leq \vec{\chi}(G)$ where $\chi(G)$ denotes the chromatic number of G. Since $\vec{\chi}(K_{n,n})$ = $2n$, $\vec{\chi}(G)$ has no upper bound as a function of $\chi(G)$. Raspaud and Sopena [17] proved that oriented coloring of a planar graph is at most 80.

A suggestion of F. Roberts to distinguish close and very close transmitters in a wireless communication system led Griggs and Yeh [9] to propose a variation of FAP as labelling the vertices of a non-oriented graph with a condition at a distance two(known as $L(2,1)$-labelling). Georges and Mauro [7] generalized this as follows.

$L(p, q)$**-Labelling** : Let $G(V, E)$ be a non-oriented graph and p, q be two positive integers. A map L : V $\longrightarrow \{0, 1, ..., k\}$ is called a k-$L(p, q)$-labelling if it satisfies the following.

(i) For any edge $xy \in E$, $\mid L(x) - L(y) \mid \geq p$.

(ii) For any pair of vertices x, y at a distance 2, $\mid L(x) - L(y) \mid \geq q$.

The *span*, $\lambda_{p,q}(G)$, of G is defined as min $\{k : G$ has a k-$L(p, q)$-labelling $\}$. For convenience, we prefer $\lambda_p(G)$ to $\lambda_{p,1}(G)$.

We cite a few known results in $L(p, q)$-labelling problems.

(1) $L(2, 1)$-labelling problem is NP-complete [9].

(2) For a tree T with maximum degree Δ, $\Delta + 1 \leq \lambda_2(T) \leq \Delta + 2$ [9] .

(3) For a graph G with maximum degree Δ, $\lambda_p(G) \leq \Delta^2 + (p - 1)\Delta - 2$ [8] .

Two oriented variations of $L(p, q)$-labelling

In this section, we extend $L(p, q)$-labelling to oriented graphs and propose a new oriented vertex partitioning problem.

$L(p, q)$**-Labelling for oriented graphs** : Let $\vec{G}(V, A)$ be an oriented graph and p, q be two positive integers. A map L : V $\longrightarrow \{0, 1, ..., k\}$ is called a k-$L(p, q)$-labelling of $\vec{G}$ if it satisfies the following.

(i) For any edge $\vec{xy} \in A$, $\mid L(x) - L(y) \mid \geq p$.

(ii) For any pair of vertices x, y at an oriented distance 2, $\mid L(x) - L(y) \mid \geq q$.

The *span*, $\lambda^o_{p,q}(\vec{G})$, of $\vec{G}$ is defined as min $\{k : \vec{G}$ has a k-$L(p, q)$-labelling $\}$. The *span* of a non-oriented graph G, $\lambda^o_{p,q}(G)$, is defined as max $\{ \lambda^o_{p,q}(\vec{G}) : \vec{G}$ is an orientation of $G \}$. For convenience (when q=1), we denote $\lambda^o_{p,1}(G)$ = $\lambda^o_p(G)$.

Oriented $L(p, q)$**-Labelling** : Let $\vec{G}(V, A)$ be an oriented graph and p be a positive integer. A map l : V $\longrightarrow \{0, 1, ..., k\}$ is called a k-oriented $L(p, q)$-labelling if it satisfies the following.

(i) For any edge $\vec{xy} \in A$, $\mid l(x) - l(y) \mid \geq p$.

(ii) For any pair of vertices x, y at an oriented distance 2, $\mid L(x) - L(y) \mid \geq q$.

(iii) There are no two edges $\vec{xy}$, $\vec{uv}$ such that $l(x) = l(v)$ and $l(y) = l(u)$.

The *span*, $\vec{\lambda}_{p,q}(\vec{G})$, of $\vec{G}$ is defined as min $\{k : \vec{G}$ has a k-oriented $L(p,q)$-labelling $\}$. The *span* of a non-oriented graph G, $\vec{\lambda}_{p,q}(G)$, is defined as max $\{ \vec{\lambda}_{p,q}(\vec{G}) : \vec{G}$ is an orientation of G $\}$. For convenience (when q=1), we denote $\vec{\lambda}_{p,1}(G) = \vec{\lambda}_p(G)$.

Remarks There is a distinction between the $L(p,q)$-labelling of oriented graphs and the oriented $L(p,q)$-labelling. Note that any two oriented $L(p,q)$-labelling color classes (i.e. set of vertices with same label) are one way oriented. But $L(p,q)$-labelling doesn't guarantee one way orientedness of its color classes. A pair of color classes in a $L(p,q)$-labelling can be viewed as a union of two (disconnected) pairs of one way oriented sets (see Figure 1). Hence, an oriented $L(p,q)$-labelling of an oriented graph $\vec{G}$ is also a $L(p,q)$-labelling of $\vec{G}$. The graph $\vec{H}$ in Figure 1, with $\lambda_1^o(\vec{H}) = 3$ and $\vec{\lambda}_1(\vec{H}) = 4$, shows that the converse is not true.

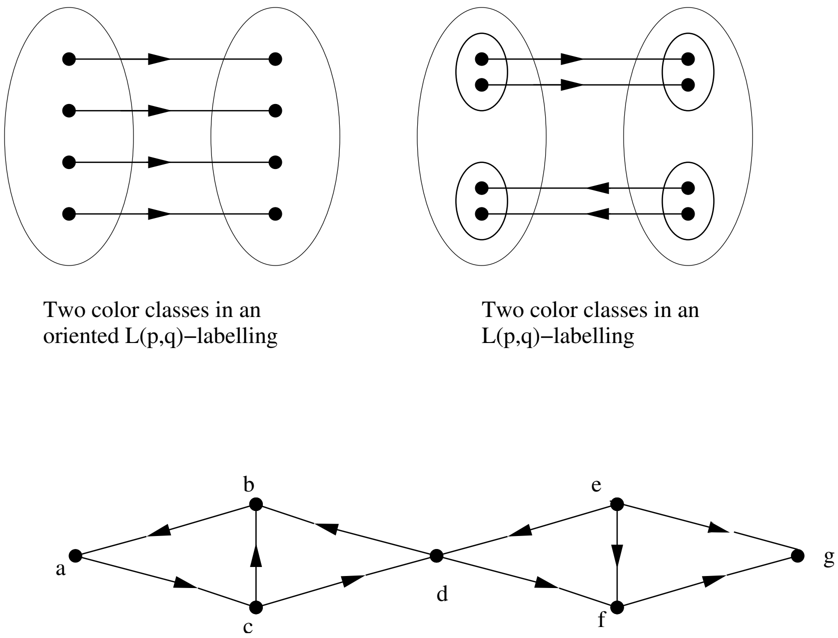

FIG. 1 – Oriented color classes and the graph $\vec{H}$

Let H be a subgraph of G. Then $\vec{\lambda_p}(H) \leq \vec{\lambda_p}(G)$. Oriented $L(p,q)$-labelling is a generalization of oriented coloring. In particular, $\vec{\lambda}_1(\vec{G}) = \vec{\chi}(\vec{G}) - 1$(we do allow '0' as a label). Let I_0, I_1,..., $I_{\vec{\chi}-1}$ be a set of oriented color classes of $\vec{G}$. We produce a $(\vec{\chi}(\vec{G})-1)p$- oriented $L(p,p)$-labelling of $\vec{G}$ by assigning the label jp to each vertex in the set I_j for $0 \leq j \leq \vec{\chi}(\vec{G}) - 1$. If there is a homomophism $h : \vec{G} \to \vec{H}$, then $\vec{\lambda}_{p,q}(\vec{G}) \leq \vec{\lambda}_{p,q}(\vec{H})$. Indeed, given a k-oriented $L(p,q)$-labelling l_H of $\vec{H}$, we define a k-oriented $L(p,q)$-labelling l_G of $\vec{G}$, by $l_G(v) = l_H(h(v))$, for all $v \in V_G$. We also note that a $L(p, q)$-labelling of a non-oriented graph G is a $L(p, q)$-labelling of any orientation of G. By definition, an oriented $L(p, q)$-labelling of an oriented graph is also its $L(p, 1)$-labelling. These remarks prove the following lemma.

Lemma 1 *Let G be a non-oriented graph. Then*

(i) For $p \geq q > 0$, $\vec{\chi}(G) - 1 \leq \vec{\lambda}_{p,q}(G) \leq (\vec{\chi}(G) - 1)p$.

(ii) $\lambda^o_{p,q}(G) \leq \lambda_{p,q}(G)$.

(iii) $\lambda^o_{p,q}(G) \leq \vec{\lambda}_{p,q}(G)$

There is no trivial relation between $\lambda_{p,q}$ and $\vec{\lambda}_{p,q}$. Indeed, the graphs H_1 and H_2 depicted in figure 2 are such that $\lambda_{p,q}(H_1) < \vec{\lambda}_{p,q}(H_1)$ and $\lambda_{p,q}(H_2) > \vec{\lambda}_{p,q}(H_2)$.

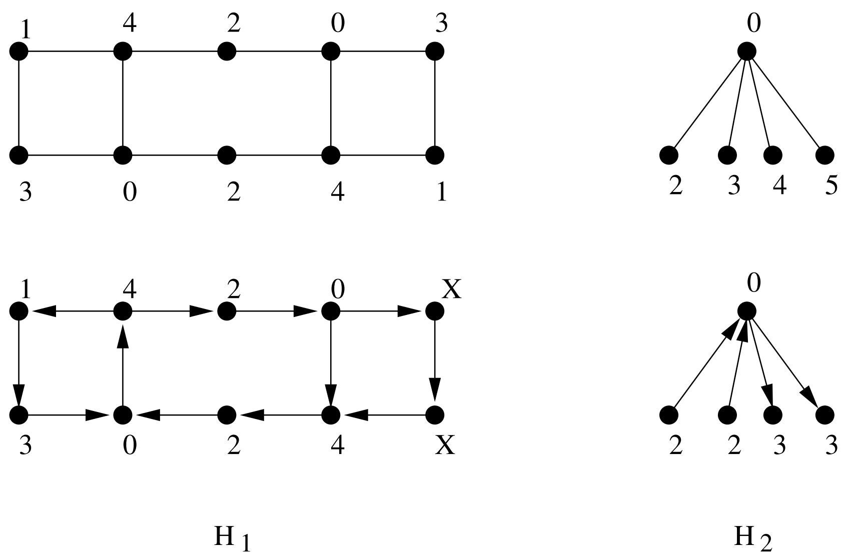

FIG. 2 – The graphs H_1 and H_2

For a compelete graph K_n, $\vec{\lambda}_p(K_n) = (\vec{\chi}(K_n) - 1)p$. Moreover, $\vec{\lambda}_2(C_5) = 4 = \vec{\chi}(C_5) - 1$. Though the bounds in the Lemma 1 (i) is tight for certain graphs, we could significantly improve this result for the class of trees.

Oriented $L(p, q)$-Labelling for Trees

A *star*, S, is a tree with a special vertex x and all other vertices of S are adjacent to x. A *double star*, D, is a tree with a special pair of adjacent vertices x, y and all other vertices of D are adjacent to either x or y (see Figure 3).

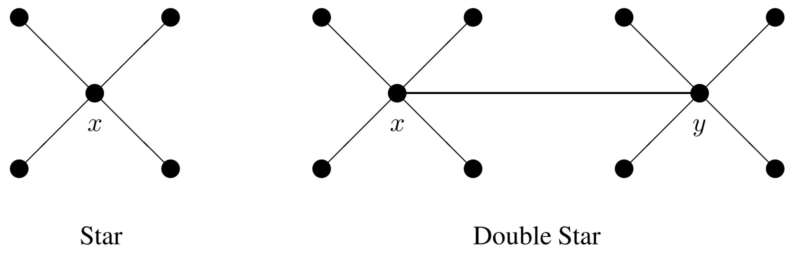

FIG. 3 – Star and Double Star

We denote P_k the paths with k vertices. Any tree $T \neq K_1, K_2$ has a P_3 as a subgraph. A tree T ($\neq K_1, K_2$) with no P_5 as a subgraph is either a star or a double star. The minimal span for the oriented $L(p,q)$-labelling of an unoriented tree T is easily computable with the following theorem.

Theorem 1 *Let T be a unoriented tree. Then*

$$\vec{\lambda}_{p,q}(T) = \begin{cases} 0 & \text{if } T = K_1, \\ p & \text{if } T = K_2, \\ p+q & \text{if } T \text{ is a star or a double star}, \\ p+2q & \text{else (i.e. } P_5 \text{ is a subgraph of } T\text{).} \end{cases}$$

The two first cases are trivial. We begin by proving that in the two last cases the span cannot be decreased. Since $\vec{\lambda}_{p,q}(H) \leq \vec{\lambda}_{p,q}(G)$ if H is a subgraph of G, we just have to note that $\vec{\lambda}_{p,q}(P_3) = p+q$ and $\vec{\lambda}_{p,q}(P_5) = p+2q$. Now we show how to label the trees.

Case 1 Let T be a star with a special vertex x. We construct an oriented $L(p,q)$-labelling of $\vec{T}$ by assigning 0 to the special vertex x, p to all vertices of $N^+_{\vec{H}}(x)$, and $p+q$ to all vertices of $N^-_{\vec{H}}(x)$.

Case 2 Let T be a double star with special vertices x, y. Without loss of generality, assume that $\vec{xy} \in E(\vec{H})$. We construct an oriented $L(p,q)$-labelling of $\vec{H}$ by assigning 0 to all vertices of $N^-_{\vec{H}}(y)$, q to all vertices of $N^+_{\vec{H}}(y)$, p to all vertices of $N^-_{\vec{H}}(x)$, and $p+q$ to all vertices of $N^+_{\vec{H}}(x)$.

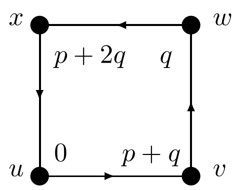

FIG. 4 – The graph $\vec{C}_4$

Case 3 Let $\vec{T}$ be an orientation of T. To prove the upperbound, we construct a homomorphism $f : \vec{T} \longrightarrow \vec{C}_4$. In Figure 4, u, v, w, and w are vertices of $\vec{C}_4$ and $0, p+q, q, p+2q$ are their respective labels in a $p+2q$-oriented $L(p,q)$-labelling of $\vec{C}_4$. First, we pick an arbitaray vertex a of $\vec{T}$ and map a to u (i.e., $f(a) = u$). Then, we map vertices of $N^+_{\vec{T}}(a)$ (respectively $N^-_{\vec{T}}(a)$) into $N^+_{\vec{C}_4}(u) = \{v\}$ (respectively $N^-_{\vec{C}_4}(u) = \{x\}$). We continue this process until all vertices of $\vec{T}$ are mapped into $V(\vec{C}_4)$.

Oriented $L(p,1)$-Labelling of Bipartite Graphs

Lemma 2 *For a complete bipartite graph $K_{m,n}$ $(m,n \geq 1)$,* $\vec{\lambda}_p(K_{m,n}) \leq m+n+p-2$.

Proof Let $V(K_{m,n}) = A \cup B$ where $A = \{u_1, u_2, ..., u_m\}$ and $B = \{v_1, v_2, ..., v_n\}$. We construct an oriented $(m+n+p-2)$-labelling of an arbitrary orientation of $K_{m,n}$ by a function $l : V(K_{m,n}) \longrightarrow \{0, 1, ..., m+n+p-2\}$ defined as $l(u_i) = i-1$ for $1 \leq i \leq m$ and $l(v_j) = m+j+p-2$ for $1 \leq j \leq n$. Hence $\vec{\lambda}_p(K_{m,n}) \leq m+n+p-2$. □

It is not hard to show that the upper bound in the Lemma is tight if $m = n$, i.e., $\vec{\lambda}_p(K_{n,n}) = 2n+p-2$. Note that every bipartite graph is a subgraph of a complete bipartite graph. Hence, for a bipartite graph G, $\vec{\lambda}_p(G) \leq \mid V(G) \mid +p-2$.

Oriented $L(p,1)$-Labelling and the Acyclic Chromatic Number

In this section, we supply an upperbound of $\vec{\lambda}_p$ of planar graphs based on a method developed by Alon, Marshall, Nesetril, Raspaud and Sopena [2, 15, 17]. In fact, they found an oriented homorphism from an oriented k-acyclic graph to a special graph, $\vec{M}_k$.

Special graph $\vec{M}_k$

Let $\vec{M}_k$ be an oriented graph with vertex set $V(M_k) = \{\ (i,\ a_1,\ a_2,...,a_{i-1},a_{i+1},...,a_k\ )\ :\ 1 \leq i \leq k$ and $a_j \in \{0,1\}\ \}$. The edge set of $\vec{M}_k$ is defined as follows. Let $x = (i,\ a_1,\ a_2,...,a_{i-1},a_{i+1},...,a_k\ )$ and $y = (l,\ b_1,\ b_2,...,b_{l-1},b_{l+1},...,b_k\ )$, $1 \leq i < l \leq k$, be two vertices. Then (i) $\vec{xy} \in E(\vec{M}_k)$ if $\overrightarrow{a_l \bar{b}_i} \in E(\vec{T})$ and (ii) $\vec{yx} \in E(\vec{M}_k)$ if $\overrightarrow{\bar{b}_i a_l} \in E(\vec{T})$ (see Figure 5 for $\vec{T}$).

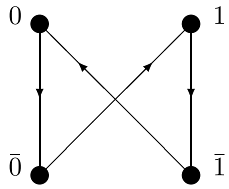

FIG. 5 – The graph $\vec{T}$

Theorem 2 *[15] Let $\vec{G}$ be an orientation of a k-acyclic graph G. Then there exists an oriented homomorphism from $\vec{G}$ to $\vec{M}_k$.*

Lemma 3 $\vec{\lambda}_p(\vec{M}_k) \leq k(2^{k-1}-1) + p(k-1)$.

Proof Let $V(\vec{M}_k) = \cup_{i=1}^{k} V_i$, where $V_i = \{\ (i,\ a_1,\ a_2,...,a_{i-1},a_{i+1},...,a_k)\ :\ a_j \in \{0,1\}\ \}$. Note that $\mid V_i \mid = 2^{k-1}$. Then we re- label the vertices of V_i as $\{\ v_{i,j} : 1 \leq j \leq 2^{k-1}\ \}$, $1 \leq i \leq k$. Now, it is easy to prove that the map $f : V(\vec{M}_k) \longrightarrow \{0, 1, ..., k2^{k-1} + (k-1)(p-1) - 1\}$ such that $f(v_{i,j})$

$= (i-1)(2^{k-1}+p-1)+j-1,\ \ 1 \le i \le k$ and $1 \le j \le 2^{k-1}$, is an oriented $L(p,1)$-labelling of $\vec{M}_k$. Hence $\vec{\lambda}_p(\vec{M}_k) \le k(2^{k-1}-1)+p(k-1)$. □

Theorem 3 *Let G be a k-acyclic graph. Then $\vec{\lambda}_p(G) \le k(2^{k-1}-1)+p(k-1)$.*

Proof Let $\vec{G}$ be an orientation of G. By Theorem 2, there exists an oriented homomorphism from $\vec{G}$ to $\vec{M}_k$. Then $\vec{\lambda}_p(\vec{G}) \le \vec{\lambda}_p(\vec{M}_k)$. Since $\vec{G}$ is an arbitrary orientation of G, $\vec{\lambda}_p(G) \le \vec{\lambda}_p(\vec{M}_k)$. By Lemma 3, $\vec{\lambda}_p(G) \le k(2^{k-1}-1)+p(k-1)$. □

A well-known result of Borodin [3] states that any planar graph is 5-acyclic colorable. In [5], the authors proved that planar graphs with girth at least 5 (resp. 7) are 4-acyclic colorable (resp. 3-acyclic colorable). Moreover it is well known that graphs with treewidth k are $(k+1)$-acyclic colorable. So we have the following corollary.

Corollary 1 *If G is a planar graph, then $\vec{\lambda}_p(G) \le 75+4p$.*
If G is a planar graph with girth at least 5, then $\vec{\lambda}_p(G) \le 28+3p$.
If G is a planar graph with girth at least 7, then $\vec{\lambda}_p(G) \le 9+2p$.
If G is a graph with treewidth k, then $\vec{\lambda}_p(G) \le (k+1)(2^k-1)+pk$.

$L(p,q)$-Labelling of Oriented Graphs

In [9], the authors conjectured that for an unoriented graph G, $\lambda_{2,1}(G) \le \Delta^2$, where Δ is the maximum degree of G. Much work [6, 13, 8] have been done on bounding $\lambda_{p,q}$ by a function of Δ. Here, we prove a similar result for oriented graphs.

Theorem 4 *For every directed graph $G=(V,A)$ with maximal degree Δ, $\lambda^o_{p,1}(G) \le \left\lfloor \frac{\Delta^2}{2} \right\rfloor + p\Delta$.*

In a directed graph $G=(V,A)$, u is a 2-*neighbor* of v if there is a directed 2-path between u and v. Given a directed graph $G=(V,A)$, its 2-*paths graph* $G^2=(V,E)$ is an unoritented graph with the same vertex set. There is an edge uv in this graph if and only if there is a directed 2-path in G linking u and v. The next lemma gives an interesting property of these graphs.

Lemma 4 *For every directed graph $G=(V,A)$ with maximal degree Δ, its 2-paths graph $G^2 = (V,E)$ is $\left\lfloor \frac{\Delta^2}{2} \right\rfloor$-degenerate.*

Proof
We prove that for any $S \subseteq V$, the induced graph $G^2[S]$ has minimal degree at most $\left\lfloor \frac{\Delta^2}{2} \right\rfloor$. We do so, using a discharging method. Let the initial charge $\gamma(v)$ of the vertices be $\frac{\Delta^2}{2}$ if $v \in S$, or 0 if $v \notin S$. The total charge of the graph is $\frac{\Delta^2}{2}|S|$. Then we proceed to the following discharging step,

every vertex of S gives the charge $\frac{\Delta}{2}$ to each of its neighbors. We denote γ^* the new charge of the vertices of G. Note that a vertex v with k neighbors in S has charge $\gamma^*(v)$ at least $\frac{k\Delta}{2}$.

Now consider the number of oriented 2-paths going through a vertex $v \in V(G)$ and linking two vertices in S. Let us denote this number $\pi_S(v)$. Note that for a vertex v with k neighbors in S, we have that $\pi_S(v) \leq \max\{i \times j, i + j = k\} = \left\lfloor \frac{k^2}{4} \right\rfloor$, where i and j are respectively the number of incoming and outgoing arcs. Since $k \leq \Delta$ we have that $\pi_S(v) \leq \left\lfloor \frac{k^2}{4} \right\rfloor \leq \frac{k\Delta}{4} \leq \frac{\gamma^*(v)}{2}$. This implies that the number of edges in $G^2[S]$, $\sum_{v \in V(G)} \pi_S$, is at most $\frac{1}{2} \sum_{v \in V(G)} \gamma^*(v)$. Since the discharging does not change the total charge of the graph, we have that $\sum_{v \in V(G)} \pi_S \leq \frac{1}{2} \sum_{v \in V(G)} \gamma(v) = \frac{1}{2} \times \frac{\Delta^2}{2}|S|$. So we have that the sum of the degrees in $G^2[S]$ is at most $\frac{\Delta^2}{2}|S|$, which implies that there is a vertex with degree at most $\left\lfloor \frac{\Delta^2}{2} \right\rfloor$ in $G^2[S]$. □

This lemma implies that there is an order $v_1, v_2, ..., v_n$ on the vertices of $G = (V, A)$, such that for every $i \leq n$, the vertex v_i has at most $\left\lfloor \frac{\Delta^2}{2} \right\rfloor$ 2-neighbors v_j, with $j < i$. Given this order on the vertices, we consider the following algorithm :

$i = 0$;
while *there are unlabelled vertices* **do**
  for $v_j = v_1$ **to** v_n **do**
    if v_j *is unlabelled* **and** v_j *can be labelled* i **then**
      let v_j be labelled i;
    end
  end
  $i = i + 1$;
end

Now consider the last vertex being labelled by this algorithm, say v with label k. What could prevent it to be labelled with the value $x < k$, when the algorithm considered the possibility (i.e. when $i = x$ and $v_j = v$) ? It is either a neighbor of v that was already labelled with the label l, with $x - p < l \leq x$, or a 2-neighbor of v that was labelled x. Note that if a 2-neighbor of v is labelled x before the possibility was offered to v, it implies that this 2-neighbor appears before v in the order. So the 2-neighbors of v posterior to v in the order cannot prohibit a value to v. Since v has at most Δ neighbors and at most $\left\lfloor \frac{\Delta^2}{2} \right\rfloor$ 2-neighbors appearing before v in the order, at most $\left\lfloor \frac{\Delta^2}{2} \right\rfloor + p\Delta$ values were refused to v. This implies that $k \leq \left\lfloor \frac{\Delta^2}{2} \right\rfloor + p\Delta$. Note that this implies that the algorithm labels the graph in time $O(\Delta^2 n)$.

Conclusion

In this article, we have explored the role of "orientation" in FAP by extending $L(p,q)$-labelling to oriented graphs and introducing oriented $L(p,q)$-labelling. We have computed upper bounds of oriented $L(p,q)$-labelling of trees, bipartite graphs and planar graphs. Note that bounds of $L(p,q)$-labelling of a tree depends on its maximum degree but bounds of oriented $L(p,q)$-labelling depends on its structure (see Theorem 1). It indicates that an oriented version of labelling may provide more structrual information of concerned network of FAP than its non-oriented version.

Références

[1] K. I. Aardal, S. P.M. V. Hossel, A.M.C. A. Koster, C. Mannino and A. Sassano, *Models and Solution Techniques for Frequency Assignment Problems*,(available at http ://fap.zib.de/survey/)

[2] N. Alon and T. H. Marshall, *Homomorphisms of edge-coloured graphs and Coxeter groups*, J. Algebraic Combin. **8** (1998) 5-13.

[3] O.V. Borodin, *On acyclic colorings of planar graphs*, Discrete Math. **26** (1979), 211-236.

[4] O. V. Borodin, A. V. Kostochka, J. Nešetřil, A. Raspaud and E. Sopena, *On the maximum average degree and the oriented chromatic number of a graph*, Discrete Math. **206** (1999) 77-89.

[5] O.V. Borodin, A.V. Kostochka, and D.R. Woodall, *Acyclic colourings of planar graphs with large girth*, J. London Math. Soc. (2). **60** (1999) 344-352.

[6] G.J. Chang, Wen-Tsai Ke, D. Kuo, D. D.F. Liu and R. K. Yeh,*On L(d,1)-labelling of graphs*, Discrete Math. **220** (2000) 57-66.

[7] J.P. Georges and D.W. Mauro, *Generalized vertex labelings with a condition at a distance two*, Congr. Numer., **109** (1994), 141-159.

[8] D. Gonçalves, *On the L(d,1)-labelling of graphs*, submitted.

[9] J.R. Griggs and R.K. Yeh, *Labelling graphs with a condition at a distance two*, SIAM J. Discrete Math., **5** (1992), 586-595.

[10] W.K. Hale, *Frequency assignment : Theory and application,* Proc. IEEE, **68** (1980), 1497-1514.

[11] J.C. M. Janssen, *Channel assignment and graph labeling*, Handbook of Wireless Networks and Mobile Computing(Editor : Ivan Stojmenvoic), John Wiley and Sons,Inc., Chapter 5, (2002) 95-117.

[12] A. V. Kostochka, E. Sopena and X. Zhu, *Acyclic and oriented chromatic numbers of graphs*, J. Graph Theory **14**, No.4 (1997) 331-340.

[13] D. Král and R. Škrekovski, A theorem about the channel assignement problem, SIAM J.Discrete Math., 16 (2003), pp. 426-43.

[14] B.H. Metzger, *Spectrum management technique*, Fall 1970, Presentation at 38th National ORSA meeting (Detroit, MI)

[15] J. Nešetřil and A. Raspaud, *Colored homomorphisms of colored mixed graphs*, J. Combin. Theory Ser. B, *80* (2000) 147-155.

[16] J. Nešetřil, A. Raspaud and E. Sopena, Colorings and girth of oriented planar graphs, Discrete Math.**165, 166**, Nos. 1-3 (1997), 519-530.

[17] A. Raspaud and E. Sopena, *Good and semi-strong colorings of oriented planar graphs*, Inform. Process. Lett. **51** (1994) 171-174.

[18] E. Sopena, *The chromatic number of oriented graphs*, J. Graph Theory **25** (1997) 191-205.