Every planar graph is the intersection graph of segments in the plane (full version)

Jérémie Chalopin[1] and Daniel Gonçalves[2]

[1] LIF, CNRS et Aix-Marseille Université, CMI, 39 rue Joliot-Curie 13453 Marseille Cedex 13, France.
[2] LIRMM, CNRS et Université Montpellier 2, 161 rue Ada 34392 Montpellier Cedex 05, France.

Abstract. Given a set S of segments in the plane, the *intersection graph* of S is the graph with vertex set S in which two vertices are adjacent if and only if the corresponding two segments intersect. We prove a conjecture of Scheinerman (*PhD Thesis, Princeton University*, 1984) that every planar graph is the intersection graph of some segments in the plane.

1 Introduction

In this paper, we consider intersection models for planar graphs. A *segment model* of a graph G maps every vertex $v \in V(G)$ to a segment $\mathbf{v}$ of the plane so that two segments $\mathbf{u}$ and $\mathbf{v}$ intersect if and only if $uv \in E(G)$. Although this graph family is simply defined, it is not easy to manipulate. Actually, even if this class of graphs is small (there are less than $2^{O(n \log n)}$ such graphs with n vertices [15]) a segment model may be long to encode (in the models of some of these graphs the endpoints of the segments need at least $2^{\sqrt{n}}$ bits to be coded [13]). There are also interesting open problems concerning this class of graphs. For example, we know that deciding whether a graph G admits a segment model is NP-hard [11] but it is still open whether this problem belongs to NP or not. Here we focus on a conjecture proposed by Scheinerman [16], stating that every planar graph has a segment model.

Many work has been done toward this conjecture. Several proofs [3,5,9] have been given for bipartite planar graphs. The case of triangle-free planar graphs was proved by de Castro *et al.* [1] and recently de Fraysseix and Ossona de Mendez [4] proved it for every planar graph that has a 4-coloring in which every induced cycle of length 4 uses at most 3 colors.

Another approach to this problem has been proposed [12,14]. Since it is known [6] that planar graphs are intersection graphs of Jordan arcs in the plane and since two non-parallel segments intersect at most once, it was asked whether planar graphs are intersection graphs of Jordan arcs in the plane if every pair of Jordan arcs $\mathbf{s}_1$ and $\mathbf{s}_2$ intersect at most once and in a non-tangent way (*i.e.* around their intersection point we successively meet $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_1$ and $\mathbf{s}_2$). It was already known when tangent intersection are allowed; indeed every planar graph is the contact graph of touching circles [10]. The authors and Ochem [2] answered positively to this question. This approach of Scheinerman's conjecture was decisive since by improving the proof of this result it yields a proof of Scheinerman's conjecture that we present here. However, the construction we give here does not exactly correspond to a stretching of the strings of the construction given in [2].

The paper is organized as follows. In Section 2 we give some definitions. In particular we define premodels and we explain how to obtain a segment model from a premodel. In Section 3 we construct premodels for 3-bounded W-triangulations, a family of plane graphs including 4-connected triangulations. Then in Section 4 we finally construct segment models for general triangulations, which implies the existence of segment models for general planar graphs.

2 Preliminaries

A *plane graph* is an embedded planar graph. Given a plane graph G, let $V(G)$, $E(G)$ and $F(G)$ be respectively the sets of vertices, edges and inner faces of G. A *near-triangulation* is a plane graph in which every inner face is a triangle. A *triangulation* is a near-triangulation with a triangular outer face. It is easy to see that every planar graph is the induced subgraph of some triangulation. This implies that it is sufficient to consider triangulations. Indeed if a planar graph G is isomorphic to the graph induced by a set $V(G) \subseteq V(T)$ of vertices in a triangulation T, then by removing the segments corresponding to $V(T) \setminus V(G)$ from a segment model of T, we clearly obtain a segment model of G.

In all the paper, the bold notations correspond to geometrical objects like points, segments or lines. For example we will usually denote by $\mathbf{v}$ the segment corresponding to a vertex v and by $(\mathbf{v})$ the line prolonging this segment. Furthermore since we consider finite planar graphs, the segment sets we consider are all finite. Given a segment set S, its set of *representative points* Rep_S is the set that contains the intersection points and the ends of the segments in S. A segment set S is *unambiguous* if every segment $\mathbf{s} \in S$ has distinct endpoints, and if parallel segments of S do not intersect. From now on we use the following definition of model.

Definition 2.1. *Given a segment set S, its* intersection graph G_S *is the graph with vertex set S and where two vertices are adjacent if and only if the corresponding segments intersect. Furthermore if (1) S is unambiguous, if (2) the intersection of any three segments of S is empty, and if (3) every endpoint belongs to exactly one segment, then S is a* model *for any graph G isomorphic to G_S.*

For the proof in Section 4 we need some geometrical structures to represent the triangular inner faces. To each triangular inner face abc we will associate a *face segment*, $\underline{\mathbf{ab}}\mathbf{c}$, $\underline{\mathbf{ac}}\mathbf{b}$ or $\underline{\mathbf{bc}}\mathbf{a}$.

Definition 2.2. *Given an unambiguous segment set S and three pairwise intersecting segments $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, a* face segment $\mathbf{f} = \underline{\mathbf{ab}}\mathbf{c}$ *is a segment $[\mathbf{p}, \mathbf{q}]$ such that:*

- *$\mathbf{p}$ is the intersection point of $\mathbf{a}$ and $\mathbf{b}$, and going around $\mathbf{p}$ we consecutively meet $\mathbf{a}$, $\mathbf{f}$ and $\mathbf{b}$,*
- *$\mathbf{q}$ is an internal point of $\mathbf{c}$ that does not belong to any other segment of S, and*
- *none of its internal points belongs to any segment of S.*

The points $\mathbf{p}$ and $\mathbf{q}$ are respectively called the cross-end *and the* flat-end *of $\underline{\mathbf{ab}}\mathbf{c}$.*

Note that the second item implies that face segments are non-trivial, *i.e.* $\mathbf{p} \neq \mathbf{q}$. Note also that in this definition $\mathbf{a}$ and $\mathbf{b}$ play the same role, so a face segment $\underline{\mathbf{ab}}\mathbf{c}$ is also a face segment $\underline{\mathbf{ba}}\mathbf{c}$ but it is not a face segment $\underline{\mathbf{ac}}\mathbf{b}$.

Definition 2.3. *Given an unambiguous segment set S, two face segments $\mathbf{f}_1$ and $\mathbf{f}_2$ on S are* non-interfering *if one of the following holds:*

- *The segments $\mathbf{f}_1$ and $\mathbf{f}_2$ do not intersect.*
- *The segments $\mathbf{f}_1$ and $\mathbf{f}_2$ have the same cross-end $\mathbf{p}$ and this point is the intersection point of exactly two segments of S, $\mathbf{a}$ and $\mathbf{b}$. Furthermore, one of the lines $(\mathbf{a})$ and $(\mathbf{b})$ separates $\mathbf{f}_1$ and $\mathbf{f}_2$ in distinct half-planes.*

Definition 2.4. *A* full model *of a near triangulation T is a couple $\mathcal{M} = (S, F)$ of segments sets such that:*

- *S is a model of T.*
- *F is a set of non-interfering face segments on S such that for each inner face abc of T, F contains one of the following face segments: $\underline{\mathbf{ab}}\mathbf{c}, \underline{\mathbf{ac}}\mathbf{b}, \underline{\mathbf{bc}}\mathbf{a}$.*
- *$S \cup F$ is unambiguous.*

The next theorem is the main result of the paper.

Theorem 2.5. *Every triangulation T has a full model $\mathcal{M} = (S, F)$.*

2.1 Premodels

In our proofs, we use a different kind of model. The main difference with full models is that more than two segments of S can intersect in a same point.

In the following, we consider a segment set S and a set F of non-interfering face segments on S, where $S \cup F$ is unambiguous. Let us denote the segments of S (resp. F) by $\mathbf{s}_1, \mathbf{s}_2, \dots$ (resp. $\mathbf{f}_1, \mathbf{f}_2, \dots$). Given a representative point $\mathbf{p}$, its *incidence sequence* $\mathcal{I}(\mathbf{p})$ is the undirected circular sequence of segments (from $S \cup F$) we meet by going around $\mathbf{p}$. This sequence is undirected because it will make no difference going clockwise or anti-clockwise. By extension, the *partial topological incidence sequence* of $\mathbf{p}$, $\mathcal{I}^*(\mathbf{p})$ is the sequence obtained in the following way. Prolong every segment that ends at $\mathbf{p}$ and consider its new incidence sequence. Then replace every occurrence of $\mathbf{s}_i$ and $\mathbf{f}_i$ that was not in $\mathcal{I}(\mathbf{p})$ before by $(\mathbf{s}_i)$ and $(\mathbf{f}_i)$. It is clear that $\mathcal{I}(\mathbf{p})$ is a subsequence of $\mathcal{I}^*(\mathbf{p})$ (*i.e.* $\mathcal{I}(\mathbf{p}) \subseteq \mathcal{I}(\mathbf{p})$). We say that $\mathcal{I}(\mathbf{p})$ is of the form $([\mathbf{r}_1], \mathbf{r}_2, \dots, \mathbf{r}_k)$ for $\mathbf{r}_i \in S \cup F$, if either $\mathcal{I}(\mathbf{p}) = (\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k)$, $\mathcal{I}(\mathbf{p}) = (\mathbf{r}_2, \dots, \mathbf{r}_k)$, or $\mathcal{I}(\mathbf{p}) \subseteq ((\mathbf{r}_1), \mathbf{r}_2, \dots, \mathbf{r}_k) \subseteq \mathcal{I}^*(\mathbf{p})$.

Let us define types for the representative points, depending on their incidence sequence. These types are not always entirely determined by the incidence sequence and we will have to assign a type (among the possible ones) to each representative point. Furthermore, these types are in correspondence with some graphs we also describe here.

- A point is a *segment end* if its incidence sequence is $(\mathbf{s}_1)$. The corresponding graph is the single vertex s_1.
- A point is a *flat face segment end* if its incidence sequence is $(\mathbf{s}_1, \mathbf{f}_1, \mathbf{s}_1)$. The corresponding graph is the single vertex s_1.
- A point may be a *crossing* if it has an incidence sequence of the form $(\mathbf{s}_1, [\mathbf{f}_1], \mathbf{s}_2, [\mathbf{f}_2], \mathbf{s}_1, [\mathbf{s}_2])$ or $(\mathbf{s}_1, [\mathbf{f}_1], \mathbf{s}_2, \mathbf{s}_1, [\mathbf{f}_2], \mathbf{s}_2)$. The corresponding graph is the edge $s_1 s_2$.
- A point may be a *path–$(s_1, s_2, \ldots, s_k)$–point* with $k \geq 2$, if it has an incidence sequence of the form $(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k, (\mathbf{s}_1), (\mathbf{s}_2))$ (See Figure 1). Such a typed point is in correspondence with *path–$(s_1, s_2, \ldots, s_k)$*, the graph with vertex set $\{s_1, \ldots, s_k\}$ and edge set $\{s_i s_{i+1} \mid 1 \leq i < k\}$.

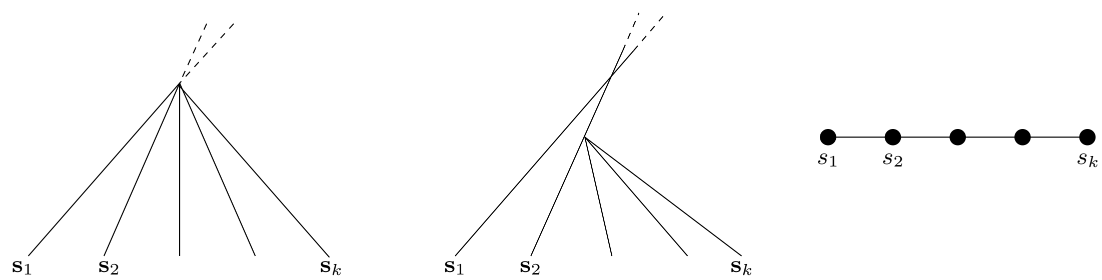

Fig. 1. A path–$(s_1, s_2, \ldots, s_k)$–point, its partial realization, and its corresponding graph

- A point may be a *fan–$s_1 \triangleleft (s_2, \ldots, s_k)$–point* with $k \geq 2$, if it has an incidence sequence of the form $(\mathbf{s}_1, [\mathbf{f}_1], \mathbf{s}_2, \ldots, \mathbf{s}_k, (\mathbf{s}_1), [\mathbf{f}_1], (\mathbf{s}_2))$ (See Figure 2), with $\mathbf{f}_1 = \underline{\mathbf{s_1 s_2}}\mathbf{x}$. Note that since $\mathbf{f}_1$ is a face segment it occurs at most once in the incidence sequence. Such a typed point is in correspondence with *fan–$s_1 \triangleleft$ $(s_2, \ldots, s_k)$*, the graph with a vertex s_1 dominating a path $(s_2, \ldots, s_k)$.

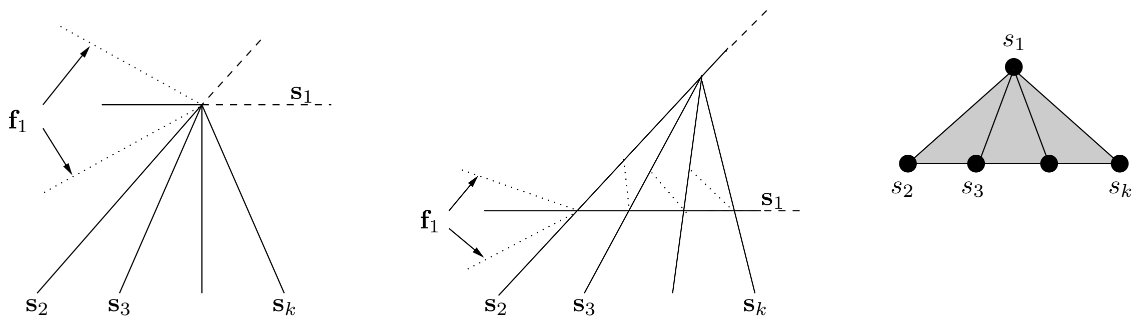

Fig. 2. A fan–$s_1 \triangleleft (s_2, \ldots, s_k)$–point, its partial realization , and its corresponding graph

- A point may be a *fan-path–$s_1 \triangleleft (s_2, \ldots, s_i) \cdot (s_i, \ldots, s_k)$–point* with $2 \leq i \leq k$, if it has an incidence sequence of the form $(\mathbf{s}_1, \ldots, \mathbf{s}_i, \ldots, \mathbf{s}_k, (\mathbf{s}_1), (\mathbf{s}_i))$ (See Figure 3). Such a typed point is in correspondence with *fan-path–$s_1 \triangleleft (s_2, \ldots, s_i) \cdot (s_i, \ldots, s_k)$*, the graph with a path $(s_2, \ldots, s_k)$ and a vertex s_1 dominating the subpath $(s_2, \ldots, s_i)$.

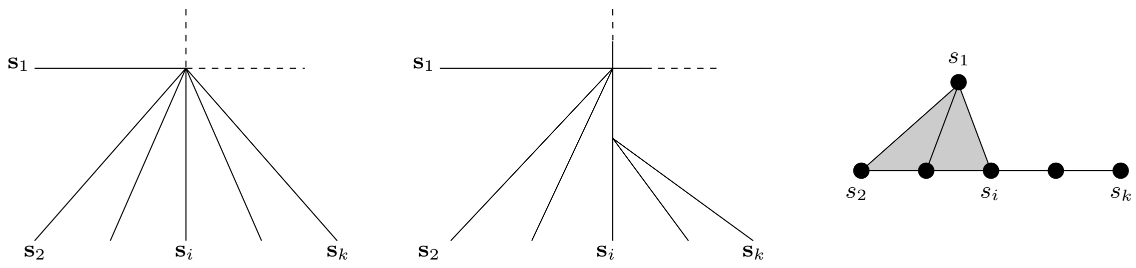

Fig. 3. A fan-path–$s_1 \triangleleft (s_2, \ldots, s_i) \cdot (s_i, \ldots, s_k)$–point, its partial realization, and its corresponding graph

- A point may be a *path-fan–*$(s_{i-1}, \ldots, s_2, s_1) \cdot s_1 \triangleleft (s_i, \ldots, s_k)$*–point* with $2 \leq i \leq k$, if it has an incidence sequence of the form $(\mathbf{s}_1, \ldots, \mathbf{s}_i, \ldots, \mathbf{s}_k, (\mathbf{s}_1), (\mathbf{s}_i))$ (See Figure 4). Such a typed point is in correspondence with *path-fan–*$(s_{i-1}, \ldots, s_2, s_1) \cdot s_1 \triangleleft (s_i, \ldots, s_k)$, the graph with two paths $(s_1, \ldots, s_{i-1})$ and $(s_i, \ldots, s_k)$, where s_1 dominates the second path.

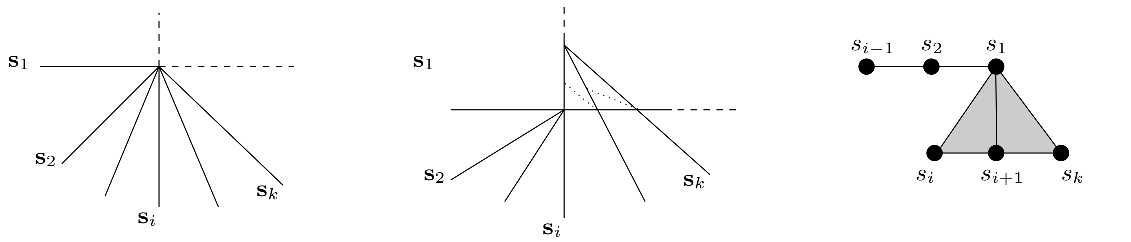

Fig. 4. A path-fan–$(s_{i-1}, \ldots, s_2, s_1) \cdot s_1 \triangleleft (s_i, \ldots, s_k)$–point, its partial realization, and its corresponding graph

- A point may be a *double-fan–*$s_1 \triangleleft (s_2, \ldots, s_i) \cdot s_i \triangleleft (s_{i+1}, \ldots, s_k, s_1)$*–point* with $2 \leq i \leq k$, if it has an incidence sequence of the form $(\mathbf{s}_1, \ldots, \mathbf{s}_i, \ldots, \mathbf{s}_k, (\mathbf{s}_1), (\mathbf{s}_i))$ (See Figure 5). Such a typed point is in correspondence with *double-fan–*$s_1 \triangleleft (s_2, \ldots, s_i) \cdot s_i \triangleleft (s_{i+1}, \ldots, s_k, s_1)$, the graph with two paths $(s_2, \ldots, s_i)$ and $(s_{i+1}, \ldots, s_k, s_1)$, where s_1 and s_i respectively dominate the first and the second path.

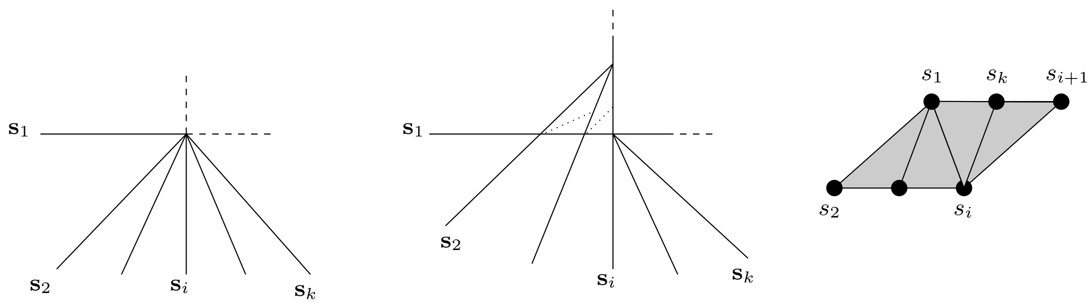

Fig. 5. A double-fan–$s_1 \triangleleft (s_2, \ldots, s_i) \cdot s_i \triangleleft (s_{i+1}, \ldots, s_k, s_1)$–point, its partial realization, and its corresponding graph

Actually, the graphs we considered here are plane graphs, and their inner faces are the grey faces in the figures. As in [4], we need a bipartite digraph to describe the constraints between segments and representative points.

Definition 2.6. *Given a segment set R, the* constraints digraph $Const_R$ *is the bipartite digraph with vertex sets R and Rep_R, and where $\mathbf{r} \in R$ and $\mathbf{p} \in Rep_R$ are linked if and only if $\mathbf{p} \in \mathbf{r}$. More precisely, there is an arc from $\mathbf{p}$ to $\mathbf{r}$ if $\mathbf{p}$ is an endpoint of $\mathbf{r}$, otherwise (when $\mathbf{p}$ is an internal point of $\mathbf{r}$) the arc goes from $\mathbf{r}$ to $\mathbf{p}$.*

Informally this graph describes the fact that the position of a segment is determined by its endpoints, and determines the position of its internal representative points.

Definition 2.7. *Given a segment set S, a set F of non-interfering face segments on S and a function τ that assigns a type to each representative point, the triple $\mathcal{M} = (S, F, \tau)$ is a premodel of a near-triangulation T if the following holds:*

- *The set $S \cup F$ is unambiguous and the digraph $Const_{S \cup F}$ is acyclic.*
- *A vertex $a \in V(T)$ if and only if $\mathbf{a} \in S$.*
- *An edge $ab \in E(T)$ if and only if $\mathbf{a}$ and $\mathbf{b}$ intersect in a point $\mathbf{p}$ such that the graph corresponding to $\tau(\mathbf{p})$ contains the edge ab.*
- *A face $abc \in F(T)$ if and only if one of the following holds:*
  - *either there exists a face segment $\underline{\mathbf{ab}}\mathbf{c}$, $\underline{\mathbf{ac}}\mathbf{b}$ or $\underline{\mathbf{bc}}\mathbf{a}$ in F,*
  - *or, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ intersect in a point $\mathbf{p}$ such that abc is an inner face of the graph corresponding to $\tau(\mathbf{p})$.*

Note that a premodel $\mathcal{M} = (S, F, \tau)$ of a near-triangulation T has a bounded number of representative points. There are at most $2|V(T)|$ segment ends, at most $F(T)$ flat face segment ends, and at most $E(T)$ points of another type (since each of them corresponds to at least one edge of T).

Remark 2.8. If a premodel $\mathcal{M} = (S, F, \tau)$ of a near-triangulation T has $2|V(T)| + |F(T)| + |E(T)|$ representative points, then (S, F) is a full model of T.

2.2 Local Perturbations

In this subsection we describe how to transform a premodel $\mathcal{M} = (S, F, \tau)$ of a near triangulation T into a full model $\mathcal{M}' = (S', F')$ of T. In the following the segments denoted by $\mathbf{r}_i$ are segments of $S \cup F$. Let us define three basic moves: prolonging, gliding and traversing.

Lemma 2.9 (prolonging). *Consider a premodel $\mathcal{M} = (S, F, \tau)$ of a near triangulation T with an intersection point $\mathbf{p}$ which is the end of a segment $\mathbf{s}_1 \in S$. If for every segment $\mathbf{s}_2 \in S$ that has an end in $\mathbf{p}$, there is no directed path from $\mathbf{s}_2$ to $\mathbf{s}_1$ in $Const_{S \cup F}$, it is possible to prolong $\mathbf{s}_1$ across $\mathbf{p}$ without creating a cycle in $Const_{S' \cup F}$ (where S' is the new segment set). Furthermore, if the type $\tau(\mathbf{p})$ is still applicable to $\mathbf{p}$ then (S', F, τ) remains a premodel of T.*

Proof. Consider a point $\mathbf{q}$ in the line $(\mathbf{s}_1)$ across $\mathbf{p}$ and let S' be as S except that we replace $\mathbf{p}$ by $\mathbf{q}$ as an endpoint for $\mathbf{s}_1$. We choose $\mathbf{q}$ in such a way that $\mathbf{s}_1$ does not intersect a new segment, and $S' \cup F$ remains unambiguous. Now it is easy to see that $Const_{S' \cup F}$ is very similar to $Const_{S \cup F}$, we just have replaced the arc $\mathbf{ps}_1$ by the arc $\mathbf{s}_1\mathbf{p}$, added a vertex for $\mathbf{q}$, and added an arc $\mathbf{qs}_1$. Since the face segments have out-degree zero in $Const_{S' \cup F}$, a cycle in this digraph should necessarily pass through $\mathbf{s}_1$, $\mathbf{p}$ and a segment $\mathbf{s}_2 \in S$ that has an end in $\mathbf{p}$. Thus, according to the conditions on $Const_{S \cup F}$, it is clear that $Const_{S' \cup F}$ is acyclic. □

Remark 2.10. Consider a premodel $\mathcal{M} = (S, F, \tau)$ with a point $\mathbf{p}$ that is the intersection of exactly two segments from S, $\mathbf{s}_1$ and $\mathbf{s}_2$. By prolonging all the segments that end at $\mathbf{p}$ we obtain a segment set S' such that $Const_{S' \cup F}$ remains acyclic.

A segment set R is *flexible* if every representative point $\mathbf{p}$ is internal for at most two segments of R. Note that according to the defined types for every premodel $\mathcal{M} = (S, F, \tau)$, the set $S \cup F$ is flexible.

Definition 2.11. *A* move *of a segment set $R = \{\mathbf{r}_i = [\mathbf{a}_i, \mathbf{b}_i] \mid 1 \le i \le |R|\}$ is a segment set R' such that $R' = \{\mathbf{r}'_i = [\mathbf{a}'_i, \mathbf{b}'_i] \mid 1 \le i \le |R|\}$. An* interpolation *of this move is a continuous function defined for $t \in [0, 1]$ that gives a move R^t of R such that $R^0 = R$ and $R^1 = R'$.*

Lemma 2.12 (gliding). *Consider a flexible and unambiguous segment set R such that $Const_R$ is acyclic, and a representative point $\mathbf{p}$ of R. If the segments $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i$ are consecutive around $\mathbf{p}$, if all the segments $\mathbf{r}_2, \ldots, \mathbf{r}_i$ have an end at $\mathbf{p}$ and are in the same half-plane delimited by $(\mathbf{s}_1)$ (See Figure 6), and if in $Const_R$ the vertex $\mathbf{r}_1$ cannot be reached from any $\mathbf{r}_j$ with $2 \le j \le i$, then there exists a move R' with an interpolation R^t such that for every $t \in]0, 1]$:*

- *The set R^t is unambiguous and $Const_{R^t}$ is acyclic.*
- *The point $\mathbf{p}$ splits into two representative points $\mathbf{p}_1^t$ and $\mathbf{p}_2^t$, which incidence sequence are respectively $(\mathbf{r}_1^t, \mathbf{r}_2^t, \ldots, \mathbf{r}_i^t, \mathbf{r}_1^t)$ and the incidence sequence of $\mathbf{p}$ without the occurrences of $\mathbf{r}_2^t, \ldots, \mathbf{r}_i^t$.*
- *For every representative point $\mathbf{q} \neq \mathbf{p}$ of R there is a representative point $\mathbf{q}^t$ in R^t with exactly the same topological incidence sequence.*
- *There is no other representative point (i.e. $|Rep_{R^t}| = |Rep_R| + 1$).*
- *Every segment $\mathbf{r}^t \in R^t$ (resp. representative point $\mathbf{q}^t \in Rep_{R^t}$) that is not reachable from any $\mathbf{p}_1^t$ in $Const_R^t$ is static, that is $\mathbf{r}^t = \mathbf{r}$ (resp. $\mathbf{q}^t = \mathbf{q}$).*

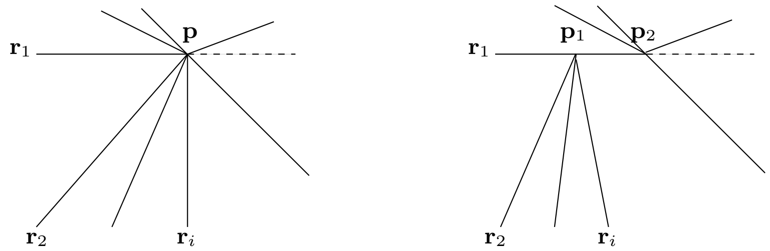

Fig. 6. gliding of $\mathbf{r}_2, \ldots, \mathbf{r}_i$ on $\mathbf{r}_1$.

Proof (of Lemma 2.12). Consider a segment $\mathbf{x} \in R$ which internal representative points have an incidence sequence of the form $(\mathbf{x}, \mathbf{y}, \mathbf{x}, \mathbf{y})$ for some $\mathbf{y} \in R$. Since $Const_R$ is acyclic, such segment necessarily exists. Now we proceed by induction on $|R|$ and consider as the initial case, the case where $i = 2$ (only one segment $\mathbf{r}_2$ is gliding on $\mathbf{r}_1$) and $\mathbf{x} = \mathbf{r}_2$. Since R is finite there exists a real $\epsilon > 0$ such that (1) every representative point $\mathbf{q} \notin \mathbf{x}$ of R verifies $dist(\mathbf{q}, \mathbf{x}) > \epsilon$ and (2) every segment $\mathbf{y} \neq x$ incident to the other end of x verifies $dist(\mathbf{p}, (\mathbf{y})) > \epsilon$ (where $dist$ is the euclidean distance). It is now clear in Figure 7 that there is a convenient move R' (with an interpolation R^t) in which only $\mathbf{x}$ is modified. Actually just one end of $\mathbf{x}$ moves continuously on $\mathbf{r}_1$ from p to p_1, a point of $\mathbf{r}_1$ such that $dist(\mathbf{p}, \mathbf{p}_1) < \epsilon$. Let us now verify that R' and R^t follow the requirements of the lemma. Consider any $t \in]0, 1]$.

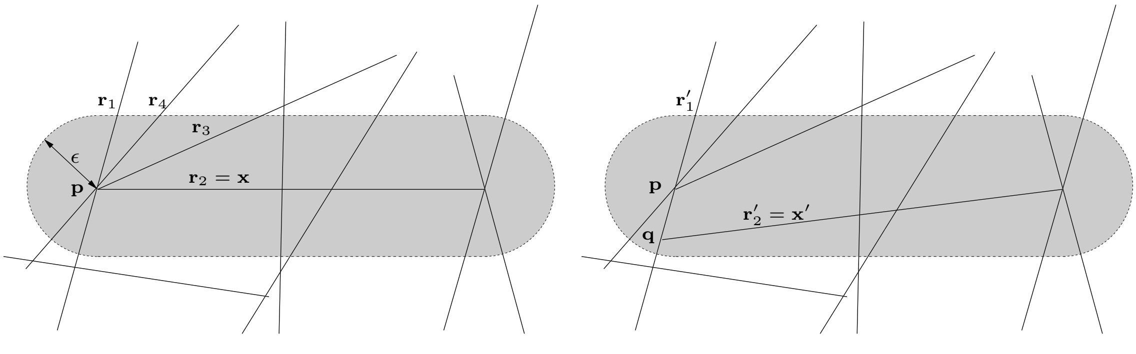

Fig. 7. Around $\mathbf{x} = \mathbf{r}_2$ when $i = 2$.

- The condition (2) in the definition of ϵ ensures us that R^t is unambiguous. Moreover, since $Const_{R^t \setminus \{\mathbf{x}^t, \mathbf{p}_2^t\}}$ is a subdigraph of $Const_R$ that is acyclic, any cycle of $Const_{R^t}$ should pass through $\mathbf{x}^t$. But since all its internal representative points have out-degree zero in $Const_{R^t}$ (because we are about to show that their incidence sequence is of the form $(\mathbf{x}^t, \mathbf{y}^t, \mathbf{x}^t, \mathbf{y}^t)$) there is no such cycle and $Const_{R^t}$ is acyclic.
- It is clear, since only one segment is moving, that the incidence sequences of $\mathbf{p}_1^t$ and $\mathbf{p}_2^t$ are as expected.

- Similarly it is clear that for every representative point $\mathbf{q} \notin \mathbf{x}$ the topological incidence sequence of $\mathbf{q}$ remains unchanged. For the representative points $\mathbf{q} \neq \mathbf{p}$ on $\mathbf{x}$, the definition of ϵ ensures us that their topological incidence sequences remain unchanged, that is of the form $(\mathbf{x}^t, \mathbf{y}^t, \mathbf{x}^t, \mathbf{y}^t)$.
- We clearly have $|Rep_{R^t}| = |Rep_R| + 1$.
- It is clear in the construction that every segment $\mathbf{y}^t \neq \mathbf{x}^t$ of R (resp. representative point $\mathbf{q}^t \neq \mathbf{p}_1^t$ that is not internal in $\mathbf{x}^t$) is static.

For the induction step (when $i > 2$ or $\mathbf{x} \neq \mathbf{r}_2$), we apply the induction hypothesis on $R_- = R \setminus \mathbf{x}$. This is possible since R_- is flexible and unambiguous, and since $Const_{R_-}$ is a subdigraph of $Const_R$, thus acyclic. Let the ends of $\mathbf{x}$ be $\mathbf{q}_1$ and $\mathbf{q}_2$, and assume here that these points are still representative points in R_- (we later explain how to proceed if it is not the case). Thus the points $\mathbf{q}_1^t$ and $\mathbf{q}_2^t$ belongs to Rep_-^t for every $t \in [0, 1]$ (if $\mathbf{q}_1 = \mathbf{p}$, let $\mathbf{q}_1^t = \mathbf{p}_1^t$ or $\mathbf{p}_2^t$ whether $\mathbf{x} \in \{\mathbf{r}_2, \ldots, \mathbf{r}_i\}$ or not) and let $\mathbf{x}^t = [\mathbf{q}_1^t, \mathbf{q}_2^t]$. Consider now the interpolation defined by $R^t = R_-^t \cup \mathbf{x}^t$.

Claim (1). Consider three points moving continuously on the plane (three continuous functions from $[0, 1]$ to the points of the plane). If these points are non-collinear for $t = 0$, then there exists a value $t_1 \in]0, 1]$ such that they are non-collinear for every $t \in [0, t_1]$.

This implies the following claims.

Claim (2). Since $R^0 = R$ is unambiguous, there is a value t_2, with $0 < t_2 \leq 1$, R^t is unambiguous for every $t \in [0, t_2]$. Furthermore, t_2 can be such that for every segment $\mathbf{y} \in R$ ($\mathbf{x}$ included) and every representative point $\mathbf{q} \in Rep_{R_-}$, if $\mathbf{q} \notin (\mathbf{y})$ then $\mathbf{q}^t \notin (\mathbf{y}^t)$ for every $t \in [0, t_2]$.

There is also an interval where $\mathbf{x}$ does not intersect undesired segments.

Claim (3). There is a value t_3, with $0 < t_3 \leq 1$, such that $|Rep_{R^t} \cap \mathbf{x}|$ is constant for every $t \in]0, t_3]$.

Now by taking $t^* = \min\{t_2, t_3\}$ we have a move R^{t^*} and an interpolation $R^{t \times t^*}$, that follows the requirements of the lemma. Indeed, for every $t \in]0, t^*]$:

- The set R^t is unambiguous (by Claim (2)), and $Const_{R^t}$ is acyclic. Indeed a cycle should necessarily pass through $\mathbf{x}$ but all its internal representative points have out-degree zero in $Const_{R^t}$.
- The incidence sequence of $\mathbf{p}_1$ and $\mathbf{p}_2$ are convenient. The only segment that could behave badly is $\mathbf{x}^t$ but this does not occur. If $\mathbf{x}$ is not incident to $\mathbf{p}$ Claim (3) ensures us that $\mathbf{p}_1^t$ and $\mathbf{p}_2^t \notin \mathbf{x}^t$. Otherwise (when $\mathbf{q}_1 = \mathbf{p}$) the definition of $\mathbf{q}_1^t$ ensures us that $\mathbf{x}^t$ is incident to the convenient point, $\mathbf{p}_1^t$ or $\mathbf{p}_2^t$, and Claim (2) ensures us that its position around this point remains correct (since $\mathbf{q}_2^t \notin (\mathbf{y}^t)$ for any $\mathbf{y}^t \neq \mathbf{x}^t$ incident to $\mathbf{p}_1^t$ or $\mathbf{p}_2^t$).
- By the induction hypothesis the only representative points, distinct from $\mathbf{p}_1^t$ and $\mathbf{p}_2^t$, that may not have the same topological incidence sequence (as in R) are the representative points on $\mathbf{x}$. Claims (2) ensures us that these sequences remain unchanged.
- We have $|Rep_{R^t}| = |Rep_R| + 1$ by the induction hypothesis and Claim (3).
- By induction hypothesis, every segment $\mathbf{r}^t \neq \mathbf{x}^t$ of R^t (resp. representative point $\mathbf{q}^t$ that is not internal in $\mathbf{x}^t$) that is not reachable from $\mathbf{p}_1^t$ in $Const_R$ is static. If $\mathbf{x}^t$ (resp. an internal point $\mathbf{q}^t$ of $\mathbf{x}^t$, at the intersection with some segment denoted $\mathbf{y}^t$) is not reachable, it is also the case of $\mathbf{q}_1^t$ and $\mathbf{q}_2^t$ (resp. $\mathbf{x}^t$ and $\mathbf{y}^t$). Thus these points (resp. segments) are static implying that $\mathbf{x}^t$ (resp. $\mathbf{q}^t$) is static.

If the point $\mathbf{q}_1$ is not a representative point of R_-, this means that $\mathbf{q}_1$ belongs to zero or one segment $\mathbf{y}$ of R_- (as an internal point). In the first case, let $\mathbf{q}_1^t$ unchanged ($\mathbf{q}_1^t = \mathbf{q}_1$), and in the second case, let $\mathbf{q}_1^t$ be the intersection point of the lines $(\mathbf{x})$ and $(\mathbf{y}^t)$. Then we put $\mathbf{q}_1^t$ in Rep_{R_-} for the computation of t_2. If $\mathbf{q}_2$ is not a representative point of R_- we proceed similarly. Then the proof would work as described above. □

Lemma 2.13 (traversing). *Consider a flexible and unambiguous segment set R such that $Const_R$ is acyclic, and a representative point $\mathbf{p}$ of R which incidence sequence is $(\mathbf{r}_1, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_j, \mathbf{r}_1, \mathbf{r}_{j+1}, \ldots, \mathbf{r}_k, \mathbf{r}_i)$ with $2 < i \leq j \leq k$ (See Figure 8). There exists a move R' with an interpolation R^t such that for every $t \in]0, 1]$:*

- *The set R^t is unambiguous and $Const_{R^t}$ is acyclic.*
- *The point $\mathbf{p}$ splits into i representative points $\mathbf{p}_l^t$, for $1 \leq l \leq i$, which incidence sequence are $(\mathbf{r}_i^t, \mathbf{r}_2^t, \ldots, \mathbf{r}_i^t)$ for $l = 1$, $(\mathbf{r}_1^t, \mathbf{r}_l^t, \mathbf{r}_1^t, \mathbf{r}_l^t)$ for $1 < l < i$, and $(\mathbf{r}_1^t, \mathbf{r}_i^t, \ldots, \mathbf{r}_j^t, \mathbf{r}_1^t, \mathbf{r}_{j+1}, \ldots, \mathbf{r}_k, \mathbf{r}_i^t)$ for $l = i$.*

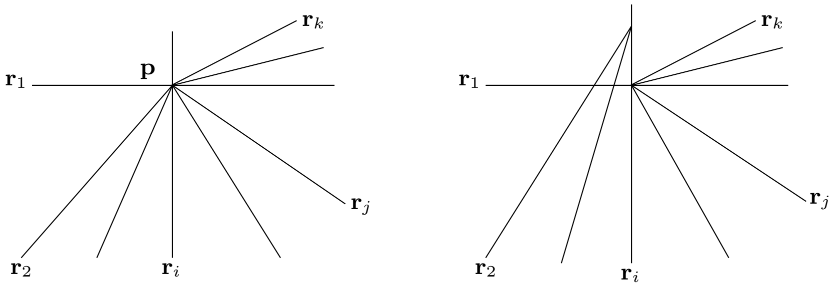

Fig. 8. traversing

- *For every representative point $\mathbf{q} \neq \mathbf{p}$ of R there is a representative point $\mathbf{q}^t$ in R^t with exactly the same topological incidence sequence.*
- *There is no other representative point (i.e. $|Rep_{R^t}| = |Rep_R| + i - 1$).*
- *Every segment $\mathbf{r} \in R$ (resp. representative point $\mathbf{q} \in Rep_R$) that is not reachable from $\mathbf{p}_i^t$ in $Const_{R^t}$ is static, that is $\mathbf{r}^t = \mathbf{r}$ (resp. $\mathbf{q}^t = \mathbf{q}$).*

Since the proof of this lemma is very similar to the proof of Lemma 2.12, we omit it here.

Given an intersection point $\mathbf{p}$ in a premodel $\mathcal{M} = (S, F, \tau)$ of T, a *partial realization* of $\mathbf{p}$ is an operation that combines a basic move at $\mathbf{p}$ and the addition of new face segments (eventually none), and that yields another premodel $\mathcal{M}' = (S', F', \tau')$ of T. A simple example of a partial realization at $\mathbf{p}$ is prolonging a segment $\mathbf{s}$ across $\mathbf{p}$, choosing $\mathbf{s}$ in such a way that $\tau(\mathbf{p})$ still applies and that the constraints digraph remains acyclic. Such a partial realization is called a *maximization* of $\mathbf{p}$, and if $\mathbf{p}$ is already internal in two segments we say that this point is *maximal*. In a premodel, we say that a point $\mathbf{p}$ is *simple* if it is either a segment end, a flat face segment end, or a maximal point without any segment of S ending here (at $\mathbf{p}$). Otherwise, we say that this point is *special*.

Proposition 2.14. *Consider a premodel $\mathcal{M} = (S, F, \tau)$ of a near-triangulation T. Every special point $\mathbf{p}$ of $\mathcal{M}$ that is maximal admits a partial realization.*

Proof. Note that since $\mathbf{p}$ is special and maximal there are at least three segments from S intersecting at $\mathbf{p}$. We distinguish five cases according to the type of $\mathbf{p}$.

If this point is a path–$(s_1, s_2, \ldots, s_k)$–point we do a gliding of $\{\mathbf{s}_3, \ldots, \mathbf{s}_k\}$ on $\mathbf{s}_2$ to a new representative point $\mathbf{q}$ (by Lemma 2.12 since $\mathbf{p}$ is not an end of $\mathbf{s}_2$). Let $\mathbf{p}$ and $\mathbf{q}$ be respectively typed as the crossing point of $\mathbf{s}_1$ and $\mathbf{s}_2$, and as a path–$(s_2, \ldots, s_k)$–point (See Figure 1). Under these conditions the gliding keeps the constraints digraph acyclic and preserves the topological incidence sequence of the other representative points (so that their type can remain unchanged). Thus, since the graph that corresponded to $\mathbf{p}$ (the path $(s_1, \ldots, s_k)$) is the union of the graphs corresponding to $\mathbf{p}$ and to $\mathbf{q}$, we are done.

If this point is a fan–$s_1 \lhd (s_2, \ldots, s_k)$–point we do a traversing of $\{\mathbf{s}_3, \ldots, \mathbf{s}_k\}$ along $\mathbf{s}_2$ and through $\mathbf{s}_1$ to a new representative point $\mathbf{q}$. We add the face segments $\underline{\mathbf{s_1 s_i}}\mathbf{s_{i-1}}$, with $3 \leq i \leq k$, and we let $\mathbf{q}$ be typed as a path–$(s_2, \ldots, s_k)$–point (See Figure 2). Under these conditions the traversing keeps the constraints digraph acyclic and preserves the topological incidence sequence of the other representative points. Thus since the graph that corresponded to $\mathbf{p}$ (the fan–$s_1 \lhd (s_2, \ldots, s_k)$) is the union of the graphs corresponding to the new crossing points, to the new face segments, to $\mathbf{p}$ and to $\mathbf{q}$, we are done.

If this point is a fan-path–$s_1 \lhd (s_2, \ldots, s_i) \cdot (s_i, \ldots, s_k)$–point with $2 \leq i \leq k$, we consider that $i < k$. Otherwise we could consider this point as a fan-point, a case we already considered. Here we do a gliding of $\{\mathbf{s}_{i+1}, \ldots, \mathbf{s}_k\}$ on $\mathbf{s}_i$ to a new representative point $\mathbf{q}$ and we let the points $\mathbf{p}$ and $\mathbf{q}$ be respectively typed as a fan–$s_1 \lhd (s_2, \ldots, s_i)$–point and as a path–$(s_i, \ldots, s_k)$–point (See Figure 3). Under these conditions the gliding keeps the constraints digraph acyclic and preserves the topological incidence sequence of the other representative points. Thus since the graph that corresponded to $\mathbf{p}$ is the union of the graphs corresponding to $\mathbf{p}$ and to $\mathbf{q}$, we are done.

If this point is a path-fan–$(s_{i-1}, \ldots, s_2, s_1) \cdot s_1 \lhd (s_i, \ldots, s_k)$–point with $2 \leq i \leq k$, we consider that $i < k$. Otherwise we could consider this point as a path-point, a case we already considered. Here we do a traversing of $\{\mathbf{s}_{i+1}, \ldots, \mathbf{s}_k\}$ through $\mathbf{s}_1$ and on $\mathbf{s}_i$ to a new representative point $\mathbf{q}$. We add the face segments $\underline{\mathbf{s_1 s_j}}\mathbf{s_{j-1}}$, with $i < j \leq k$, and we respectively let $\mathbf{p}$ and $\mathbf{q}$ be respectively typed as a path–$(\mathbf{s}_i, \mathbf{s}_1, \ldots, \mathbf{s}_{i-1})$–point and as a path–$(\mathbf{s}_i, \ldots, \mathbf{s}_k)$–point (See Figure 4). Under these conditions the traversing keeps the constraints digraph acyclic and preserves the topological incidence sequence of the other representative points. Thus since the graph that corresponded to $\mathbf{p}$ is the union of the graphs corresponding to the new crossing points, to the new face segments, to $\mathbf{p}$ and to $\mathbf{q}$, we are done.

If this point is a double-fan–$s_1 \triangleleft (s_2, \ldots, s_i) \cdot s_i \triangleleft (s_{i+1}, \ldots, s_k, s_1)$–point with $2 \leq i \leq k$, we consider that $2 < i$. Otherwise we could consider this point as a fan-point, a case we already considered. Here we do a traversing of $\{\mathbf{s}_2, \ldots, \mathbf{s}_{i-1}\}$ along $\mathbf{s}_i$ and through $\mathbf{s}_1$ to a new representative point $\mathbf{q}$. We add the face segments $\underline{\mathbf{s_1 s_j}}\mathbf{s_{j+1}}$, with $2 \leq j < i$, and we respectively let $\mathbf{p}$ and $\mathbf{q}$ be typed as a path–$(\mathbf{s}_i, \ldots, \mathbf{s}_2)$–point and as a fan–$\mathbf{s}_i \triangleleft (\mathbf{s}_1, \mathbf{s}_k, \ldots, \mathbf{s}_{i+1})$–point (See Figure 5). Under these conditions the traversing keeps the constraints digraph acyclic and preserves the topological incidence sequence of the other representative points. Thus since the graph that corresponded to $\mathbf{p}$ is the union of the graphs corresponding to the new crossing points, to the new face segments, to $\mathbf{p}$ and to $\mathbf{q}$, we are done.

This concludes the proof of the proposition. □

Given a special point $\mathbf{p}$ in a premodel $\mathcal{M} = (S, F, \tau)$ of T, a *total realization* of $\mathbf{p}$ is a sequence of partial realizations such that every edge (resp. face) of the graph corresponding to $\tau(\mathbf{p})$ corresponds now to a crossing point (resp. to a face segment).

Definition 2.15. *Consider a special point* $\mathbf{p}$ *of a premodel* $\mathcal{M} = (S, F, \tau)$ *and let* $\{\mathbf{s}_1, \ldots, \mathbf{s}_k\} \subseteq S$ *be the set of segments that have an end at* $\mathbf{p}$*. This special point is* free *if for any pair of segments* $\mathbf{s}_i$ *and* $\mathbf{s}_j$ *with* $1 \leq i < j \leq k$*, there is no path in the constraints digraph of* $\mathcal{M}$ *linking* $\mathbf{s}_i$ *and* $\mathbf{s}_j$*.*

It is clear that a free special point can be maximized (*cf.* Lemma 2.9). In the proof above one can observe that if the point $\mathbf{p}$ is free, then the new special points (after the partial realization) are also free, thus we have that:

Remark 2.16. In a premodel $\mathcal{M}$, every free special point admits a total realization.

Since the constraints digraph of a premodel is acyclic we have that:

Remark 2.17. If a premodel has $k > 0$ special points, then one of them is free, and thus partially (totally) realizable.

Now let us note that any partial realization increases the number of representative points. Since a premodel with the maximum number of representative points is a full model (*cf.* Remark 2.8), we have the following corollary.

Corollary 2.18. *Any premodel* $\mathcal{M} = (S, F, \tau)$ *of a near-triangulation* T *admits a sequence of partial realizations that yield a full model* $\mathcal{M}' = (S', F')$ *of* T*.*

The total realizations preserve the freeness of special points.

Lemma 2.19. *Consider a premodel* $\mathcal{M} = (S, F, \tau)$ *with a special point* $\mathbf{p}$*. There exists a total realization of* $\mathbf{p}$ *such that in the obtained premodel* $\mathcal{M}' = (S', F', \tau')$*, every special point* $\mathbf{q} \neq \mathbf{p}$ *of* $\mathcal{M}$ *is preserved (i.e. there is no partial realization at* $\mathbf{q}$*) and every free special point* $\mathbf{q} \neq \mathbf{p}$ *of* $\mathcal{M}$ *remains free.*

Proof. It is clear that a total realization of $\mathbf{p}$ minimizing the number of partial realization preserves every representative point $\mathbf{p}' \neq \mathbf{p}$ of $Rep_{S \cup F}$. Now to prove that $\mathbf{q}$ is still free we show that for every pair of segments $\mathbf{r}_1$ and $\mathbf{r}_2$ from $S \cup F$ there is a path from $\mathbf{r}_1$ to $\mathbf{r}_2$ in $Const_{S' \cup F'}$ only if there was one in $Const_{S \cup F}$.

Since every $\mathbf{p}' \neq \mathbf{p}$ of $Rep_{S \cup F}$ is preserved, for every segment $\mathbf{r} \in S \cup F$, the arc $\mathbf{p}'\mathbf{r}$ (resp. $\mathbf{r}\mathbf{p}'$) belongs to $Const_{S \cup F}$ if and only it belongs to $Const_{S' \cup F'}$. Thus a new path from $\mathbf{r}_1$ to $\mathbf{r}_2$ should necessarily pass through one of the new representative points, say $\mathbf{p}^* \in Rep_{S' \cup F'} \setminus (Rep_{S \cup F} \setminus \mathbf{p})$. Since $\mathbf{p}^*$ is simple (otherwise the realization would not be total) we consider three cases according to $\tau'(\mathbf{p}^*)$.

- If $\mathbf{p}^*$ is a segment end, it has no in-neighbor in $Const_{S' \cup F'}$, and thus it cannot be part of a path from $\mathbf{r}_1$ to $\mathbf{r}_2$.

- If $\mathbf{p}^*$ is a flat face segment end, it has a unique out-neighbor in $Const_{S' \cup F'}$ and it is a face segment $\mathbf{f}$. Being a face segment $\mathbf{f}$ has no out-neighbor, thus we just have to show that $\mathbf{f} \neq \mathbf{r}_2$. According to the descriptions of maximization and the realizations used in the proof of Proposition 2.14 it is clear that $\mathbf{f}$ is new ($\mathbf{f} \in F' \setminus F$) and thus $\mathbf{f} \neq \mathbf{r}_2$.
- If $\mathbf{p}^*$ is a maximal crossing point, all its out-neighbors in $Const_{S' \cup F'}$ are face segments. Being face segments none of them has an out-neighbor, thus we just have to show that they are distinct from $\mathbf{r}_2$. If one of them is $\mathbf{r}_2$, since $\mathbf{p}^*$ is the cross end of this face segment, the other end of $\mathbf{r}_2$ is a flat face segment end, and thus $\mathbf{q}$ is not free in $\mathcal{M}$.

This concludes the proof of the lemma. □

This lemma and Remark 2.16 imply the following corollary.

Corollary 2.20. *Consider a premodel $\mathcal{M} = (S, F, \tau)$ with a set $P \subset Rep_{S \cup F}$ of free special points. There exists a sequence of total realizations that totally realizes every $\mathbf{p} \in P$ and preserves every point of $Rep_{S \cup F} \setminus P$.*

2.3 Global transformations

It is folklore that under a linear transformation of the plane, collinear points remain collinear. Furthermore if this linear transformation is injective, the image of an half-plane remains an half-plane. Thus we have the following lemma.

Lemma 2.21. *For any premodel $\mathcal{M} = (S, F, \tau)$ of a near triangulation T and any injective linear transformation of the plane ϕ, the triple $\mathcal{M}' = (\phi(S), \phi(F), \tau)$ remains a premodel of T.*

This is useful since the plane admits many such transformations.

Lemma 2.22. *For any two triplets of points in general position (i.e. non-collinear points), $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ and $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, there is an injective linear transformation of the plane ϕ such that $\phi(\mathbf{p}_i) = \mathbf{q}_i$, for every $i \in \{1, 2, 3\}$.*

3 The case of 4-connected triangulations.

3.1 Particular Premodels

Let T be a near-triangulation. A *chord* of T is an edge not incident to the outer face but which ends are on the outer face. A *separating 3-cycle* C is a cycle of length 3 such that some vertices of T lie inside C whereas other vertices are outside. It is well known that a triangulation is 4-connected if and only if it contains no separating 3-cycle.

Definition 3.1. *A* W-triangulation *T is a 2-connected near-triangulation containing no separating 3-cycle. Such a W-triangulation is* 3-bounded *if its outer boundary is the union of three paths, $(a_1, \ldots, a_p)$, $(b_1, \ldots, b_q)$, and $(c_1, \ldots, c_r)$, that satisfy the following conditions (see Figure 9):*

- *$a_1 = c_r$, $b_1 = a_p$, and $c_1 = b_q$.*
- *the paths are non-trivial (*i.e. *$p \geq 2$, $q \geq 2$, and $r \geq 2$).*
- *there exists no chord $a_i a_j$, $b_i b_j$, or $c_i c_j$.*

This 3-boundary *of T will be denoted by $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$.*

In the following, we will use the order on the three paths and their directions, *i.e.* $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ will be different from $(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$-$(a_1, \ldots, a_p)$ and $(a_p, \ldots, a_1)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_1)$.

Lemma 3.2. *Let T be a W-triangulation and consider a cycle C of T. The subgraph defined by C and the edges inside C (according to the embedding of T) is a W-triangulation.*

Proof. Consider the near-triangulation T' induced by some cycle C of T and the edges inside C. By definition, T has no separating 3-cycle and consequently T' does not have any separating 3-cycle. It is then sufficient to show that T' is 2-connected, i.e. T does not have any cut vertex. Consider a vertex v of T, all the faces incident to v are triangles, except at most one (the outer face). Consequently, there exists a path that contains all the neighbors of v, and so $T \setminus v$ is connected. □

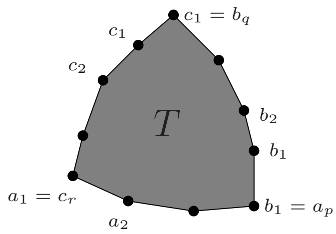

Fig. 9. A 3-bounded W-triangulation T.

Property 1 *Consider any W-triangulation T 3-bounded by $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$.*

(1) If $p = 2$ (see Figure 10, left), for any triangle $\mathbf{BCD}$, there exists a premodel $\mathcal{M} = (S, F, \tau)$ of T contained in the triangle $\mathbf{BCD}$ such that
- *every special point $\mathbf{p}$ of $\mathcal{M}$ is a point of $b_q = c_1 = [\mathbf{BC}]$, $a_2 = b_1 = [\mathbf{BD}]$ or $c_r = a_1 = [\mathbf{CD}]$,*
- *$\mathbf{B}$ is a path–$(b_1, b_2, \ldots, b_q)$–point,*
- *$\mathbf{C}$ is a path–$(c_1, c_2, \ldots, c_r)$–point,*
- *$\mathbf{D}$ is a fan–$a_2 \triangleleft (d_1, \ldots, d_s, a_1)$–point (where $d_1, d_2, \ldots, d_s$ are inner vertices of T) such that there is a face segment incident only if $s = 0$ (i.e., $\mathbf{D}$ is a fan–$a_2 \triangleleft (a_1)$).*

(2) If $p > 2$ (see Figure 10, right), for any triangle $\mathbf{ABC}$ there exists a point $\mathbf{D}$ inside this triangle and a premodel $\mathcal{M} = (S, F, \tau)$ of T contained in the polygon $\mathbf{ABCD}$ such that
- *every special point $\mathbf{p}$ of $\mathcal{M}$ is a point of $a_p = b_1 = [\mathbf{AB}]$, $b_q = c_1 = [\mathbf{BC}]$, $[\mathbf{CD}]$ (that is contained in $a_1 = c_r$) or $[\mathbf{AD}]$ (that is contained in a_2),*
- *$\mathbf{A}$ is a path–$(a_2, \ldots, a_p)$–point.*
- *$\mathbf{B}$ is a path–$(b_1, b_2, \ldots, b_q)$–point,*
- *$\mathbf{C}$ is a path–$(c_1, c_2, \ldots, c_r)$–point,*
- *$\mathbf{D}$ is the crossing point of $\mathbf{a}_1$ and $\mathbf{a}_2$ (with possibly one face segment incident to it corresponding to the inner face of T incident to $a_1 a_2$),*

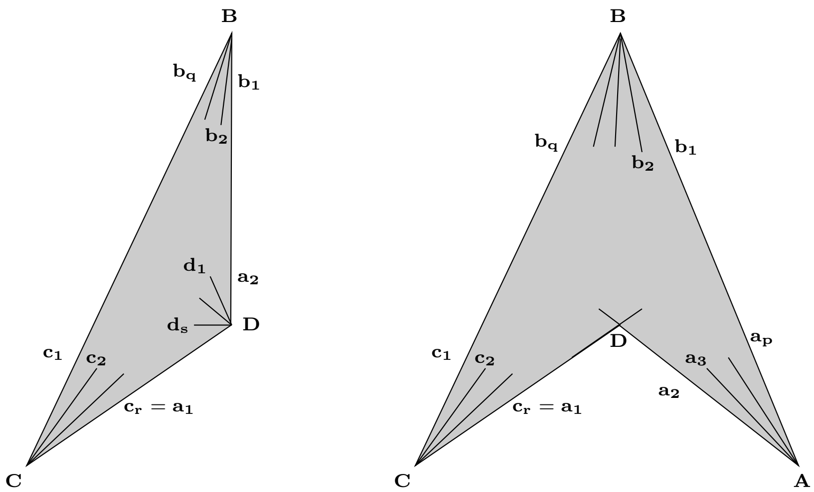

Fig. 10. Property 1 for one W-triangulation T with $p = 2$ and one with $p > 2$.

Note that in both cases, at most one face segment is incident to $\mathbf{D}$, since $a_1 a_2$ is incident to exactly one inner face of T. Furthermore since path–points cannot have incident face segments, there is no face segment incident to $\mathbf{A}, \mathbf{B}, \mathbf{C}$ (resp. $\mathbf{B}, \mathbf{C}$) when $p > 2$ (resp. $p = 2$).

Given the description of $\mathcal{M}$ we can deduce that almost every special point is free. A special point $\mathbf{p}$ that is not free has two incident segments $\mathbf{s}_1$ and $\mathbf{s}_k$ of S such that there is a directed path in $Const_{S\cup F}$ from $\mathbf{s}_1$ to $\mathbf{s}_k$. By a geometrical argument this path passes through some other segments of $S\cup F$. But since face segments have out-degree zero in this digraph, these other segments also belong to S and let us denote $(\mathbf{s}_1, \mathbf{p}_1, \mathbf{s}_2, \mathbf{p}_2, \ldots, \mathbf{s}_k)$ with $k \geq 3$ the considered path. Then since the points $\mathbf{p}_i$ are on the polygon bounding $\mathcal{M}$ (since they are special), and since $\mathbf{p}_i$ is an internal point of $\mathbf{s}_i$ and the end of $\mathbf{s}_{i+1}$ we have when $p=2$ (resp. $p>2$) that $\{\mathbf{s}_1,\ldots,\mathbf{s}_{k-1}\} \subseteq \{\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1\}$ and $\mathbf{s}_i \notin \{\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1\}$ for $i>1$ (resp. $\{\mathbf{s}_1,\ldots,\mathbf{s}_{k-1}\} \subseteq \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{c}_1\}$ and $\mathbf{s}_i \notin \{\mathbf{b}_1, \mathbf{c}_1\}$ for $i>1$). This implies the following remark.

Remark 3.3. When $p=2$ (resp. $p>2$), every special point $\mathbf{p}$ of $\mathcal{M}$ (resp. $\mathbf{p} \neq \mathbf{B}$ of $\mathcal{M}$) is free. Furthermore, if $\mathbf{B}$ is not free (when $p>2$) then there is a path in $Const_{S\cup F}$ of the form $(\mathbf{b}_1, \mathbf{p}_1, \mathbf{a}_1, \mathbf{p}_2, \mathbf{b}_i)$ or of the form $(\mathbf{b}_q, \mathbf{p}_1, \mathbf{a}_2, \mathbf{p}_2, \mathbf{b}_i)$

Property 1 is sufficient to prove Theorem 2.5. However, in our proof of Property 1, we need Property 2 (defined below) that is defined for some particular W-triangulations.

Consider a W-triangulation $T \neq K_3$ that is 3-bounded by $(a_1,\ldots,a_p)$-$(b_1,\ldots,b_q)$-$(c_1,\ldots,c_r)$ such that T does not contain any chord a_ib_j or a_ic_j. Let $D \subseteq V_i(T)$ be the set of inner vertices of T that are adjacent to some vertex a_i with $i>1$. Since T is a 3-bounded W-triangulation, the set D induces a connected graph. Since T has at least 4 vertices, no separating 3-cycle, and no chord a_ia_j, a_ib_j, or a_ic_j, then a_1 and a_2 (resp. b_1 and b_2) have exactly one common neighbor in $V(T)\setminus\{c_1\}$ (resp. $V(T)\setminus\{a_1\}$) that will be denoted a (resp. d_1). Since a is in D, the set $D\cup\{a_1\}$ also induces a connected graph. The *adjacent path* of T with respect to the 3-boundary $(a_1,\ldots,a_p)$-$(b_1,\ldots,b_q)$-$(c_1,\ldots,c_r)$ is the shortest path linking d_1 and a_1 in $T[D\cup\{a_1\}]$ (the graph induced by $D\cup\{a_1\}$). This path will be denoted $(d_1, d_2, \ldots, d_s, a_1)$. Note that, by definition of the adjacent path, there exists no edge $d_id_j \in E(T)$ with $2 \leq i+1 < j \leq s$, and no edge $a_1d_i \in E(T)$ with $1 \leq i < s$ (See Figure 11).

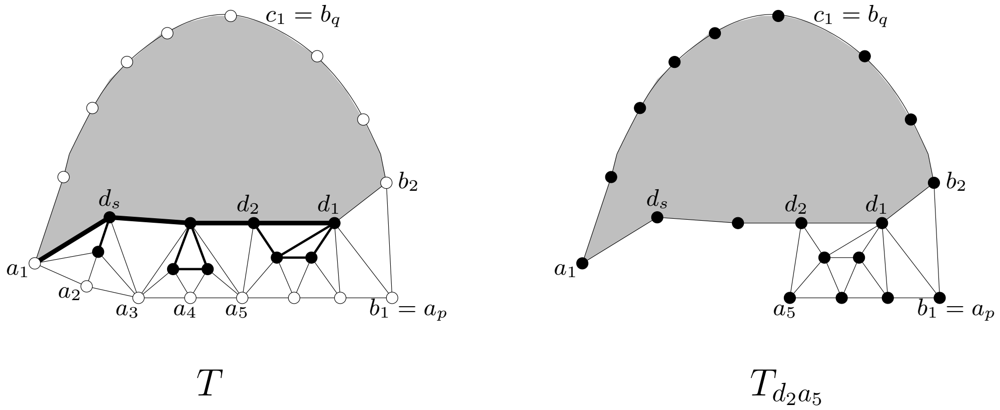

Fig. 11. the adjacent path of T and the graph $T_{d_2a_5}$.

For each edge $d_xa_y \in E(T)$ with $x\in[1,s]$ and $y\in[2,p]$, we define $T_{d_xa_y}$ as the W-triangulation lying inside the cycle $C=(a_1, d_s, \ldots, d_x, a_y, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_r)$. We now state the property on such particular triangulations that we use to prove Property 1.

Property 2 *Consider a 3-bounded W-triangulation T with a 3-boundary $(a_1,\ldots,a_p)$-$(b_1,\ldots,b_q)$-$(c_1,\ldots,c_r)$, without any chord a_ib_j or a_ic_j, and which adjacent path is $(d_1, d_2, \ldots, d_s, a_1)$. Consider the W-triangulation $T_{d_xa_y}$ for some edge d_xa_y of T.*

1. *If $y=p$ (see Figure 12 left), for any triangle* $\mathbf{BCD}$, *there exists a premodel $\mathcal{M}=(S,F,\tau)$ of $T_{d_xa_p}$ contained in the triangle* $\mathbf{BCD}$ *such that*
   - *every special point $\mathbf{p}$ of $\mathcal{M}$ is a point of $b_q=c_1=[\mathbf{BC}]$, $a_p=b_1=[\mathbf{BD}]$ or $c_r=a_1=[\mathbf{CD}]$,*
   - $\mathbf{B}$ *is a path–$(b_1, b_2, \ldots, b_q)$–point,*
   - $\mathbf{C}$ *is a path–$(c_1, c_2, \ldots, c_r)$–point,*

– $\mathbf{D}$ *is a fan-path–*$a_p \triangleleft (d_1, \ldots, d_x) \cdot (d_x, \ldots, d_s, a_1)$*–point.*

2. *If* $y < p$ *(see Figure 12 right), for any triangle* $\mathbf{ABC}$ *there exists a point* $\mathbf{D}$ *inside this triangle and a premodel* $\mathcal{M} = (S, F, \tau)$ *of* $T_{d_x a_y}$ *contained in the polygon* $\mathbf{ABCD}$ *such that*
   – *every special point* $\mathbf{p}$ *of* $\mathcal{M}$ *is a point of* $a_p = b_1 = [\mathbf{AB}]$, $b_q = c_1 = [\mathbf{BC}]$, $a_1 = c_r = [\mathbf{CD}]$ *or* $[\mathbf{AD}]$ *(that is contained in* a_y*),*
   – $\mathbf{A}$ *is a path–*$(a_y, \ldots, a_p)$*–point,*
   – $\mathbf{B}$ *is a path–*$(b_1, b_2, \ldots, b_q)$*–point,*
   – $\mathbf{C}$ *is a path–*$(c_1, c_2, \ldots, c_r)$*–point,*
   – $\mathbf{D}$ *is a path–*$(a_y, d_x, \ldots, d_s, a_1)$*–point whose incidence sequence is* $(\mathbf{a}_y, \mathbf{d}_x, \ldots, \mathbf{d}_s, \mathbf{a}_1, \mathbf{a}_y, \mathbf{d}_x)$

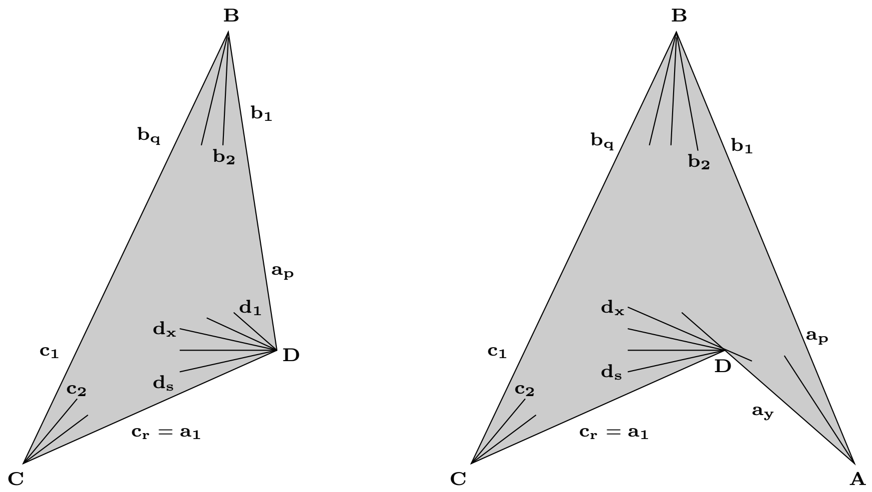

Fig. 12. Property 2 for one W-triangulation $T_{d_x a_y}$ with $y = p$ and one with $y < p$.

Note that if $p > y$ (resp. $p = y$), there is no face segment incident to $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ (resp. $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$). Note that when $y > p$, in a premodel (M, S, τ) of $T_{d_x a_y}$ satisfying conditions of Property 2, $\mathbf{D}$ is an internal point of the segments $\mathbf{d}_x$ and $\mathbf{a}_y$.

With a similar argument as for Remark 3.3 we obtain the following remark.

Remark 3.4. Consider a premodel $\mathcal{M}$ satisfying Property 2. If $y = p$, any special point of $\mathcal{M}$ is free. If $y < p$, any special point of $[bA\mathbf{D}]$ or $[\mathbf{DC}]$ is free.

Remark 3.5. According to Lemmas 2.21 and 2.22, it is sufficient to show that there exists a set of points $\mathbf{B}, \mathbf{C}, \mathbf{D}$ (or $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$) such that conditions of Property 1 (resp. Property 2) hold.

Let us now prove these two properties by doing a “crossed” induction.

Theorem 3.6. *Property 1 (resp. Property 2) holds for any W-triangulation* T *(resp.* $T_{d_x a_y}$*).*

3.2 Proof of Theorem 3.6

We prove Theorem 3.6 by induction on the number of edges of T (for Property 1) or $T_{d_x a_y}$ (for Property 2). Our proof is based on a decomposition of 4-connected triangulations already used in [7,18].

The following lemma proves the initial step of the induction.

Lemma 3.7. *Property 1 (resp. Property 2) holds for any W-triangulation* T *(resp.* $T_{d_x a_y}$*) with at most three edges.*

Proof. There is only one W-triangulation with so few edges, the graph K_3.

This implies that there is no W-triangulation $T_{d_x a_y}$ with at most 3 edges, so Property 2 obviously holds by vacuity.

For Property 1, we have to consider all the possibles 3-boundaries of K_3. All these 3-boundaries are equivalent. Let $V(K_3) = \{a, b, c\}$ and consider the 3-boundary (a, b)-(b, c)-(c, a). Given any triangle **BCD**, let **a** = **CD**, **b** = **DB** and **c** = **BC**. We add a face segment **$\underline{\mathbf{ab}}$c** from **D** to an internal point of [**BC**]. The types of **B**, **C**, **D** are as follows: **B** is a path–(b, c)–point, **C** is a path–(c, a)–point and **D** is a fan–$b\lhd(a)$–point, with the face segment **$\underline{\mathbf{ab}}$c** incident to it.

It is easy to check that we have defined a premodel of K_3 that satisfies Property 1. □

We prove the inductive step for Property 1 with the following lemma.

Lemma 3.8. *For any integer $m > 3$, if Property 1 holds for any W-triangulation T such that $|E(T)| < m$ and Property 2 holds for any W-triangulation $T_{d_x a_y}$ such that $|E(T_{d_x a_y})| < m$, then Property 1 holds for any W-triangulation T such that $|E(T)| = m$.*

Case 1: Proof of Property 1 for a W-triangulation T such that $|E(T)| = m$.

Let $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ be the 3-boundary of T considered. We distinguish different cases according to the existence of a chord $a_i b_j$ or $a_i c_j$ in T:

(Case 1.1) either there exists a chord $a_1 b_j$, $j \in [2, q-1]$,
(Case 1.2) or there exists a chord $a_i b_j$, with $i \in [2, p-1]$ and $j \in [2, q]$,
(Case 1.3) or there exists a chord $a_i c_j$, with $i \in [2, p]$ and $j \in [2, r-1]$,
(Case 1.4) or there is no chord $a_i b_j$ or $a_i c_k$, with $i \in [1, p], j \in [1, q], k \in [1, r]$.

Note that all the cases are considered since there is no chord $a_1 b_q = c_r c_1$, $a_i b_1 = a_i a_p$, $a_p b_j = b_1 b_j$, $a_1 c_j = c_r c_j$, $a_p c_1 = b_1 b_q$ or $a_i c_r = a_i a_1$ and since a chord $a_i c_1$ is a chord $a_i b_q$.

Case 1.1: There is a chord $a_1 b_j$, with $1 < j < q$ (see Figure 13).

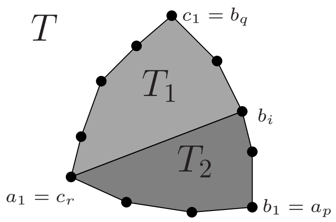

Fig. 13. Case 1.1: Chord $a_1 b_i$.

Let T_1 (resp. T_2) be the subgraph of T that lies inside the cycle $(a_1, b_i, \ldots, b_q, c_2, \ldots, c_r)$ (resp. $(a_1, a_2, \ldots, b_1, \ldots, b_i, a_1)$). By Lemma 3.2, T_1 and T_2 are W-triangulations. Since T has no chord $a_x a_y$, $b_x b_y$, or $c_x c_y$, (b_i, c_r)-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_i)$ (resp. $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_i)$-$(b_i a_1)$) is a 3-boundary of T_1 (resp. T_2). Furthermore, since $a_1 a_2 \notin E(T_1)$ (resp. $c_1 c_2 \notin E(T_2)$), T_1 (resp. T_2) has less edges than T and Property 1 holds for T_1 and T_2 with the mentioned 3-boundaries.

If $p = 2$ we want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of T contained in a triangle **BCD** while if $p > 2$ we want it to be contained in a concave polygon **ABCD**. In both cases, consider three points **B**, **C** and **D** and let **E** be an inner-point of the segment [**CD**].

Consider a premodel $\mathcal{M}_1 = (S_1, F_1, \tau_1)$ of T_1 satisfying Property 1 contained in **BCE** where the points **B**, **C** and **E** are respectively a path–$(b_i, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point, and a fan–$b_i\lhd(a_1, \ldots)$–point (if **E** is a fan–$b_i\lhd(a_1)$–point, there can be face segment incident to it).

If $p = 2$ (see Figure 14 left), consider a premodel $\mathcal{M}_2 = (S_2, F_2, \tau_2)$ of T_2 satisfying Property 1 contained in **BED** where the points **B**, **E** and **D** are respectively a path–$(b_1, \ldots, b_i)$–point, a path–(b_i, a_1)–point, and a fan–$b_1\lhd(a_1, \ldots,)$–point.

If $p > 2$ (see Figure 14 right), there exists a point **A** and a premodel $\mathcal{M}_2 = (S_2, F_2, \tau_2)$ of T_2 satisfying Property 1 contained in **ABED** and where the points **A**, **B**, **E** and **D** are respectively a path–$(a_2, \ldots, a_p)$–point, a path–$(b_1, \ldots, b_i)$–point, a path–(b_i, a_1)–point, and the crossing-point of $\mathbf{a_1}$ and $\mathbf{a_2}$.

By using Lemma 2.12, if necessary, we can ensure that except $\mathbf{B}, \mathbf{E}$, there is no representative point $\mathbf{p}_1$ of $\mathcal{M}_1$ and $\mathbf{p}_2$ of $\mathcal{M}_2$ that are exactly at the same position on $\mathbf{b}_i$.

Note that in both cases ($p = 2$ and $p > 2$) the two segments $\mathbf{a_1}$ (resp. $\mathbf{b_i}$) of S_1 and S_2 form now a single segment $\mathbf{a_1}$ (resp. $\mathbf{b_i}$). Consider now $\mathcal{M} = (S, F, \tau)$ where $S = S_1 \cup S_2$ (up to the identification of the two $\mathbf{a}_1$s and of the two $\mathbf{b}_i$s), $F = F_1 \cup F_2$, $\tau(p) = \tau_1(p)$ (resp. $\tau(p) = \tau_2(p)$) for any point $p \in Rep_{S_1 \cup F_1} \setminus \{\mathbf{B}, \mathbf{E}\}$ (resp. $p \in Rep_{S_2 \cup F_2} \setminus \{\mathbf{B}, \mathbf{E}\}$), and where $\tau(E)$ and $\tau(B)$ are defined as follows: **B** is now a path–$(b_1, \ldots, b_q)$–point and **E** remains a fan–$b_i \triangleleft (a_1, \ldots)$–point (as in $\mathcal{M}_1$); this is possible since around **E**, we just have prolonged $\mathbf{a}_1$.

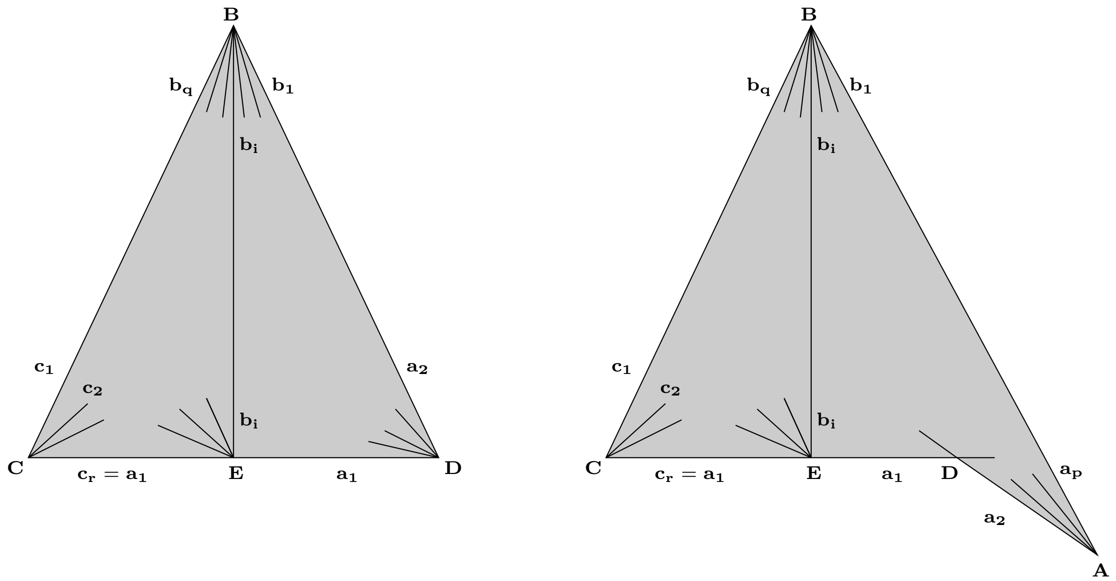

Fig. 14. Case 1.1: when $p = 2$ (left) or $p > 2$ (right).

Since $V(T) = V(T_1) \cup V(T_2)$ and $V(T_1) \cap V(T_2) = \{a_1, b_i\}$, every vertex $v \in V(T)$ corresponds to exactly one segment $\mathbf{v}$ in S. Note that $E(T) = E(T_1) \cup E(T_2)$ and that $E(T_1) \cap E(T_2) = \{a_1 b_i\}$. Note also that an edge uv is in the graph corresponding to **E** (resp. **B**) in $\mathcal{M}$ if and only if uv is an edge of the graph corresponding to **E** (resp. **B**) in $\mathcal{M}_1$ (resp. in $\mathcal{M}_1$ or in $\mathcal{M}_2$). Thus the edges of T are exactly the edges represented (either by a face segment or in a special point) in $\mathcal{M}$. Since $F(T) = F(T_1) \cup F(T_2)$, since $F(T_1) \cap F(T_2) = \emptyset$, since no face segment has been added or removed, since $\tau(\mathbf{E})$ has not been modified and since **B** is a path–point (and thus no face is represented in **B**), the faces represented in $\mathcal{M}$ are exactly the union of the faces represented in $\mathcal{M}_1$ and $\mathcal{M}_2$, i.e., the faces of T.

We know that $Const_{S_1 \cup F_1}$ and $Const_{S_2 \cup F_2}$ are acyclic. Let $Const'_1$ (resp. $Const'_2$) be the digraph $Const_{S_1 \cup F_1}$ (resp. $Const_{S_2 \cup F_2}$) where the arc from **E** to $\mathbf{a}_1$ has been replaced by an arc from $\mathbf{a}_1$ to **E** (this corresponds to the fact that **E** is no longer an end of $\mathbf{a}_1$). Since **E** is free in $\mathcal{M}_1$, it is easy to see that $Const'_1$ is acyclic. Moreover, the internal special points of $\mathbf{b}_i$ remain free (there is no directed path from any segment ending on $\mathbf{b}_i$ to $\mathbf{b}_i$ or **E** since **E** is free). Since **E** is also free in $\mathcal{M}_2$, $Const'_2$ is acyclic and the internal special points of $\mathbf{b}_i$ remain free.

The digraph $Const_{S \cup F}$ is the union of $Const'_1$ and $Const'_2$ where the two vertices corresponding to $\mathbf{a}_1$ (resp. $\mathbf{b}_i$, **B**, **E**) have been identified. Since $Const'_1$ and $Const'_2$ are acyclic, any cycle of $Const_{S \cup F}$ must contain at least two vertices among $\mathbf{a}_1, \mathbf{b}_i, \mathbf{B}, \mathbf{E}$. Note that **B** has no predecessor and thus is not in any cycle. Moreover, $\mathbf{a}_1$ has no predecessor except **C** (that has no predecessor) in $Const'_1$ and any cycle containing **E** contains $\mathbf{a}_1$ and any cycle containing $\mathbf{b}_i$ contains **E** or **B**. Consequently, there is no cycle containing a directed path going from $Const'_1$ to $Const'_2$ through $\mathbf{a}_1, \mathbf{b}_i, \mathbf{B}$ or **E** and thus, $Const_{S \cup F}$ is acyclic. For any internal special point $\mathbf{p}$ of $\mathbf{b}_i$ that is in $\mathcal{M}_1$ (resp. $\mathcal{M}_2$), the segments ending in $\mathbf{p}$ are all in $\mathcal{M}_1$ (resp. all in $\mathcal{M}_2$); thus they remain free in $\mathcal{M}$, since they were free in $\mathcal{M}_1$ (resp. $\mathcal{M}_2$).

In order to obtain a premodel of T satisfying Property 1, we just realize the special points of $\mathcal{M}$ that are some inner points of $\mathbf{b}_i$ (this is possible by Corollary 2.20 since they are free).

Case 1.2: There is a chord a_ib_j, with $1 < i < p$ and $1 < j \leq q$ (see Figure 15).

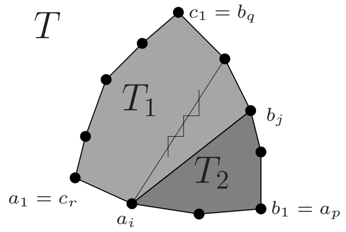

Fig. 15. Case 1.2: Chord a_ib_j.

If there are several chords a_ib_j, we consider one which maximizes j, *i.e.*, there is no chord a_ib_k with $j < k \leq q$. Let T_1 (resp. T_2) be the subgraph of T that lies inside the cycle $(a_1, a_2, \ldots, a_i, b_j, \ldots, b_q, c_2, \ldots, c_r)$ (resp. $(a_i, \ldots, a_p, b_2, \ldots, b_j, a_i)$). By Lemma 3.2, T_1 and T_2 are W-triangulations. Since T has no chord a_xa_y, b_xb_y, c_xc_y, or a_ib_k with $k > j$, $(a_1, \ldots, a_i)$-$(a_i, b_j, \ldots, b_q)$-$(c_1, \ldots, c_r)$ (resp. (a_i, b_j)-$(b_j, \ldots, b_1)$-$(a_p, \ldots, a_i)$) is a 3-boundary of T_1 (resp. T_2). Furthermore, since $b_1b_2 \notin E(T_1)$ (resp. $a_1a_2 \notin E(T_2)$), T_1 (resp. T_2) has less edges than T and Property 1 holds for T_1 and T_2 with the mentioned 3-boundaries. We know that $p > 2$ and we want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of T contained in some concave polygon **ABCD**.

If $i = 2$, let $\mathcal{M}_1 = (S_1, F_1, \tau_1)$ be a premodel of T_1 satisfying Property 1 that is contained in a triangle **BCD** where the points **B**, **C** and **D** are respectively a path–$(a_i, b_j, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point, and a fan–$a_2 \triangleleft (a_1, \ldots)$–point.

If $i > 2$ let $\mathcal{M}_1 = (S_1, F_1, \tau_1)$ be a premodel of T_1 satisfying Property 1 that is contained in a concave polygon **ABCD** and where the points **A**, **B**, **C** and **D** are respectively a $(a_2, \ldots, a_i)$-point, a $(a_i, b_j, \ldots, b_q)$-point, a $(c_1, \ldots, c_r)$-point, and the crossing-point of $\mathbf{a_1}$ and $\mathbf{a_2}$.

In both cases ($i = 2$ or $i > 2$), we want to do a gliding of $\mathbf{a}_i$ along $\mathbf{b}_j$. If $\mathbf{b}_j$ has no end on $\mathbf{a}_1$ or if $\mathbf{a}_1$ has no end on $\mathbf{a}_i$, the conditions of Lemma 2.12 are satisfied and we can do a gliding of $\mathbf{a}_i$ on $\mathbf{b}_j$ inside the polygon (See Figure 16).

Otherwise, we cannot use Lemma 2.12, since there exists a directed path from $\mathbf{a}_i$ to $\mathbf{b}_j$ in $Const_{S_1 \cup F_1}$. However, consider the intersection point $\mathbf{I}$ of $\mathbf{a}_1$ and $\mathbf{a}_i$ ($\mathbf{I}$ is an end of $\mathbf{a}_1$). It is easy to see that any segment $\mathbf{s} \neq \mathbf{a}_1$ ending in $\mathbf{I}$ does not have any internal special point. Note also that only $\mathbf{a}_i$ appears twice in the incidence sequence of $\mathbf{I}$. Consequently, we can prolong $\mathbf{a}_1$ after $\mathbf{I}$ and keep a flexible segment set $S \cup F$ with an acyclic constraints digraph. Once we have prolonged $\mathbf{a}_1$, we can apply Lemma 2.12 to do a gliding of $\mathbf{a}_i$ on $\mathbf{b}_j$. After that, we erase the part of $\mathbf{a}_1$ that is outside the polygon (at this moment, the constraints digraph is no longer acyclic). Let $\mathbf{E}$ be the new intersection of $\mathbf{a}_i$ and $\mathbf{b}_j$. If $j < q$, we do a prolonging of $\mathbf{a}_i$ after $\mathbf{E}$ (on the other side of $\mathbf{b}_j$).

We know that $Const_{S_1 \cup F_1}$ is acyclic. Let $Const'_1$ be the new constraints digraph obtained after the previous transformation. If $j = q$, the ends of $\mathbf{b}_q$ are not internal points of $\mathbf{a}_1$ and thus $Const'_1$ is still acyclic (we have done a gliding according to Lemma 2.12). If $j < q$, $Const_{S_1 \cup F_1}$ differs from $Const_{S_1 \cup F_1}$ by the facts that the arc from $\mathbf{B}$ to $\mathbf{a}_i$ has been removed and that an arc from $\mathbf{b}_j$ to the new point $\mathbf{E}$, an arc from $\mathbf{a}_i$ to $\mathbf{E}$ and an arc from $\mathbf{a}_i$ to its new end have been created. Since $\mathbf{E}$ has no successor and since the new end of $\mathbf{a}_i$ has no predecessor, we have not created any cycle. Then, if $i = 2$, we extend the segment $\mathbf{a}_i$ after $\mathbf{D}$ to a new endpoint $\mathbf{A}$ (See Figure 16, top right). Otherwise $\mathbf{A}$ is unchanged.

Let $\mathcal{M}_2 = (S_2, F_2, \tau_2)$ be a premodel of T_2 contained in **ABE** and where the points **A**, **B** and **E** are respectively a path–$(a_i, \ldots, a_p)$–point, a path–$(b_1, \ldots, b_j)$–point, and a fan–$b_j \triangleleft (a_i, \ldots)$–point. By using Lemma 2.12, we can ensure that except $\mathbf{B}, \mathbf{E}, \mathbf{A}$, there is no representative points $\mathbf{p}_1$ of $\mathcal{M}_1$ and $\mathbf{p}_2$ of $\mathcal{M}_2$ exactly at the same position on $\mathbf{a}_i$ or $\mathbf{b}_j$.

Note that in both cases ($i = 2$ or $i \geq 2$), the two segments $\mathbf{a}_i$ (resp. $\mathbf{b}_j$) of S_1 and S_2 form now a single segment $\mathbf{a}_i$ (resp. $\mathbf{b}_j$). Consider now $\mathcal{M} = (S, F, \tau)$ where $S = S_1 \cup S_2$ (up to the identifications of the $\mathbf{a}_i$s and of the $\mathbf{b}_j$s), $F = F_1 \cup F_2$, $\tau(p) = \tau_1(p)$ (resp. $\tau(p) = \tau_2(p)$) for any point $p \in Rep_{S_1 \cup F_1} \setminus \{\mathbf{A}, \mathbf{B}, \mathbf{E}\}$ (resp. $p \in Rep_{S_2 \cup F_2} \setminus \{\mathbf{A}, \mathbf{B}, \mathbf{E}\}$), and where $\tau(A)$, $\tau(E)$ and $\tau(B)$ are defined as follows: $\mathbf{A}$ is now a path–$(a_2, \ldots, a_p)$–point, $\mathbf{B}$ is now a path–$(b_1, \ldots, b_q)$–point and $\mathbf{E}$ remains a fan–$b_j \triangleleft (a_i, \ldots)$–point (as in $\mathcal{M}_2$); this is possible since around $\mathbf{E}$, we just have prolonged $\mathbf{b}_j$ (resp. $\mathbf{a}_i$ and $\mathbf{b}_j$) when $i = q$ (resp. $i < q$).

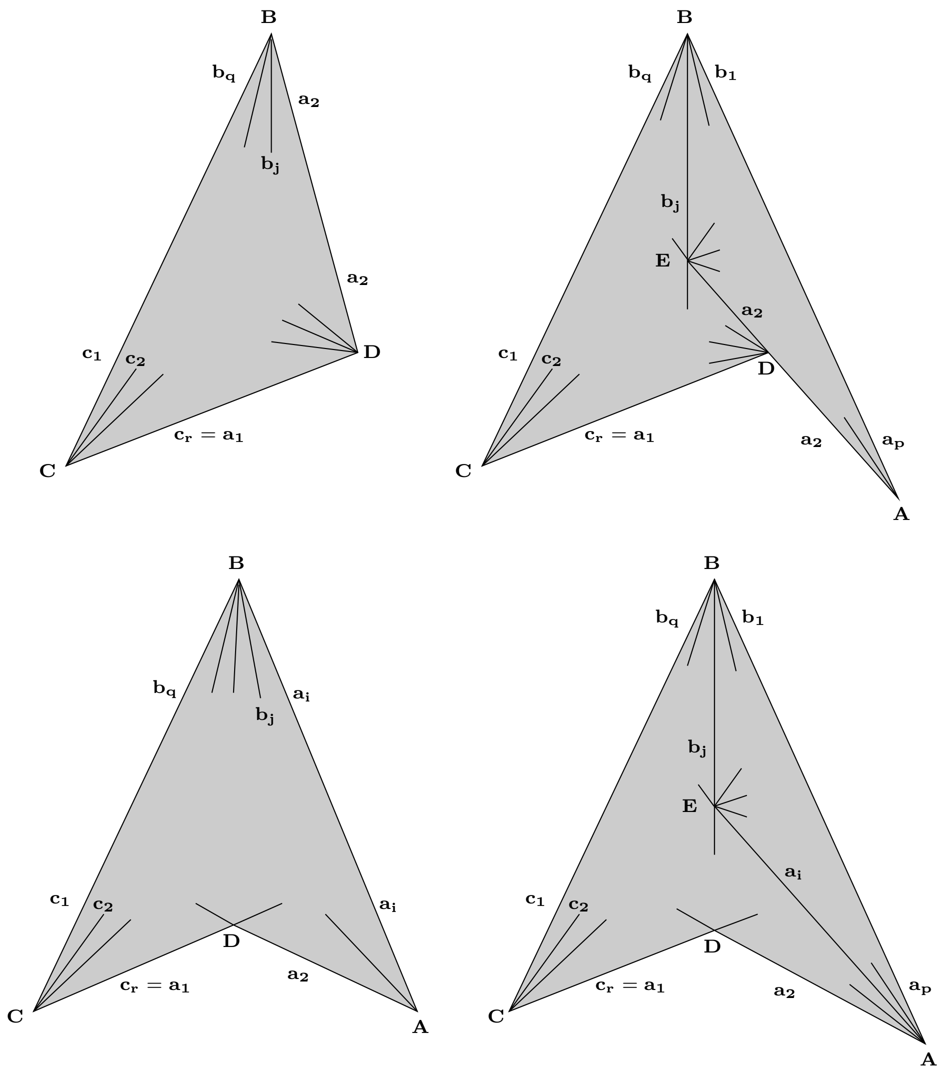

Fig. 16. Case 1.2: when $i = 2$ (top) or $i > 2$ (bottom); in both cases, a model of T_1 is represented on the left and a model of T (obtained from the model of T_1 and from a model of T_2) is represented on the right.

Since $V(T) = V(T_1) \cup V(T_2)$ and $V(T_1) \cap V(T_2) = \{a_i, b_j\}$, every vertex $v \in V(T)$ corresponds to exactly one segment $\mathbf{v}$ in S. Note that $E(T) = E(T_1) \cup E(T_2)$ and that $E(T_1) \cap E(T_2) = \{a_i b_j\}$. Note also that an edge uv is in the graph corresponding to $\mathbf{E}$ (resp. $\mathbf{A}$) in $\mathcal{M}$ if and only if uv is an edge of the graph corresponding to $\mathbf{E}$ (resp. $\mathbf{A}$) in $\mathcal{M}_1$ (resp. in $\mathcal{M}_1$ or $\mathcal{M}_2$). Note that an edge $uv \neq a_i b_j$ is represented in $\mathbf{B}$ in $\mathcal{M}$ if and only if uv is represented in $\mathbf{B}$ in $\mathcal{M}_1$ or $\mathcal{M}_2$. Thus the edges of T are exactly the edges represented (either by a face segment or in a special point) in $\mathcal{M}$. Since $F(T) = F(T_1) \cup F(T_2)$, since $F(T_1) \cap F(T_2) = \emptyset$, since no face segment has been added or removed, since $\tau(\mathbf{E})$ has not been modified and since $\mathbf{A}$ and $\mathbf{B}$ are path points (and thus no face is represented in $\mathbf{A}, \mathbf{B}$), the faces represented in $\mathcal{M}$ are exactly the union of the faces represented in $\mathcal{M}_1$ and $\mathcal{M}_2$, i.e., the faces of T.

We know that $Const_{S_1 \cup F_1}$ and $Const_{S_2 \cup F_2}$ are acyclic. Recall that $Const'_1$ is the digraph corresponding to (S_1, F_1, τ_1) once we have glided $\mathbf{a}_i$ on $\mathbf{b}_j$.

When $i = 2$, we also replace the arc from $\mathbf{D}$ to $\mathbf{a}_i$ by an arc from $\mathbf{a}_i$ to $\mathbf{D}$ (since $\mathbf{D}$ is no longer an end of $\mathbf{a}_i$). If $j < q$, both ends of $\mathbf{a}_i$ are segment ends, there is no cycle going through $\mathbf{a}_i$ and thus there is no cycle going through $\mathbf{D}$, since $\mathbf{D}$ was free in $\mathcal{M}_1$; consequently, $Const'_1$ is acyclic. If $j = q$, the digraph $Const'_1$ corresponds to the digraph obtained from $Const_{S_1 \cup F_1}$ if we extend $\mathbf{b}_q$ after $\mathbf{B}$ and $\mathbf{a}_2$ after $\mathbf{D}$. Since, $\mathbf{B}$ and $\mathbf{D}$ are free in $\mathcal{M}_1$ (we are in the cases where $\mathcal{M}_1$ is contained in the triangle $\mathbf{BCD}$), it is easy to see that $Const'_1$ is acyclic.

Let $Const'_2$ be the digraph obtained from $Const_{S_2 \cup F_2}$ where the arc from $\mathbf{E}$ to $\mathbf{a}_i$ and the arc from $\mathbf{E}$ to $\mathbf{b}_j$ have been replaced by an arc from $\mathbf{a}_i$ to $\mathbf{E}$ and an arc from $\mathbf{E}$ to $\mathbf{b}_j$. Since $\mathbf{E}$ is free in $\mathcal{M}_2$, $Const'_2$ is acyclic.

The digraph $Const_{S \cup F}$ is the union of $Const'_1$ and $Const'_2$ where the two vertices corresponding to $\mathbf{a}_i$ (resp. $\mathbf{b}_j$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{E}$) have been identified. Since $Const'_1$ and $Const'_2$ are acyclic, any cycle of $Const_{S \cup F}$ must contain at least two vertices among $\mathbf{a}_i, \mathbf{b}_j, \mathbf{A}, \mathbf{B}, \mathbf{E}$. Note that $\mathbf{A}$ and $\mathbf{B}$ have no predecessors, that the predecessors of $\mathbf{a}_i$ are $\mathbf{A}$ and a segment end, that the predecessor of $\mathbf{E}$ is $\mathbf{a}_i$. Consequently, there is no cycle going through $\mathbf{a}_i, \mathbf{A}, \mathbf{B}$ or $\mathbf{E}$ and thus $Const_{S \cup F}$ is acyclic. For the same reasons as in Case 1.1, the special points belonging to $\mathbf{a}_i$ and $\mathbf{b}_j$ remain free.

In order to obtain a premodel of T satisfying Property 1, we have to realize some special points of $\mathcal{M}$. When $i > 2$, we realize the special points appearing on $\mathbf{a}_i$ and $\mathbf{b}_j$ except $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{E}$; this is possible since they are free by Corollary 2.20. If $j < q$, we realize $\mathbf{E}$ (if $j = q$, $\mathbf{E}$ is on the border of the polygon).

When $i = 2$, we first realize the special points appearing on $\mathbf{b}_j$ except $\mathbf{B}$ and the special points appearing on $[\mathbf{DE}]$ (that is contained in $\mathbf{a}_i$), except $\mathbf{D}$ and $\mathbf{E}$; this is possible since they are free by Corollary 2.20. If $j < q$, we realize $\mathbf{E}$. If there is a face segment incident to $\mathbf{D}$, then $\mathbf{D}$ is a fan–$a_2 \triangleleft (a_1)$–point and then it is sufficient to prolong $\mathbf{a}_1$ to realize it (it is easy to see it keeps $Const_{S \cup F}$ acyclic, since the predecessors of $\mathbf{a}_1$ are its new endpoint and $\mathbf{C}$). Otherwise, since $\mathbf{D}$ is a fan–$a_2 \triangleleft (a_1, d'_1, \ldots, d'_{s'})$–point, the first step of the realization of $\mathbf{D}$ (as explained in Proposition 2.14) is done by making a traversing of $\mathbf{a}_2$ by the segments $\mathbf{d}'_1, \ldots, \mathbf{d}'_{s'}$ along $\mathbf{a}_1$ (that has been prolonged). Thus, we realize $\mathbf{D}$ inside the polygon $\mathbf{ABCD}$.

Once these realizations have been done, we have obtained a premodel contained in a concave polygon $\mathbf{ABCD}$ that satisfy Property 1.

Case 1.3: There is a chord $a_i c_j$, with $1 < i \leq p$ and $1 < j < r$ (see Figure 17).

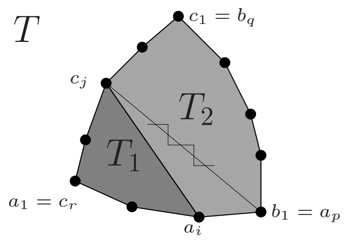

Fig. 17. Case 1.3: Chord $a_i c_j$.

If there are several chords $a_i c_j$, we consider one which maximizes i, *i.e.*, there is no chord $a_k c_j$ with $i < k < r$. Let T_1 (resp. T_2) be the subgraph of T that lies inside the cycle $(a_1, a_2, \ldots, a_i, c_j, \ldots, c_r)$ (resp. $(c_j, a_i, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_j)$). By Lemma 3.2, T_1 and T_2 are W-triangulations. Since T has no chord $a_x a_y$, $b_x b_y$, $c_x c_y$ or $a_k c_j$ with $k > i$, $(a_1, \ldots, a_i)$-(a_i, c_j)-$(c_j, \ldots, c_r)$ (resp. $(c_j, a_i, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_j)$) is a 3-boundary of T_1 (resp. T_2). Furthermore, since $b_1 b_2 \notin E(T_1)$ (resp. $a_1 a_2 \notin E(T_2)$), T_1 (resp. T_2) has less edges than T and Property 1 holds for T_1 and T_2 with the mentioned 3-boundaries.

We distinguish different cases depending on the values of i and p.

Case 1.3.1: $i = p$ (See Figure 18, top left for $i = p = 2$ and top right for $i = p > 2$)

If $i = p = 2$,we want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of T contained in some triangle $\mathbf{BCD}$. Consider three non collinear points $\mathbf{B}, \mathbf{C}, \mathbf{D}$ and let $\mathbf{E}$ be an inner point of the segment $[\mathbf{BD}]$. Let $\mathcal{M}_1 =$

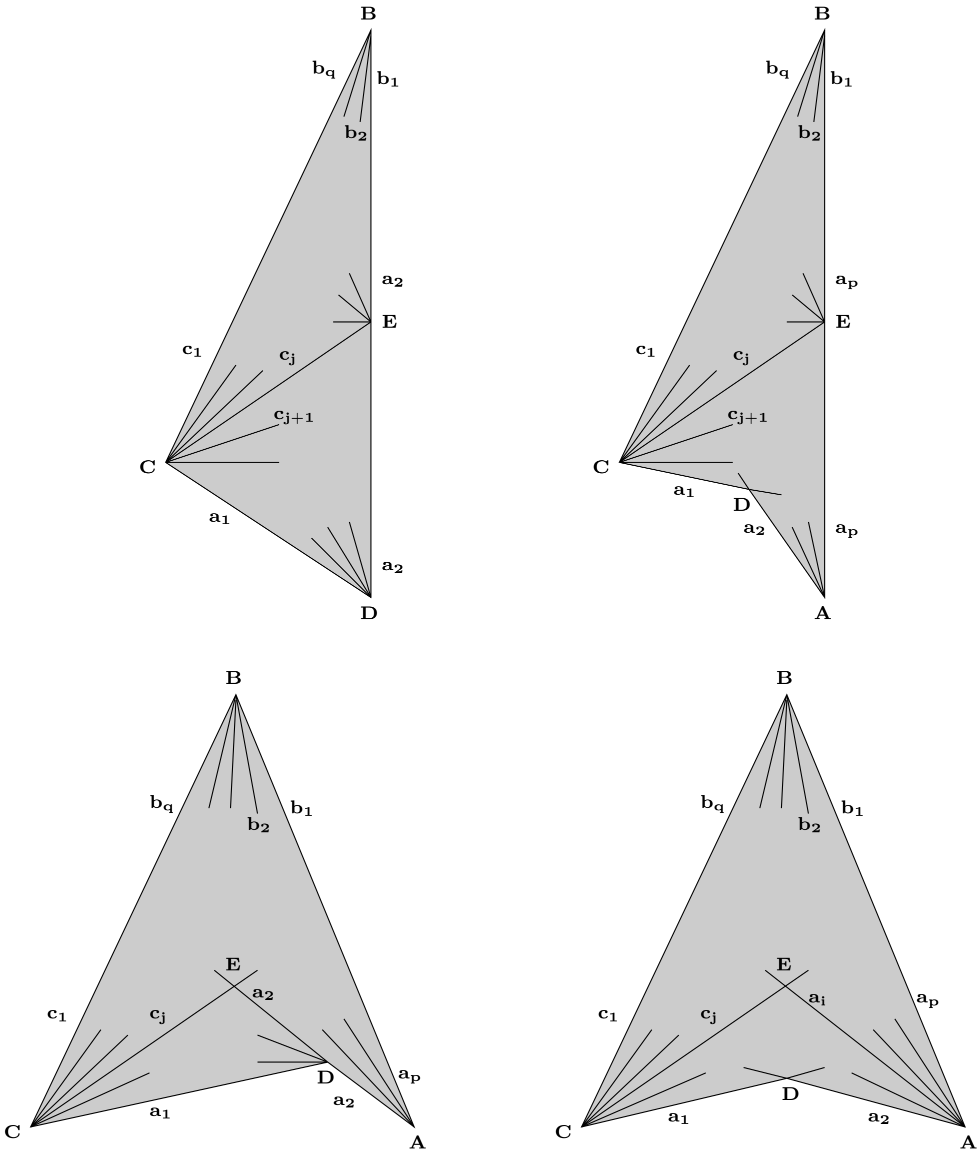

Fig. 18. Case 1.3: when $i = p$ (top) or $i < p$ (bottom) and when $i = 2$ (left) or $i > 2$ (right).

(S_1, F_1, τ_1) be a premodel of T_1 satisfying Property 1 that is contained in the triangle **ECD** where the points **E**, **C** and **D** are respectively a path–(a_p, c_j)–point, a path–$(c_j, \ldots, c_r)$–point, and a fan–$a_p \triangleleft (a_1, \ldots)$–point.

If $i = p > 2$, we want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of T contained in some concave polygon **ABCD**. Consider three non collinear points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and let **E** be an inner point of the segment $[\mathbf{AB}]$. Let $\mathcal{M}_1 = (S_1, F_1, \tau_1)$ be a premodel of T_1 satisfying Property 1 that is contained in some concave polygon **ECAD** for some point **D** where the points **E**, **C**, **A** and **D** are respectively a path–(a_p, c_j)–point, a path–$(c_j, \ldots, c_r)$–point, a path–$(a_2, \ldots, a_p)$–point and the crossing of $\mathbf{a}_1$ and $\mathbf{a}_2$.

In both cases, let $\mathcal{M}_2 = (S_2, F_2, \tau_2)$ be a premodel of T_1 satisfying Property 1 that is contained in the triangle **BCE** where the points **B**, **C** and **E** are respectively a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_j)$–point, and a fan–$c_j \triangleleft (a_p, \ldots)$–point.

By using Lemma 2.12, we can ensure that except $\mathbf{C}, \mathbf{E}$, there is no representative points $\mathbf{p}_1$ of $\mathcal{M}_1$ and $\mathbf{p}_2$ of $\mathcal{M}_2$ exactly at the same position on $\mathbf{c}_j$.

Note that the two segments $\mathbf{c}_j$ (resp. $\mathbf{a}_p$) of S_1 and S_2 form now a single segment $\mathbf{c}_j$ (resp. $\mathbf{a}_p$). Consider now $\mathcal{M} = (S, F, \tau)$ where $S = S_1 \cup S_2$ (up to the identification of the $\mathbf{c}_j$s and of the $\mathbf{a}_p$s), $F = F_1 \cup F_2$, $\tau(\mathbf{p}) = \tau_1(\mathbf{p})$ (resp. $\tau(\mathbf{p}) = \tau_2(\mathbf{p})$) for any point $\mathbf{p} \in Rep_{S_1 \cup F_1} \setminus \{\mathbf{C}, \mathbf{E}\}$ (resp. $\mathbf{p} \in Rep_{S_2 \cup F_2} \setminus \{\mathbf{C}, \mathbf{E}\}$), and where $\tau(\mathbf{C})$ and $\tau(\mathbf{E})$ are defined as follows: $\mathbf{C}$ is now a a path–$(c_1,\ldots,c_r)$–point and $\mathbf{E}$ remains a fan–$a_p\triangleleft(c_j,\ldots)$–point (as in $\mathcal{M}_2$): this is possible since around $\mathbf{E}$, we just have prolonged $\mathbf{a}_p$.

Since $V(T) = V(T_1) \cup V(T_2)$ and $V(T_1) \cap V(T_2) = \{a_p, c_j\}$, every vertex $v \in V(T)$ corresponds to exactly one segment $\mathbf{v}$ in S. Note that $E(T) = E(T_1) \cup E(T_2)$ and that $E(T_1) \cap E(T_2) = \{a_pc_j\}$. Note also that an edge uv is in the graph corresponding to $\mathbf{E}$ (resp. $\mathbf{C}$) in $\mathcal{M}$ if and only if uv is an edge of the graph corresponding to $\mathbf{E}$ (resp. $\mathbf{C}$) in $\mathcal{M}_2$ (resp. in $\mathcal{M}_1$ or in $\mathcal{M}_2$). Thus the edges of T are exactly the edges represented (either by a face segment or in a special point) in $\mathcal{M}$. Since $F(T) = F(T_1) \cup F(T_2)$, since $F(T_1) \cap F(T_2) = \emptyset$, since no face segment has been added or removed, since $\tau(\mathbf{E})$ has not been modified and since $\mathbf{C}$ is a path point (and thus no face is represented in $\mathbf{C}$), the faces represented in $\mathcal{M}$ are exactly the union of the faces represented in $\mathcal{M}_1$ and $\mathcal{M}_2$, i.e., the faces of T.

We know that $Const_{S_1\cup F_1}$ and $Const_{S_2\cup F_2}$ are acyclic. Let $Const'_1$ (resp. $Const'_2$) be the digraph $Const_{S_1\cup F_1}$ (resp. $Const_{S_2\cup F_2}$) where the arc from $\mathbf{E}$ to $\mathbf{a}_p$ has been replaced by an arc from $\mathbf{a}_p$ to $\mathbf{E}$ (this corresponds to the fact that $\mathbf{E}$ is no longer an end of $\mathbf{a}_p$). For the same reasons as in the proof of Case 1.1, $Const'_1$ and $Const'_2$ are acyclic and the internal special points of $\mathbf{c}_j$ remain free.

The digraph $Const_{S\cup F}$ is the union of $Const'_1$ and $Const'_2$ where the two vertices corresponding to $\mathbf{a}_p$ (resp. $\mathbf{c}_j$, $\mathbf{C}$, $\mathbf{E}$) have been identified. Since $Const'_1$ and $Const'_2$ are acyclic, any cycle of $Const_{S\cup F}$ must contain at least two vertices among $\mathbf{a}_p, \mathbf{c}_j, \mathbf{C}, \mathbf{E}$. Note that $\mathbf{C}$ has no predecessor and that any cycle containing $\mathbf{c}_j$ (resp. $\mathbf{E}$) must contain $\mathbf{E}$ (resp. $\mathbf{a}_p$). Since $\mathbf{a}_p$ is not in any cycle, $Const_{S\cup F}$ is acyclic. For the same reasons as in the proof of Case 1.1, the internal special points of $\mathbf{c}_j$ remain free in $\mathcal{M}$.

In order to obtain a premodel of T satisfying Property 1, we just realize the special points of $\mathcal{M}$ that are some inner points of $\mathbf{c}_i$ (this is possible by Corollary 2.20 since they are free).

Case 1.3.2: $p > i$ and $i = 2$ (See Figure 18, bottom left)

Since $p > i = 2$, we want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of T contained in some concave polygon $\mathbf{ABCD}$. Consider three non collinear points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and let $\mathcal{M}_2 = (S_2, F_2, \tau_2)$ be a premodel of T_2 satisfying Property 1 that is contained in some concave polygon $\mathbf{ABCE}$ for some point $\mathbf{E}$ and where the points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{E}$ are respectively a path–$(a_2,\ldots,a_p)$–point, a path–$(b_1,\ldots,b_q)$–point, a path–$(\mathbf{c}_1,\ldots,\mathbf{c}_j)$–point and the crossing of $\mathbf{a}_2$ and $\mathbf{c}_j$.

Let $\mathbf{D}$ be an inner point of $[\mathbf{AE}]$ and let $\mathcal{M}_1 = (S_1, F_1, \tau_1)$ be a premodel of T_1 satisfying Property 1 that is contained in the triangle $\mathbf{ECD}$ where the points $\mathbf{E}$, $\mathbf{C}$ and $\mathbf{D}$ are respectively a path–(a_2, c_j)–point, a path–$(c_j,\ldots,c_r)$–point, and a fan–$a_2\triangleleft(a_1,\ldots)$–point.

By using Lemma 2.12, we can ensure that except $\mathbf{C}, \mathbf{E}$ (note that $\mathbf{D}$ is not a representative point of $\mathcal{M}_2$), there is no representative points $\mathbf{p}_1$ of $\mathcal{M}_1$ and $\mathbf{p}_2$ of $\mathcal{M}_2$ exactly at the same position on $\mathbf{c}_j$ or $\mathbf{a}_2$.

Note that the two segments $\mathbf{c}_j$ (resp. $\mathbf{a}_2$) of S_1 and S_2 form now a single segment $\mathbf{c}_j$ (resp. $\mathbf{a}_2$). Consider now $\mathcal{M} = (S, F, \tau)$ where $S = S_1 \cup S_2$ (up to the identification of the $\mathbf{c}_j$s and of the $\mathbf{a}_2$s), $F = F_1 \cup F_2$, $\tau(\mathbf{p}) = \tau_1(\mathbf{p})$ (resp. $\tau(\mathbf{p}) = \tau_2(\mathbf{p})$) for any point $\mathbf{p} \in Rep_{S_1\cup F_1} \setminus \{\mathbf{C}, \mathbf{E}\}$ (resp. $\mathbf{p} \in Rep_{S_2\cup F_2} \setminus \{\mathbf{C}, \mathbf{E}\}$), and where $\tau(\mathbf{C})$ and $\tau(\mathbf{E})$ are defined as follows: $\mathbf{C}$ is now a a path–$(c_1,\ldots,c_r)$–point and $\mathbf{E}$ remains the crossing point of $\mathbf{C}_j$ and $\mathbf{a}_2$ (as in $\mathcal{M}_2$). Note that $\mathbf{D}$ remains a fan–$a_2\triangleleft(a_1,\ldots)$–point (as in $\mathcal{M}_1$): this is possible, since around $\mathbf{D}$, we just have prolonged $\mathbf{a}_2$.

Since $V(T) = V(T_1) \cup V(T_2)$ and $V(T_1) \cap V(T_2) = \{a_2, c_j\}$, every vertex $v \in V(T)$ corresponds to exactly one segment $\mathbf{v}$ in S. Note that $E(T) = E(T_1) \cup E(T_2)$ and that $E(T_1) \cap E(T_2) = \{a_2c_j\}$. Note also that an edge uv is in the graph corresponding to $\mathbf{C}$ in $\mathcal{M}$ if and only if uv is an edge of the graph corresponding to $\mathbf{C}$ in $\mathcal{M}_1$ or in $\mathcal{M}_2$. Note that the edge a_2c_j is represented by the crossing of $\mathbf{a}_2$ and $\mathbf{c}_j$ in $\mathbf{E}$. Thus the edges of T are exactly the edges represented in $\mathcal{M}$. Since $F(T) = F(T_1) \cup F(T_2)$, since $F(T_1) \cap F(T_2) = \emptyset$, since no face segment has been added or removed and since $\mathbf{C}$ is a path point (and thus no face is represented in $\mathbf{C}$), the faces represented in $\mathcal{M}$ are exactly the union of the faces represented in $\mathcal{M}_1$ and $\mathcal{M}_2$, i.e., the faces of T.

We know that $Const_{S_1\cup F_1}$ and $Const_{S_2\cup F_2}$ are acyclic. Let $Const'_1$ be the digraph obtained from $Const_{S_1\cup F_1}$ where the arc from $\mathbf{D}$ to $\mathbf{a}_2$, the arc from $\mathbf{E}$ to $\mathbf{a}_2$ and the arc from $\mathbf{E}$ to $\mathbf{c}_j$ have been respectively replaced by an arc from $\mathbf{a}_2$ to $\mathbf{D}$, an arc from $\mathbf{a}_2$ to $\mathbf{E}$ and an arc from $\mathbf{c}_j$ to $\mathbf{E}$ (this corresponds to the fact that $\mathbf{D}$ is not longer an end of $\mathbf{a}_2$ and that $\mathbf{E}$ is not longer an end of $\mathbf{a}_2$ or $\mathbf{c}_j$). Since $Const_{S_1\cup F_1}$ is acyclic and since $\mathbf{D}$ and $\mathbf{E}$ are free, it is easy to see that $Const'_1$ is acyclic.

The digraph $Const_{S\cup F}$ is the union of $Const'_1$ and $Const_{S_2\cup F_2}$ where the two vertices corresponding to $\mathbf{a}_2$ (resp. $\mathbf{c}_j$, $\mathbf{C}$, $\mathbf{E}$) have been identified. Since $Const_{1'}$ and $Const_{S_2\cup F_2}$ are acyclic, any cycle in $Const_{S\cup F}$ must contain vertices of $Const_{S_2\cup F_2}$ and of $Const'_1$ and thus, there must be at least two vertices among $\mathbf{a}_2, \mathbf{c}_j, \mathbf{C}, \mathbf{E}$ in any cycle of $Const_{S\cup F}$.

Note that **C** has no predecessor and that **E** has no successor, except possibly a face segment (that has no successor); thus none of them is in any cycle. The predecessors of $\mathbf{c}_j$ and $\mathbf{a}_2$ different from **C** are both in $Const_{S_2 \cup F_2}$ (but not in $Const'_1$). Any cycle containing $\mathbf{c}_j$ and $\mathbf{a}_2$ would be a cycle in $Const_{S_2 \cup F_2}$, which is impossible. Consequently, $Const_{S \cup F}$ is acyclic and thus $\mathcal{M}$ is a premodel of T. For the same reasons as in the proof of Case 1.1, the internal special points of $\mathbf{c}_j$ and $\mathbf{a}_2$ remain free in $\mathcal{M}$.

In order to obtain a premodel of T satisfying Property 1, we have to realize some special points of $\mathcal{M}$. We first realize the special points appearing on $\mathbf{c}_j$ except **C** (they are all on [**CE**]) and the special points appearing on **DE** (that is contained in $\mathbf{a}_2$), except **D** (note that **E** is not a special point). This is possible by Corollary 2.20.

If there is a face segment incident to **D**, then **D** is a fan–$a_2 \triangleleft (a_1)$–point and then it is sufficient to extend $\mathbf{a}_1$ to realize it. Otherwise, since **D** is a fan–$a_2 \triangleleft (a_1, d'_1, \ldots, d'_s)$–point, the first step of the realization of **D** (according to the proof of Proposition 2.14) is done by making a traversing of $\mathbf{a}_2$ by the segments $\mathbf{d}'_1, \ldots, \mathbf{d}'_{s'}$ along $\mathbf{a}_1$ (that has been prolonged) to create a path–$(a_1, d'_1, \ldots, d'_{s'})$–point. Thus, we realize **D** inside the polygon **ABCD** (this is possible since **D** is free).

Once these realizations have been done, we have obtained a premodel contained in a concave polygon **ABCD** that satisfy Property 1.

Case 1.3.3: $p > i$ and $i > 2$ (See Figure 18, bottom right)

Since $p > i > 2$, we want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of T contained in some concave polygon **ABCD**. Consider three non collinear points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and let $\mathcal{M}_2 = (S_2, F_2, \tau_2)$ be a premodel of T_2 satisfying Property 1 that is contained in the concave polygon **ABCE** for some point **E** where the points **A**, **B**, **C**, **E** are respectively a path–$(a_i, \ldots, a_p)$–point, a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_j)$–point and the crossing of $\mathbf{a}_i$ and $\mathbf{c}_j$.

Let $\mathcal{M}_1 = (S_1, F_1, \tau_1)$ be a premodel of T_1 satisfying Property 1 that is contained in the concave polygon **AECD** for some point **D** where the points **A**, **E**, **C** and **D** are respectively a path–$(a_2, \ldots, a_i)$–point, a path–(a_i, c_j)–point, a path–$(c_j, \ldots, c_r)$–point, and the crossing of $\mathbf{a}_1$ and $\mathbf{a}_2$. By using Lemma 2.12, we can ensure that except $\mathbf{C}, \mathbf{E}, \mathbf{A}$, there is no representative points $\mathbf{p}_1$ of $\mathcal{M}_1$ and $\mathbf{p}_2$ of $\mathcal{M}_2$ exactly at the same position on $\mathbf{c}_j$ or $\mathbf{a}_i$.

Note that the two segments $\mathbf{c}_j$ (resp. $\mathbf{a}_i$) of S_1 and S_2 form now a single segment $\mathbf{c}_j$ (resp. $\mathbf{a}_i$). Consider now $\mathcal{M} = (S, F, \tau)$ where $S = S_1 \cup S_2$ (up to the identification of the $\mathbf{c}_j$s and of the $\mathbf{a}_i$s), $F = F_1 \cup F_2$, $\tau(\mathbf{p}) = \tau_1(\mathbf{p})$ (resp. $\tau(\mathbf{p}) = \tau_2(\mathbf{p})$) for any point $\mathbf{p} \in Rep_{S_1 \cup F_1} \setminus \{\mathbf{A}, \mathbf{C}, \mathbf{E}\}$ (resp. $\mathbf{p} \in Rep_{S_2 \cup F_2} \setminus \{\mathbf{A}, \mathbf{C}, \mathbf{E}\}$), and where $\tau(\mathbf{A})$, $\tau(\mathbf{C})$ and $\tau(\mathbf{E})$ are defined as follows: **C** is now a a path–$(c_1, \ldots, c_r)$–point, **A** is now a a path–$(a_2, \ldots, a_p)$–point, and $\tau(\mathbf{E})$ remains the crossing of $\mathbf{a}_i$ and $\mathbf{c}_j$ (as in $\mathcal{M}_2$).

Since $V(T) = V(T_1) \cup V(T_2)$ and $V(T_1) \cap V(T_2) = \{a_i, c_j\}$, every vertex $v \in V(T)$ corresponds to exactly one segment $\mathbf{v}$ in S. Note that $E(T) = E(T_1) \cup E(T_2)$ and that $E(T_1) \cap E(T_2) = \{a_i c_j\}$. Note also that an edge uv is in the graph corresponding to **A** (resp. **C**) in $\mathcal{M}$ if and only if uv is an edge of the graph corresponding to **A** (resp. **C**) in $\mathcal{M}_1$ or in $\mathcal{M}_2$. Note that the edge $a_i c_j$ is represented by the crossing of $\mathbf{a}_i$ and $\mathbf{c}_j$ in **E**. Thus the edges of T are exactly the edges represented in $\mathcal{M}$. Since $F(T) = F(T_1) \cup F(T_2)$, since $F(T_1) \cap F(T_2) = \emptyset$, since no face segment has been added or removed and since **A** and **C** are path points (and thus no face is represented in **A** or **C**), the faces represented in $\mathcal{M}$ are exactly the union of the faces represented in $\mathcal{M}_1$ and $\mathcal{M}_2$, i.e., the faces of T.

We know that $Const_{S_1 \cup F_1}$ and $Const_{S_2 \cup F_2}$ are acyclic. Let $Const'_1$ be the digraph obtained from $Const_{S_1 \cup F_1}$ where the arc from **E** to $\mathbf{a}_i$ and the arc from **E** to $\mathbf{c}_j$ have been respectively replaced by an arc from $\mathbf{a}_i$ to **E** and an arc from $\mathbf{c}_j$ to **E** (this corresponds to the fact that **E** is not longer an end of $\mathbf{a}_i$ or $\mathbf{c}_j$). Since $Const_{S_1 \cup F_1}$ is acyclic and since **E** is free in $\mathcal{M}_1$, $Const'_1$ is acyclic.

The digraph $Const_{S \cup F}$ is the union of $Const'_1$ and $Const_{S_2 \cup F_2}$ where the two vertices corresponding to $\mathbf{a}_i$ (resp. $\mathbf{c}_j$, **A**, **C**, **E**) have been identified. Since $Const_{1'}$ and $Const_{S_2 \cup F_2}$ are acyclic, any cycle in $Const_{S \cup F}$ must contain vertices of $Const_{S_2 \cup F_2}$ and of $Const'_1$ and thus, there must be at least two vertices among $\mathbf{a}_i, \mathbf{c}_j, \mathbf{A}, \mathbf{C}, \mathbf{E}$ in any cycle of $Const_{S \cup F}$. Note that $\mathbf{A}, \mathbf{C}$ have no predecessor and that **E** has no successor, except possibly a face segment (that has no successor); thus none of them is in any cycle. The predecessors of $\mathbf{c}_j$ and $\mathbf{a}_i$ different from $\mathbf{A}, \mathbf{C}$ are both in $Const_{S_2 \cup F_2}$ (but not in $Const'_1$). Any cycle containing $\mathbf{c}_j$ and $\mathbf{a}_2$ would be a cycle in $Const_{S_2 \cup F_2}$, which is impossible. Consequently, $Const_{S \cup F}$ is acyclic and thus $\mathcal{M}$ is a premodel of T. For the same reasons as in the proof of Case 1.1, the internal special points of $\mathbf{c}_j$ and $\mathbf{a}_2$ remain free in $\mathcal{M}$.

In order to obtain a premodel of T satisfying Property 1, we realize the special points appearing on $\mathbf{c}_j$ (resp. $\mathbf{a}_i$) except **C** (resp. **A**); this is possible by Corollary 2.20, since they are free.

Case 1.4: There is no chord a_ib_j, with $1 \leq i \leq p$ and $1 \leq j \leq q$, and no chord a_ic_j, with $1 \leq i \leq p$ and $1 \leq j \leq r$ (see Figure 19).

In this case we consider the adjacent path $(d_1, \ldots, d_s, a_1)$ (see Figure 11) of T with respect to its 3-boundary, $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. Consider the edge d_sa_y, with $1 < y \leq p$ and which minimizes y. This edge exists since, by definition of the adjacent path, d_s is adjacent to some vertex a_y with $y > 1$. The W-triangulation $T_{d_sa_y}$ has less edges than T ($a_1a_2 \notin E(T_{d_sa_y})$), and thus Property 2 holds for $T_{d_sa_y}$.

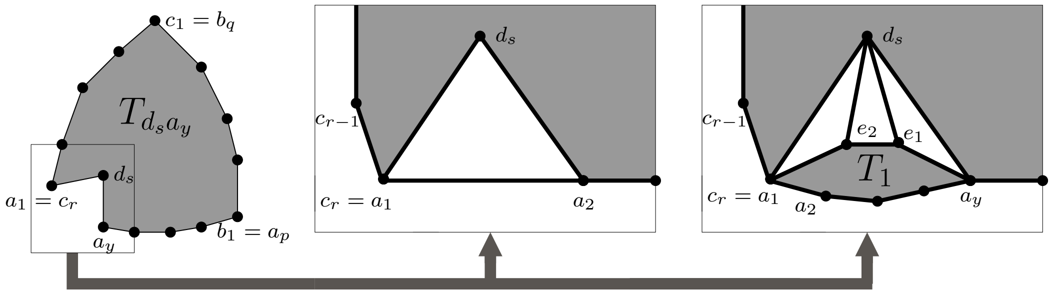

Fig. 19. Case 1.4: No chord a_ib_j or a_ic_j.

Now we distinguish two cases according to the position of a_y, the first is when $y = 2$ and the second is when $y > 2$.

Case 1.4.1: $y = 2$.

In that case, $E(T) = E(T_{d_sa_2}) \cup \{a_1a_2\}$ and $F(T) = F(T_{d_sa_2}) \cup \{a_1a_2d_s\}$.

If $p = y = 2$, for any non-collinear points $\mathbf{B}, \mathbf{C}, \mathbf{D}$, there exists a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_sa_y}$ contained in the triangle **BCD** that satisfies Property 2 and where **B**, **C**, **D** are respectively a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point and a fan-path–$a_2 \triangleleft (d_1, \ldots, d_s) \cdot (d_s, a_1)$–point.

Now, we only change the type of **D** that is now a fan–$a_2 \triangleleft (a_1, d_s, \ldots, d_1)$–point. This is possible since the incidence sequence of **D** is $(a_2, a_1, d_s, \ldots, d_1)$. Note that this modification only adds the edge a_1a_2 to the set of represented edges and the face $a_1a_2d_s$ to the set of represented faces. Consequently, $\mathcal{M}$ is a premodel of T and since there is no face segment incident to **D** (since it was a fan-path point), $\mathcal{M}$ satisfies Property 1.

If $p > 2$, for any non-collinear points $\mathbf{A}, \mathbf{B}, \mathbf{C}$, consider a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_sa_y}$ contained in the concave polygon **ABCE** for some point **E** that satisfies Property 2 and where **A**, **B**, **C**, **E** are respectively a path–$(b_1, \ldots, b_q)$–point, a path–$(a_2, \ldots, a_p)$–point, a path–$(c_1, \ldots, c_r)$–point and a path–(a_2, d_s, a_1)–point.

We do a traversing of $\mathbf{a_2}$ by $\mathbf{a_1}$ along $\mathbf{d_s}$ and then we prolong $\mathbf{a_1}$ (See Figure 20); this is possible by Lemma 2.13, since $Const_{S' \cup F'}$ is aylic. Let **D** be the crossing of $\mathbf{a_1}$ and $\mathbf{a_2}$ and $\mathbf{D'}$ be the crossing of $\mathbf{a_1}$ and $\mathbf{d_s}$. After this move, **E** is the crossing of $\mathbf{d_s}$ and $\mathbf{a_2}$ and is no longer a special point. We add a face segment $\underline{\mathbf{a_1d_sa_2}}$ from $\mathbf{D'}$ to an inner point of [**DE**]. Let S (resp. F, τ) denotes the new segment set, (resp. the new face segment set, the new type function). Note that $S \cup F$ is contained in the concave polygon **ABCD**.

Note that this modification only adds the edge a_1a_2 to the set of represented edges and the face $a_1a_2d_s$ to the set of represented faces. Indeed, there was no face represented in **E** and the edges d_sa_1 and d_sa_2 that were previously represented in **E** are now respectively realized in $\mathbf{D'}$ and in **E**. Note that since we have transformed a special point into different simple points (that cannot belong to any cycle), $Const_{S \cup F}$ is acyclic and thus $\mathcal{M} = (S, F, \tau)$ is a premodel of T that satisfies Property 1 (since **D** is now the crossing of $\mathbf{a_1}$ and $\mathbf{a_2}$).

Case 1.4.2: $y > 2$ (see Figure 21).

Let us denote $e_1, e_2, \ldots, e_t$ the neighbors of d_s strictly inside the cycle $(d_s, a_1, a_2, \ldots, a_y)$, going "from right to left" (see Figure 19). Since y is minimal we have $e_i \neq a_j$, for all $1 \leq i \leq t$ and $1 \leq j \leq y$.

Let T_1 be the subgraph of T that lies inside the cycle $(a_1, \ldots, a_y, e_1, \ldots, e_t, a_1)$. By Lemma 3.2, T_1 is a W-triangulation. Since the W-triangulation T has no separating 3-cycle (d_s, a_1, e_i), (d_s, a_y, e_i) or (d_s, e_i, e_j), there exists no chord a_1, e_i, a_ye_i or e_ie_j in T_1. So (a_2, a_1)-$(a_1, e_t, \ldots, e_1, a_y)$-$(a_y, \ldots, a_2)$ is a 3-boundary of

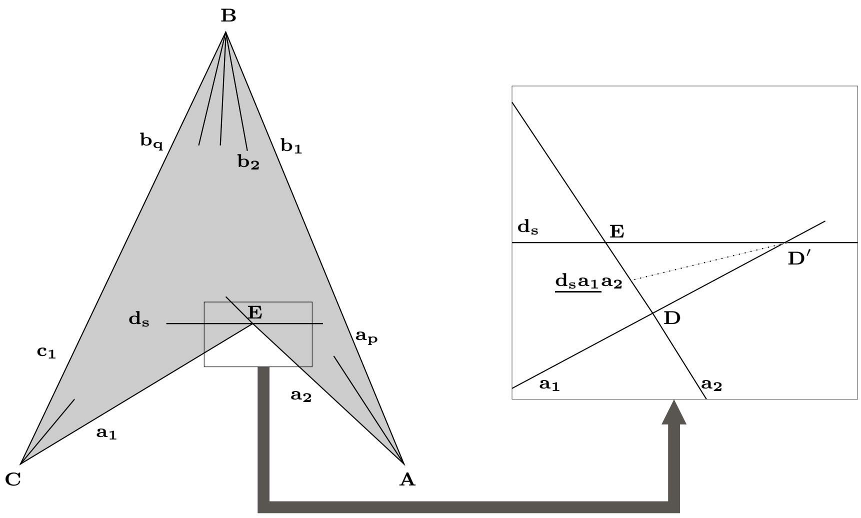

Fig. 20. Case 1.4.1: when $y = 2$ and $p > 2$.

T_1. Finally, since T_1 has less edges than T ($a_1d_s \notin E(T_1)$), Property 1 holds for T_1 with respect to the mentioned 3-boundary.

Since $p \geq y > 2$, we want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of T contained in some concave polygon **ABCD**. Consider three non collinear points $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

If $p = y$ (see Figure 21, left), let $\mathbf{E}$ be an inner point of $[\mathbf{AB}]$. Consider a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_s a_y}$ satisfying Property 2 that is contained in **BCE** and where $\mathbf{B}, \mathbf{C}, \mathbf{E}$ are respectively a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point and a fan-path–$a_p \triangleleft (d_1, \ldots, d_s) \cdot (d_s, a_1)$–point.

If $p > y$ (see Figure 21, right), there exists a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_s a_y}$ satisfying Property 2 that is contained in a concave polygon **ABCE** for some $\mathbf{E}$ and where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{E}$ are respectively a path–$(a_y, \ldots, a_p)$–point, a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point and a path–(a_y, d_s, a_1)–point. Note that there is no face segment incident to $\mathbf{E}$, since it is a path–point.

In both cases, let $\mathbf{D}$ be an inner point of $[\mathbf{EC}]$ and consider a premodel $\mathcal{M}_1 = (S_1, F_1, \tau_1)$ of T_1 that is contained in **AED** and where $\mathbf{A}, \mathbf{E}, \mathbf{D}$ are respectively a path–$(a_2, \ldots, a_y)$–point, path–$(a_y, e_1, \ldots, e_t, a_1)$–point and a fan–$a_1 \triangleleft (a_2, \ldots)$–point. By using Lemma 2.12, when $y = p$ (resp. $y > p$) we can ensure that except $\mathbf{C}, \mathbf{E}$ (resp. $\mathbf{A}, \mathbf{C}, \mathbf{E}$), there is no representative points $\mathbf{p}$ of $\mathcal{M}'$ and $\mathbf{p}_1$ of $\mathcal{M}_1$ exactly at the same position on $\mathbf{a}_1$ (resp. $\mathbf{a}_1, \mathbf{a}_y$).

Note that the two segments $\mathbf{a}_1$ (resp. $\mathbf{a}_y$) of S' and S_1 form now a single segment $\mathbf{a}_1$ (resp. $\mathbf{a}_y$). Consider now $\mathcal{M} = (S, F, \tau)$ where $S = S' \cup S_1$ (up to the identification of the $\mathbf{a}_1$s and of the $\mathbf{a}_y$s), $F = F' \cup F_1$, $\tau(\mathbf{p}) = \tau'(\mathbf{p})$ (resp. $\tau(\mathbf{p}) = \tau_1(\mathbf{p})$) for any point $\mathbf{p} \in Rep_{S' \cup F'} \setminus \{\mathbf{A}, \mathbf{E}\}$ (resp. $\mathbf{p} \in Rep_{S_1 \cup F_1} \setminus \{\mathbf{A}, \mathbf{D}, \mathbf{E}\}$) and where $\tau(\mathbf{A}), \tau(\mathbf{D}), \tau(\mathbf{E})$ are defined as follows.

If $p = y$, $\mathbf{A}$ remains a path–$(a_2, \ldots, a_p)$–point as in $\mathcal{M}_1$, $\mathbf{D}$ remains a fan–$a_1 \triangleleft (a_2, \ldots)$–point as in $\mathcal{M}_1$ (this is possible since around $\mathbf{D}$ we have only prolonged $\mathbf{a}_1$) and $\mathbf{E}$ is a double-fan–$a_p \triangleleft (d_1, \ldots, d_s) \cdot d_s \triangleleft (a_1, e_t, \ldots, e_1, a_p)$–point (this is possible, since the incidence sequence of $\mathbf{E}$ is $(a_p, d_1, \ldots, d_s, a_1, e_t, \ldots, e_1, a_p)$ and since there is no face-segment incident to $\mathbf{E}$).

If $p > y$, $\mathbf{A}$ is now a path–$(a_2, \ldots, a_p)$–point, $\mathbf{D}$ remains a fan–$a_1 \triangleleft (a_2, \ldots)$–point as in $\mathcal{M}_1$ (this is possible since around $\mathbf{D}$ we have only prolonged $\mathbf{a}_1$) and $\mathbf{E}$ is a fan–$d_s \triangleleft (a_y, e_1, \ldots, e_t, a_1)$–point (this is possible, since the incidence sequence of $\mathbf{E}$ is $(d_s, a_y, e_1, \ldots, e_t, a_1, d_s, a_y)$).

Since $V(T) = V(T_1) \cup V(T_{d_s a_y})$ and $V(T_1) \cap V(T_{d_s a_y}) = \{a_1, a_y\}$, every vertex $v \in V(T)$ corresponds to exactly one segment $\mathbf{v}$ in S. Note that $E(T_1) \cap E(T_{d_s a_y}) = \emptyset$ and that $E(T) = E(T_1) \cup E(T_{d_s a_y}) \cup \{d_s e_i \mid i \in [1, t]\}$ (See Figure 21). Any edge uv is represented in $\mathbf{D}$ (resp. $\mathbf{A}$) in $\mathcal{M}$ if and only if uv is represented in $\mathbf{D}$ (resp. $\mathbf{A}$) in $\mathcal{M}_1$ (resp. in $\mathcal{M}'$ or in $\mathcal{M}_1$). In both cases ($y = p$ or $y < p$, see Figure 22), the edges represented in $\mathbf{E}$ in $\mathcal{M}$ are exactly the edges represented in $\mathbf{E}$ in $\mathcal{M}'$, the edges represented in $\mathbf{E}$ in $\mathcal{M}_1$ and the edges in $\{d_s e_i \mid i \in [1, t]\}$. Consequently, the edges represented in $\mathcal{M}$ are exactly the edges of T.

Note that $F(T) = F(T_1) \cup F(T_{d_s a_y} \cup \{d_s a_1 e_t, d_s a_y e_1\} \cup \{d_s e_i e_{i+1} \mid i \in [1, t-1]\}$ (See Figure 21). Since the type of $\mathbf{D}$ has not been changed, the faces represented in $\mathbf{D}$ in $\mathcal{M}$ are exactly the faces represented in $\mathbf{D}$

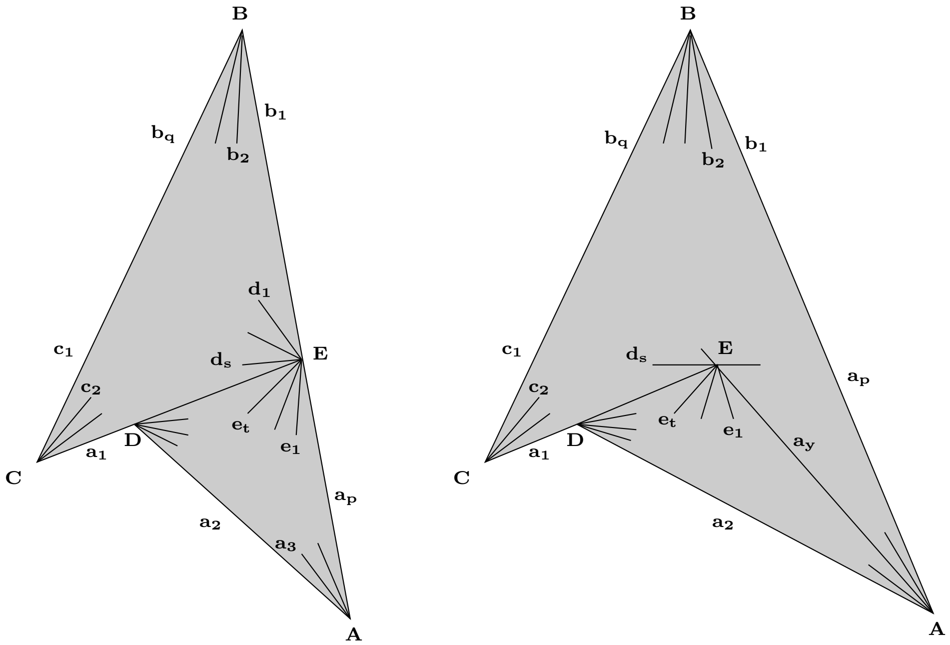

Fig. 21. Case 1.4.2: when $y = p$ (left) or when $y < p$ (right).

in $\mathcal{M}_1$. Since **A** is a path–point in $\mathcal{M}, \mathcal{M}'$ and $\mathcal{M}_1$, no face is represented in **A** in $\mathcal{M}, \mathcal{M}'$ or $\mathcal{M}_1$. In both cases ($y = p$ or $y < p$, see Figure 22), the faces represented in **E** in $\mathcal{M}$ are exactly the faces represented in **E** in $\mathcal{M}'$, the faces represented in **E** in $\mathcal{M}_1$ and the faces in $\{d_s e_i e_{i+1} \mid i \in [1, t-1]\} \cup \{d_s a_y e_1, d_s a_1 e_T\}$. Consequently, the edges represented in $\mathcal{M}$ are exactly the edges of T.

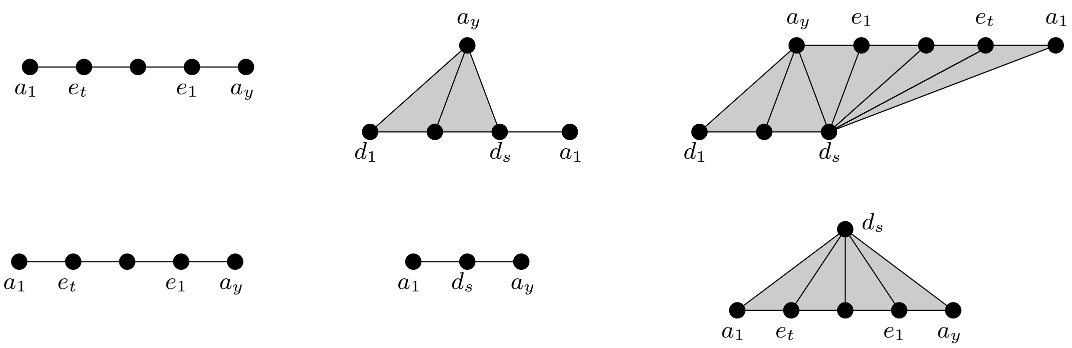

Fig. 22. Case 1.4.2: the graph represented by **E** in $\mathcal{M}_1$ (left), $\mathcal{M}'$ (middle) and $\mathcal{M}$ (right) when $y = p$ (top) or $y < p$ (bottom).

We know that $Const_{S_1 \cup F_1}$ and $Const_{S' \cup F'}$ are acyclic. Let $Const'_1$ be the digraph $Const_{S_1 \cup F_1}$ where the arc from **E** to $\mathbf{a}_y$ and the arc from **D** to $\mathbf{a}_1$ have been respectively replaced by an arc from $\mathbf{a}_y$ to **E** and an arc from $\mathbf{a}_1$ to **D**. If $y = p$, let $Const'_2$ be the digraph $Const_{S' \cup F'}$ where the arc from **E** to $\mathbf{a}_y$ has been replaced by an arc from $\mathbf{a}_y$ to **E**. If $y > p$, $Const'_2 = Const_{S' \cup F'}$. For the same reasons as in the proof of Case 1.1, $Const'_1$ and $Const'_2$ are acyclic and the internal special points of $\mathbf{a}_1$ (resp. $\mathbf{a}_1$ and $\mathbf{a}_y$) remain free if $y = p$ (resp. $y > p$).

The digraph $Const_{S \cup F}$ is the union of $Const'_1$ and $Const'_2$ where the two vertices corresponding to $\mathbf{a}_1$ (resp. $\mathbf{a}_y$, **A**, **C**, **E**) have been identified. Since $Const'_1$ and $Const'_2$ are acyclic, any cycle of $Const_{S \cup F}$ must contain at least two vertices among $\mathbf{a}_1, \mathbf{a}_y, \mathbf{A}, \mathbf{C}, \mathbf{E}$. It is easy to see that $\mathbf{a}_1$ (resp. $\mathbf{a}_y, \mathbf{E}$) has no predecessor in $Const'_1$ except $\mathbf{E}$ (resp. $\mathbf{A}$, $\mathbf{a}_y$). Thus, since $\mathbf{A}$ and $\mathbf{C}$ have no predecessor, $Const_{S \cup F}$ is acyclic.

In order to obtain a premodel of T satisfying Property 1, we have to realize some special points. If $y = p$ (resp. $y < p$), we first realize the special points of $\mathbf{a}_1$ (resp. $\mathbf{a}_1$ and $\mathbf{a}_y$) except $\mathbf{D}$ and $\mathbf{E}$; this is possible since these points are free. If $\mathbf{D}$ is a fan–$a_1 \triangleleft (a_2)$–point, then it is sufficient to prolong a_2 to realize it. If $\mathbf{D}$ is a fan–$a_1 \triangleleft (a_2, d'_1, \ldots, d'_{s'})$–point, then we realize it according to Proposition 2.14. The first step is a traversing of $\mathbf{a}_1$ by $\mathbf{d}'_1, \ldots, \mathbf{d}'_{s'}$ along $\mathbf{a}_2$; thus $\mathbf{D}$ is realized inside $\mathbf{ABCD}$.

If $y > p$, we still have to realize the point $\mathbf{E}$ that is not necessary free (there may be an intersection between one of the e_i and a_2). Since $\mathbf{E}$ is a fan–$d_s \triangleleft (a_y, e_1, \ldots, e_t, a_1)$–point, we first do a traversing of $\mathbf{d}_s$ by $(e_1, \ldots, e_t, a_1)$ to obtain a path–$(a_y, a_1, e_t, \ldots, e_1)$–point $\mathbf{E}'$. We can prolong $\mathbf{a}_1$ without changing the type of $\mathbf{E}'$; it is possible since we know that $\mathbf{a}_1$ has no predecessor in $Const'_1$. Since $\mathbf{E}$ was free in $Const'_1$, $\mathbf{E}'$ is a free point and then it can be realized.

Once all these realizations have been done, we have obtained a premodel contained in a concave polygon $\mathbf{ABCD}$ satisfying Property 1.

This completes the study of Case 1 and ends the proof of Lemma 3.8. □

We now prove the inductive step for Property 2 with the following lemma.

Lemma 3.9. *For any integer $m > 3$, if Property 1 holds for any W-triangulation T such that $|E(T)| < m$ and Property 2 holds for any W-triangulation $T_{d_x a_y}$ such that $|E(T_{d_x a_y})| < m$, then Property 2 holds for any W-triangulation $T_{d_x a_y}$ such that $|E(T)| = m$.*

Case 2: Proof of Property 2 for any W-triangulation $T_{d_x a_y}$ such that $|E(T_{d_x a_y})| = m$.

Recall that the W-triangulation $T_{d_x a_y}$ is a subgraph of a W-triangulation T with a 3-boundary $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. Moreover, T has no chord $a_i b_j$ or $a_i c_j$ and its adjacent path is $(d_1, \ldots, d_s, a_1)$, with $s \geq 1$. We distinguish two cases: either $d_x a_y = d_1 a_p$ or $d_x a_y \neq d_1 a_p$.

Case 2.1: $d_x a_y = d_1 a_p$ (see Figure 23).

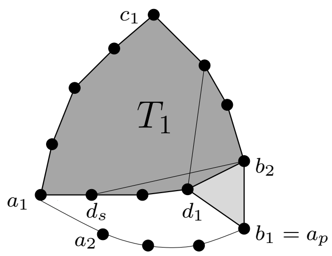

Fig. 23. Case 2.1: $T_{d_x a_y} = T_{d_1 a_p}$.

Let T_1 be the subgraph of $T_{d_1 a_p}$ that lies inside the cycle $(a_1, d_s, \ldots, d_1, b_2, \ldots, b_q, c_2, \ldots, c_r)$. By Lemma 3.2, T_1 is a W-triangulation. This W-triangulation has no chord $b_i b_j$, $c_i c_j$, $d_i d_j$, or $a_1 d_j$. We consider two cases according to the existence of an edge $d_1 b_i$ with $2 < i \leq q$.

(1) If T_1 has no chord $d_1 b_i$ then $(d_1, b_2, \ldots, b_q)$-$(c_1, \ldots, c_r)$-$(a_1, d_s, \ldots, d_1)$ is a 3-boundary of T_1.
(2) If T_1 has a chord $d_1 b_i$, with $2 < i \leq q$, note that $q > 2$ and that there cannot be a chord $b_2 a_1$ or $b_2 d_j$, with $1 < j \leq s$ (this would violate the planarity of $T_{d_x a_y}$, see Figure 23). So in this case, $(b_2, d_1, \ldots, d_s, a_1)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_2)$ is a 3-boundary of T_1.

Finally, since T_1 is a W-triangulation with less edges than $T_{d_1 a_p}$ ($b_1 b_2 \notin E(T_1)$), Property 1 holds for T_1 with respect to at least one of the two mentioned 3-boundaries.

We want to construct a premodel $\mathcal{M}$ of $T_{d_1 a_p}$ contained in a triangle $\mathbf{BCD}$. Consider three non-collinear points $\mathbf{B}, \mathbf{C}, \mathbf{D}$.

If we consider the 3-boundary mentioned in (1) and if $q = 2$, consider a inner point $\mathbf{E}$ of $[\mathbf{BD}]$ and consider a premodel $\mathcal{M}' = (S', F', \tau')$ contained in $\mathbf{CDE}$ satisfying Property 1 where $\mathbf{C}, \mathbf{D}, \mathbf{E}$ are respectively a path–$(c_1, \ldots, c_r)$–point, a path–$(a_1, d_s, \ldots, d_1)$–point and a fan–$b_2 \lhd (d_1, \ldots)$–point. In that case, we prolong $\mathbf{b}_2$ so that its new end is $\mathbf{B}$ (See Figure 24, left).

Otherwise, consider a premodel $\mathcal{M}' = (S', F', \tau')$ satisfying Property 1 contained in a concave polygon $\mathbf{BCDE}$ for some point $\mathbf{E}$ where $\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$ are respectively a path–$(b_2, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point, a path–$(a_1, d_s, \ldots, d_1)$–point and the crossing point of $\mathbf{d}_1$ and $\mathbf{b}_2$ (See Figure 24, right).

In both cases, we add a new segment $\mathbf{b}_1$ from $\mathbf{D}$ to $\mathbf{B}$ and a new face segment $\underline{\mathbf{b_2 d_1}}\mathbf{b_1}$ going from $\mathbf{E}$ to an inner point of $\mathbf{b}_1$.

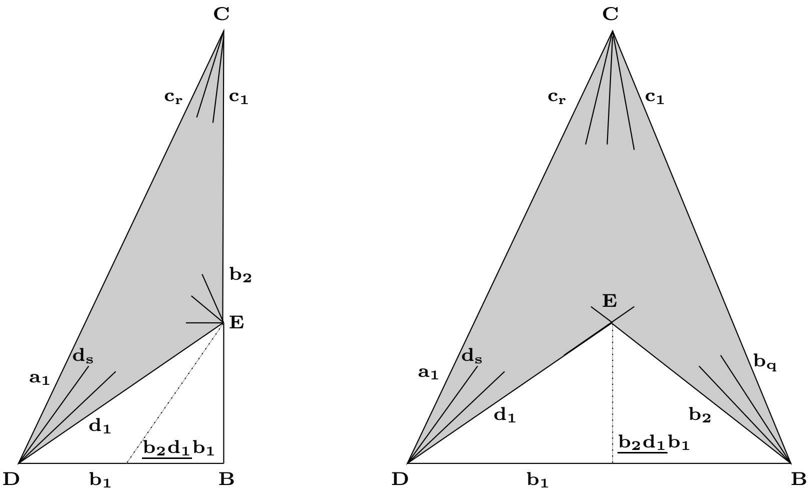

Fig. 24. Case 2.1.

Consider now $\mathcal{M} = (S, F, \tau)$ with $S = S' \cup \{\mathbf{b}_1\}$, $F = F' \cup \{\underline{\mathbf{b_2 d_1}}\mathbf{b_1}\}$, $\tau(\mathbf{p}) = \tau'(\mathbf{p})$ for any $\mathbf{p} \in Rep_{S' \cup F'} \setminus \{\mathbf{B}, \mathbf{D}, \mathbf{E}\}$ and where $\tau(\mathbf{B}), \tau(\mathbf{D}), \tau(\mathbf{E})$ are defined as follows. $\mathbf{B}$ is a path–$(b_1, \ldots, b_q)$–point; this is possible since its incidence sequence is $(\mathbf{b}_1, \ldots, \mathbf{b}_q)$. $\mathbf{D}$ is a fan-path–$b_1 \lhd (d_1) \cdot (d_1, \ldots, d_s, a_1)$–point; this is possible since its incidence sequence is $(\mathbf{b}_1, \mathbf{d}_1, \ldots, \mathbf{d}_s, \mathbf{a}_1)$. If in $\mathcal{M}'$, $\mathbf{E}$ is the crossing of $\mathbf{a}_1$ and $\mathbf{a}_2$ or is a fan–$b_2 \lhd (d_1)$–point, then in $\mathcal{M}$, $\mathbf{E}$ is the crossing of $\mathbf{a}_1$ and $\mathbf{a}_2$; it is possible since if there is a face segment incident to $\mathbf{E}$ in $\mathcal{M}'$, then $\mathbf{d}_1$ or $\mathbf{d}_2$ separates it from $\underline{\mathbf{b_2 d_1}}\mathbf{b_1}$. If in $\mathcal{M}'$, $\mathbf{E}$ is a fan–$b_2 \lhd (d_1, d'_1, \ldots, d'_{s'})$–point (with $s' \geq 1$), then it remains a fan–$b_2 \lhd (d_1, d'_1, \ldots, d'_{s'})$–point; this is possible since its incidence sequence is $(\mathbf{b}_2, \underline{\mathbf{b_2 d_1}}\mathbf{b_1}, \mathbf{d}_1, \mathbf{d}'_1, \ldots, \mathbf{d}'_{s'}, \mathbf{b}_2)$. In both cases, it is easy to see that the edges and the faces represented in $\mathbf{E}$ have not been modified.

Since $V(T_{d_1 a_p}) = V(T) \cup \{b_1\}$, every vertex $v \in V(T)$ corresponds to exactly one segment $\mathbf{v}$ in S. Note that $E(T_{d_1 a_p}) = E(T) \cup \{b_1 d_1, b_1 b_2\}$. It is easy to see that the edges represented in $\mathbf{B}$ (resp. $\mathbf{D}$) in $\mathcal{M}$ are exactly the edges represented in $\mathbf{B}$ (resp. $\mathbf{D}$) in $\mathcal{M}'$ and the edge $b_1 b_2$ (resp. $b_1 d_1$). Since we have not modified the edges represented in $\mathbf{E}$, the edges represented in $\mathcal{M}$ are exactly the edges of $T_{d_1 a_p}$. Note that $F(T_{d_1 a_p}) = F(T) \cup \{b_2 d_1 b_1\}$. Since we have added a face segment $\underline{\mathbf{b_2 d_1}}\mathbf{b_1}$ and since we have not changed the faces represented in $\mathbf{B}, \mathbf{D}, \mathbf{E}$, the edges represented in $\mathcal{M}$ are exactly the faces of $T_{d_1 a_p}$.

Since all the special points of $\mathcal{M}$ appear on $[\mathbf{BC}]$, $[\mathbf{CD}]$ or $[\mathbf{BD}]$, it is easy to see that $Const_{S \cup F}$ is acyclic and thus, $\mathcal{M}$ is a premodel of $T_{d_1 a_p}$ that satisfies Property 2.

Case 2.2: $T_{d_x a_y} \neq T_{d_1 a_p}$.

In this case we consider an edge $d_z a_w \in E(T_{d_x a_y})$ such that $d_z a_w \neq d_x a_y$. Among all the possible edges $d_z a_w$ we choose the one that first maximizes z and then minimizes w. Such an edge necessarily exists and actually one can see that $d_z = d_x$ or $d_z = d_{x+1}$. Indeed, if $d_x = d_1$ there is at least one edge $d_1 a_w$ with $w > y$, the edge $d_1 a_p$. If $x > 1$, it is clear by definition of the adjacent path that the vertex d_{x-1} is adjacent to at least one vertex a_w with $w \geq y$. By Lemma 3.2, $T_{d_z a_w}$ is a W-triangulation. Since $d_x a_y \notin E(T_{d_z a_w})$, the W-triangulation $T_{d_z a_w}$ has less edges than $T_{d_x a_y}$, and so Property 2 holds for $T_{d_z a_w}$.

We distinguish 4 cases according to the values of z and w.

(Case 2.1) $z = x$ and $w = y + 1$,
(Case 2.2) $z = x - 1$ and $w = y$,
(Case 2.3) $z = x$ and $w > y + 1$,
(Case 2.4) $z = x - 1$ and $w > y$.

Case 2.2.1: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x$ and $w = y + 1$ (see Figure 25).

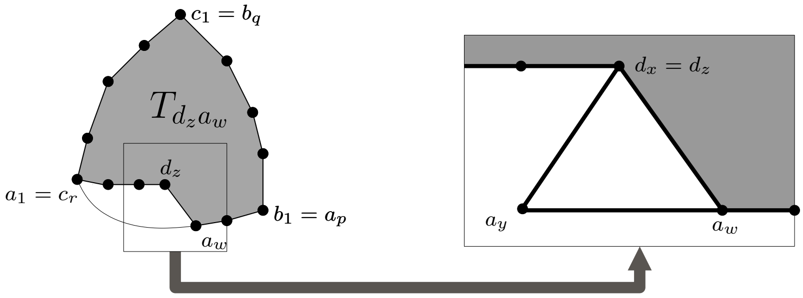

Fig. 25. Case 2.2.1: $z = x$ and $w = y + 1$.

We want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of $T_{d_x a_y}$ contained in some concave polygon $\mathbf{ABCD}$. Consider three non-collinear points $\mathbf{B}, \mathbf{C}, \mathbf{E}$.

If $w = p$ (See Figure 26, top left), consider a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_x a_w}$ satisfying Property 2 that is contained in $\mathbf{BCE}$ and where the points $\mathbf{B}, \mathbf{C}, \mathbf{E}$ are respectively a path–$(b_1, \dots, b_q)$–point, a path–$(c_1, \dots, c_r)$–point and a fan-path–$a_w \lhd (d_1, \dots, d_x) \cdot (d_x, \dots, d_s, a_1)$–point. We then prolong $\mathbf{a}_w$ after $\mathbf{E}$ to a new point $\mathbf{A}$ (since $\mathbf{E}$ is free, it keeps the constraints digraph acyclic).

If $w < p$ (See Figure 26, bottom left), consider a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_x a_w}$ satisfying Property 2 that is contained in a concave polygon $\mathbf{ABCE}$ for some point $\mathbf{A}$ and where the points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{E}$ are respectively a path–$(a_w, \dots, a_p)$–point, a path–$(b_1, \dots, b_q)$–point, a path–$(c_1, \dots, c_r)$–point and a path–$(a_w, d_x, \dots, d_s, a_1)$–point.

In both cases, we do a gliding of $(\mathbf{d}_{x+1}, \dots, \mathbf{d}_s, \mathbf{a}_1)$ on $\mathbf{d}_x$; this is possible and it keeps the constraints digraph acyclic from Lemma 2.12 since $\mathbf{E}$ is free. Let $\mathbf{D}$ be the new intersection point of $\mathbf{d}_x$ and $\mathbf{d}_{x+1}, \dots, \mathbf{d}_s, \mathbf{a}_1$. Then, we add a segment $\mathbf{a}_y$ from $\mathbf{A}$ to $\mathbf{D}$ and we prolong it after $\mathbf{D}$. Then, we add a face segment $\underline{\mathbf{d_x a_w a_y}}$ from $\mathbf{E}$ to an inner point of $[\mathbf{AE}]$. One can easily check that adding this segment and this face segment keeps the constraints digraph acyclic.

Consider now $\mathcal{M} = (S, F, \tau)$ where $S = S' \cup \{\mathbf{a}_y\}$, $F = F' \cup \{\underline{\mathbf{d_x a_w a_y}}\}$, $\tau(\mathbf{p}) = \tau'(\mathbf{p})$ for any $\mathbf{p} \in Rep_{S' \cup F'} \setminus \{\mathbf{E}, \mathbf{A}\}$ and where $\tau(\mathbf{A}), \tau(\mathbf{D})$ and $\tau(\mathbf{E})$ are defined as follows. $\mathbf{A}$ is now a path–$(a_y, a_w, \dots, a_p)$–point. $\mathbf{D}$ is a path–$(a_y, d_x, \dots, d_s, a_1)$–point; this is possible since its incidence sequence is $(\mathbf{a}_y, \mathbf{d}_x, \dots, \mathbf{d}_s, \mathbf{a}_1, \mathbf{a}_y, \mathbf{d}_x)$. If $w = p$, $\mathbf{E}$ is now a fan–$a_w \lhd (d_x, \dots, d_1)$–point; this is possible since its incidence sequence is $(\mathbf{a}_w, \underline{\mathbf{d_x a_w a_y}}, \mathbf{d}_x, \dots, \mathbf{d}_1, \mathbf{a}_w)$. If $w < p$, $\mathbf{E}$ is now the crossing point of a_w and d_x; if there is a face segment incident to $\mathbf{E}$ in $\mathcal{M}'$, either $\mathbf{d}_x$ or $\mathbf{a}_w$ separates it from $\underline{\mathbf{d_x a_w a_y}}$.

Note that in both cases, the edges represented in $\mathbf{D}$ and $\mathbf{E}$ in $\mathcal{M}$ are exactly the edges represented in $\mathbf{E}$ in $\mathcal{M}'$ and the edge $d_x a_y$. Note that no face is represented in $\mathbf{D}$ in $\mathcal{M}$ and that the faces represented in $\mathbf{E}$ in $\mathcal{M}$ are exactly the faces represented in $\mathbf{E}$ in $\mathcal{M}'$.

Since $V(T_{d_x a_y}) = V(T_{d_z a_w}) \cup \{a_y\}$, every vertex $v \in V(T)$ corresponds to exactly one segment $\mathbf{v}$ in S. Note that $E(T_{d_x a_y}) = E(T_{d_z a_w}) \cup \{d_x a_y, a_w a_y\}$. Since $d_x a_y$ (resp. $a_w a_y$) are now represented in $\mathbf{D}$ (resp. $\mathbf{A}$) and since the other edges represented in $\mathcal{M}$ are exactly the edges represented in $\mathcal{M}'$, the edges represented in $\mathcal{M}$ are exactly the edges of $T_{d_x a_y}$. Note that $F(T_{d_x a_y}) = F(T_{d_z a_y}) \cup \{d_x a_w a_y\}$. Since we have added a face segment $\underline{\mathbf{d_x a_w a_y}}$ and since we have preserved the faces represented in $\mathcal{M}'$, the faces represented in $\mathcal{M}$ are exactly the faces of $T_{d_x a_y}$.

If $w < p$, we realize all the special points appearing on $\mathbf{a}_w$ (they are on $[\mathbf{AE}]$). Then, in both cases, we have constructed a premodel $\mathcal{M}$ of $T_{d_x a_y}$ that satisfies Property 2.

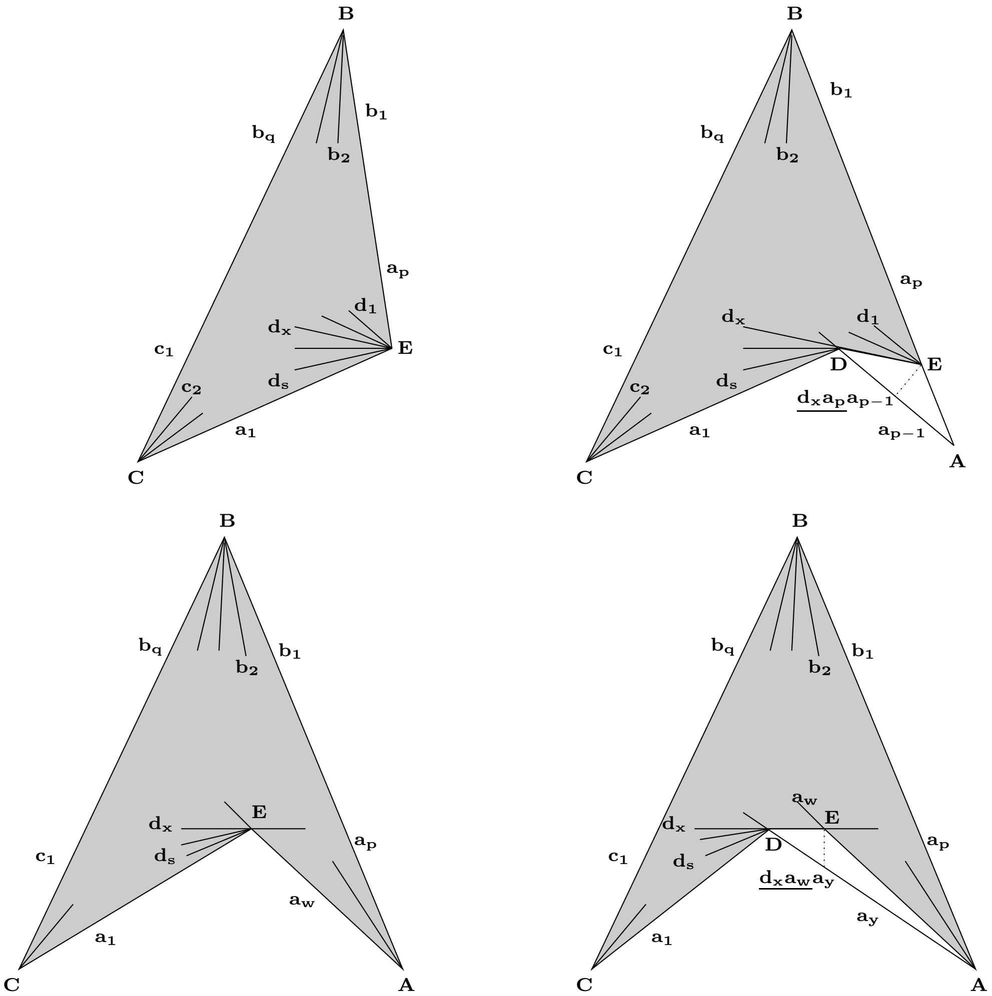

Fig. 26. Case 2.2.1: when $w = p$ (top) or $w < p$ (bottom

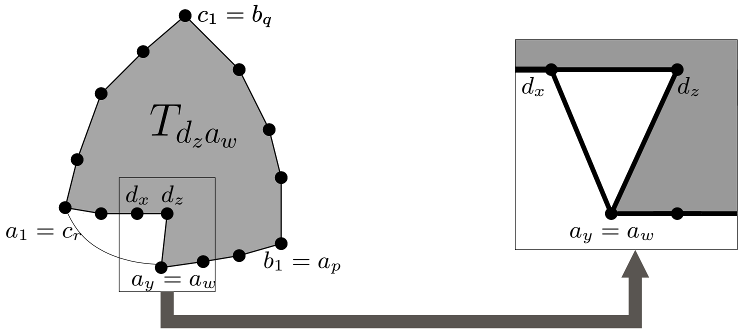

Fig. 27. Case 2.2.2: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x - 1$ and $w = y$.

Case 2.2.2: $z = x - 1$ and $w = y$ (see Figure 27).

If $w = p$, we want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of $T_{d_x a_y}$ contained in a triangle **BCD**. Consider a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_z a_y}$ satisfying Property 2 that is contained in **BCD** and where the points $\mathbf{B}, \mathbf{C}, \mathbf{A}$ are respectively a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point and a fan-path–$a_y \triangleleft (d_1, \ldots, d_z) \cdot (d_z, d_x, \ldots, d_s, a_1)$–point.

Let $\mathcal{M} = (S', F', \tau)$ where $\tau(\mathbf{p}) = \tau'(\mathbf{p})$ for any $\mathbf{p} \in Rep_{S' \cup F'} \setminus \mathbf{D}$ and let $\mathbf{D}$ be a fan-path–$a_y \triangleleft (d_1, \ldots, d_z, d_x) \cdot (d_x, \ldots, d_s, a_1)$–point.

By changing the type of $\mathbf{D}$, we have added the edge $d_x a_y$ to the set of represented edges and the face $d_x d_z a_y$ to the set of represented faces. Since $V(T_{d_x a_y}) = V(T_{d_z a_y})$, $E(T_{d_x a_y}) = E(T_{d_z a_y}) \cup \{d_x a_y\}$ and $F(T_{d_x a_y}) = F(T_{d_z a_y}) \cup \{d_x d_z a_y\}$, $\mathcal{M}$ is a premodel of $T_{d_x a_y}$.

If $w > p$ (See Figure 28), we want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of $T_{d_x a_y}$ contained in a concave polygon $\mathbf{ABCD}$. Consider three non-collinear points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_z a_y}$ satisfying Property 2 that is contained in a concave polygon $\mathbf{ABCE}$ for some point $\mathbf{E}$ and where the points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{E}$ are respectively a path–$(a_y, \ldots, a_p)$–point, a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point and a path–$(a_y, d_z, d_x, \ldots, d_s, a_1)$–point.

We do a gliding of $(\mathbf{d}_x, \ldots, \mathbf{d}_s, \mathbf{a}_1)$ on $\mathbf{a}_y$; by Lemma 2.12, this is possible and it keeps the constraints digraph acyclic, since $\mathbf{E}$ is free. Let $\mathbf{D}$ be the new intersection point of $\mathbf{a}_y$ and $\mathbf{d}_x, \ldots, \mathbf{d}_s, \mathbf{a}_1$ (note that $\mathbf{D}$ is free). Note that since $\mathbf{E}$ is not an end of $\mathbf{d}_z$, by choosing $\mathbf{D}$ close enough from $\mathbf{E}$, one can ensure that $(\mathbf{d}_x)$ and $\mathbf{d}_z$ intersect. We prolong $\mathbf{d}_x$ after $\mathbf{D}$ such that $\mathbf{d}_x$ and $\mathbf{d}_z$ intersect in some point $\mathbf{D}'$. If necessary, we extend $\mathbf{d}_z$ and $\mathbf{d}_x$ in such a way that $\mathbf{D}'$ is not an end of $\mathbf{d}_z$ or $\mathbf{d}_x$. Note that since $\mathbf{D}$ is free and since the crossing between $\mathbf{d}_x$ and $\mathbf{d}_z$ is not a special point, when extending $\mathbf{d}_x$, we keep the constraints digraph acyclic. Then, we add a face segment $\underline{\mathbf{d_x d_z}}\mathbf{a_y}$ from $\mathbf{D}'$ to an inner point of $[\mathbf{ED}]$ (that is contained in $\mathbf{a}_y$).

Let $\mathcal{M} = (S, F, \tau)$ with $S = S'$, $F = F' \cup \underline{\mathbf{d_x d_z}}\mathbf{a_y}$, where for any representative point $\mathbf{p} \in Rep_{S' \cup F'} \setminus \{\mathbf{E}\}$, $\tau(\mathbf{p}) = \tau(\mathbf{p}')$ and where $\tau(\mathbf{D}), \tau(\mathbf{D}')$ and $\tau(\mathbf{E})$ are defined as follows: $\mathbf{D}$ is a path–$(a_y, d_x, \ldots, d_s, a_1)$–point, $\mathbf{D}'$ is the crossing point of $\mathbf{d}_x$ and $\mathbf{d}_z$ and $\mathbf{E}$ is now the crossing point of $\mathbf{d}_z$ and $\mathbf{a}_y$.

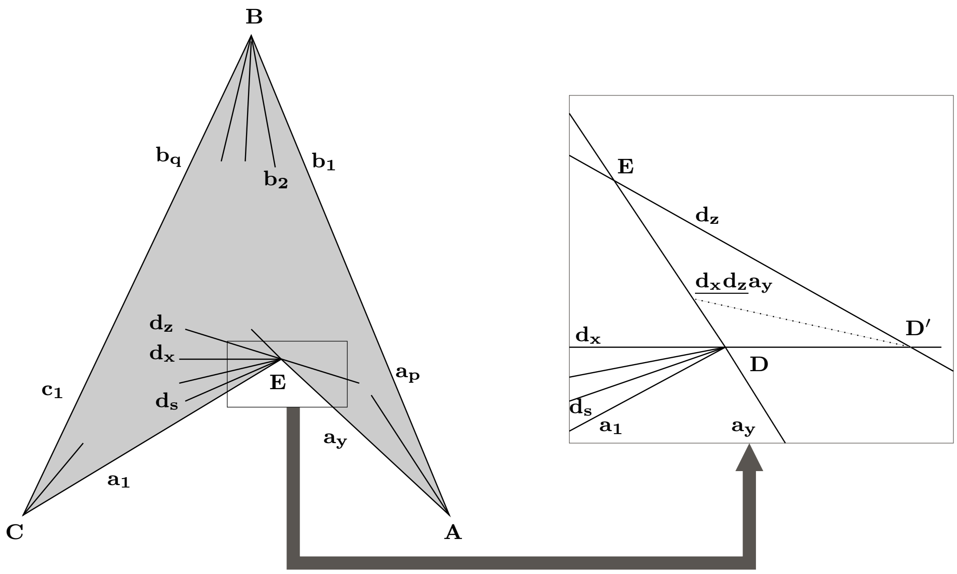

Fig. 28. Case 2.2.2: when $y < p$.

Since $V(T_{d_x a_y}) = V(T_{d_z a_y})$, every vertex $v \in V(T_{d_x a_y})$ corresponds to exactly one segment $\mathbf{v}$ in S. Note that $E(T_{d_x a_y}) = E(T_{d_z a_y}) \cup \{d_x a_y\}$. In $\mathcal{M}'$, the edges $\{d_i d_{i+1} \mid i \in [x, s-1]\} \cup \{d_x d_z, d_z a_y, d_s a_1\}$ are represented in $\mathbf{E}$. In $\mathcal{M}$, the edges represented in $\mathbf{D}$ are $\{d_i d_{i+1} \mid i \in [x, s-1]\} \cup \{d_x a_y, d_s a_1\}$. Since the edges $d_x d_z$ and $d_z a_y$ are represented respectively in $\mathbf{D}'$ and $\mathbf{E}$ in $\mathcal{M}$, the edges represented in $\mathcal{M}$ are exactly the edges of $T_{d_x a_y}$. Note that $F(T_{d_x a_y}) = F(T_{d_z a_y}) \cup \{d_x d_z a_y\}$. Since no face is represented in $\mathbf{E}$ in $\mathcal{M}'$ or in $\mathbf{D}$ in $\mathcal{M}$ and since we have added a face segment $\underline{\mathbf{d_x d_z}}\mathbf{a_y}$, the faces represented in $\mathcal{M}$ are exactly the faces of $T_{d_x a_y}$.

Since all the special points of $\mathcal{M}$ appear on $\mathbf{AC}$, $\mathbf{BC}$, $\mathbf{CD}$ or $\mathbf{BD}$, $\mathcal{M}$ is a premodel of $T_{d_1 a_p}$ that satisfy Property 2.

Case 2.2.3: $z = x$ and $w > y + 1$ (see Figure 29).

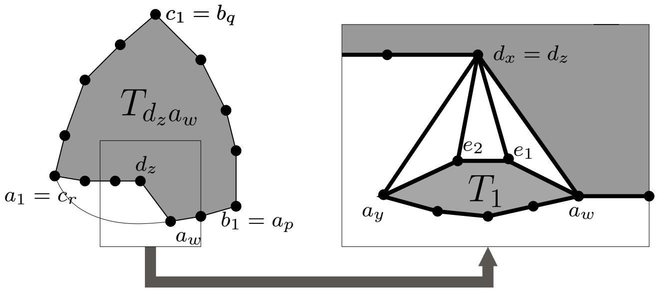

Fig. 29. Case 2.2.3: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x$ and $w > y + 1$.

Let us denote $e_1, e_2, \ldots, e_t$ the neighbors of d_x strictly inside the cycle $(d_x, a_y, \ldots, a_w)$, going "from right to left" (see Figure 29). Since there is no chord $a_i a_j$ we have $t \geq 1$. Furthermore w being minimal we have $e_i \neq a_j$, for all $1 \leq i \leq t$ and $y \leq j \leq w$. Let T_1 be the subgraph of $T_{d_x a_y}$ that lies inside the cycle $(a_y, \ldots, a_w, e_1, \ldots, e_t, a_y)$. By Lemma 3.2, T_1 is a W-triangulation. Since the W-triangulation $T_{d_x a_y}$ has no separating 3-cycle (d_x, a_w, e_i) or (d_x, e_i, e_j), there exists no chord $a_w e_i$ or $e_i e_j$ in T_1. With the fact that $t \geq 1$, we know that (e_t, a_y)-$(a_y, \ldots, a_w)$-$(a_w, e_1, \ldots, e_t)$ is a 3-boundary of T_1. Finally, since T_1 has less edges than $T_{d_x a_y}$ ($d_x a_y \notin E(T_1)$), Property 1 holds for T_1 with respect to the mentioned 3-boundary.

We want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of $T_{d_x a_y}$ contained in some concave polygon **ABCD**. Consider three non-collinear points **B**, **C**, **E**.

If $w = p$ (See Figure 30, top left), consider a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_x a_w}$ satisfying Property 2 that is contained in **BCE** and where the points **B**, **C**, **E** are respectively a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point and a fan-path–$a_w \triangleleft (d_1, \ldots, d_x) \cdot (d_x, \ldots, d_s, a_1)$–point. We then prolong $\mathbf{a}_w$ after **E** to a new point **A** (since **E** is free, it keeps the constraints digraph acyclic).

If $w < p$ (See Figure 30, bottom left), consider a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_x a_w}$ satisfying Property 2 that is contained in a concave polygon **ABCE** for some point **A** and where the points **A**, **B**, **C**, **E** are respectively a path–$(a_w, \ldots, a_p)$–point, a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point and a path–$(a_w, d_x, \ldots, d_s, a_1)$–point.

In both cases, as in Case 2.2.1, we do a gliding of $(\mathbf{d}_{x+1}, \ldots, \mathbf{d}_s, \mathbf{a}_1)$ on $\mathbf{d}_x$. Let **D** be the new intersection point of $\mathbf{d}_x$ and $\mathbf{d}_{x+1}, \ldots, \mathbf{d}_s, \mathbf{a}_1$. Since we have done exactly the same moves as in Case 2.2.1, for the same reasons as before, the constraints digraph is still acyclic after these modifications.

Consider now an inner point **F** of [**AD**] and a premodel $\mathcal{M}_1 = (S_1, F_1, \tau_1)$ of T_1 satisfying Property 1 that is contained in **AEF** and where the points **A**, **E**, **F** are respectively a path–$(a_y, \ldots, a_w)$–point, a path–$(a_w, e_1, \ldots, e_t)$–point and a fan–$a_y \triangleleft (e_t, \ldots)$–point. By using Lemma 2.12, we can ensure that when $w < p$, there are no representative points $\mathbf{p}_1$ of $\mathcal{M}_1$ and $\mathbf{p}_2$ of $\mathcal{M}'$ exactly at the same position on $\mathbf{a}_w$, except **A** and **E**.

Then, we prolong $\mathbf{a}_y$ after **F** in such a way that **D** is now an inner point of $\mathbf{a}_y$ (See Figure 30, right). We now add a face segment $\underline{\mathbf{a_y e_t d_x}}$ from **F** to an inner point of [**DE**] (that is contained in $\mathbf{d}_x$).

Note that the two segments $\mathbf{a}_w$ of S_1 and S' form now a single segment $\mathbf{a}_w$. Consider now $\mathcal{M} = (S, F, \tau)$ where $S = S' \cup S_1$ (up to the identification of the $\mathbf{a}_w$s), $F = F' \cup F_1 \cup \{\underline{\mathbf{a_y e_t d_x}}\}$, $\tau(\mathbf{p}) = \tau'(\mathbf{p})$ (resp. $\tau(\mathbf{p}) = \tau_1(\mathbf{p})$) for any $\mathbf{p} \in Rep_{S' \cup F'} \setminus \{\mathbf{A}, \mathbf{E}\}$ (resp. $\mathbf{p} \in Rep_{S_1 \cup F_1} \setminus \{\mathbf{A}, \mathbf{E}, \mathbf{F}\}$) and where $\tau(\mathbf{A}), \tau(\mathbf{D}), \tau(\mathbf{E})$ and $\tau(\mathbf{F})$ are defined as follows. **A** is now a path–$(a_y, \ldots, a_p)$–point; this is possible, since its incidence sequence is $(\mathbf{a}_y, \ldots, \mathbf{a}_w, \ldots, \mathbf{a}_p)$. As in Case 2.2.1, **D** is now a path–$(a_y, d_x, \ldots, d_s, a_1)$–point.

If $w < p$, **E** is a fan–$d_x \triangleleft (a_w, e_1, \ldots, e_t)$–point; this is possible since its incidence sequence is $(\mathbf{d}_x, \mathbf{a}_w, \mathbf{e}_1, \ldots, \mathbf{e}_t, \mathbf{d}_x, \mathbf{a}_w)$. If $w = p$, **E** is a double-fan–$a_w \triangleleft (d_1, \ldots, d_x) \cdot d_x \triangleleft (e_t, \ldots, e_1, a_w)$–point; this is possible since its incidence sequence is $(\mathbf{a}_w, \mathbf{d}_1, \ldots, \mathbf{d}_x, \mathbf{e}_t, \ldots, \mathbf{e}_1, \mathbf{a}_w)$.

If **F** is a fan–$a_y \triangleleft (e_t)$–point in $\mathcal{M}_1$, then **F** is the crossing point of $\mathbf{a}_y$ and $\mathbf{e}_t$ in $\mathcal{M}$; this is possible since if there was a face segment incident to **F** in $\mathcal{M}_1$, then $\mathbf{e}_t$ separates it from $\underline{\mathbf{a_y e_t d_x}}$ in $\mathcal{M}$. Otherwise, there is no face segment incident to **F** and **F** remains a fan–$a_y \triangleleft (e_t, \ldots)$–point in $\mathcal{M}$ (as in $\mathcal{M}_1$); this is possible since its incidence sequence is $(\mathbf{a}_y, \underline{\mathbf{a_y e_t d_x}}, e_t, \ldots, a_y)$.

Since $V(T_{d_x a_y}) = V(T_{d_x a_w}) \cup V(T_1)$, every vertex $v \in V(T_{d_x a_y})$ corresponds to exactly one segment $\mathbf{v}$ in S. Note that $E(T_{d_x a_y}) = E(T_{d_x a_w}) \cup E(T_1) \cup \{d_x a_y\} \cup \{d_x e_i \mid i \in [1, t]\}$ (See Figure 29). Note that the edges

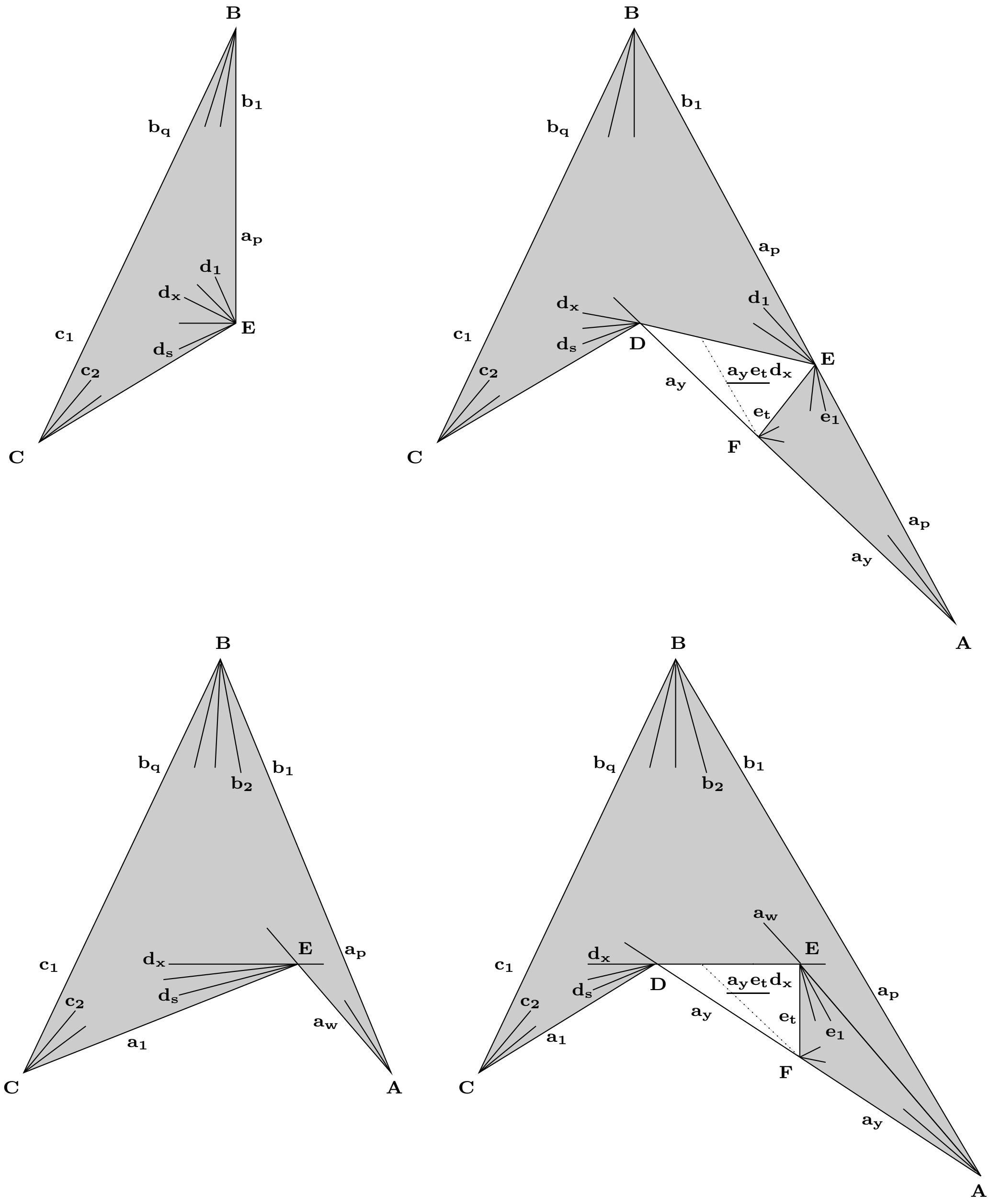

Fig. 30. Case 2.2.3 ($z = x$ and $y > w + 1$): when $w = p$ (top) or $w < p$ (bottom).

represented in **F** in $\mathcal{M}$ are the edges represented in **F** in $\mathcal{M}_1$ and that an edge uv is represented in **A** in $\mathcal{M}$ if and only if uv is represented in **A** in $\mathcal{M}'$ or $\mathcal{M}_1$. One can check that in both cases ($w = p$ or $w < p$), the edges represented in **D** and **E** in $\mathcal{M}$ are exactly the edges represented in **E** in $\mathcal{M}'$, in **E** in $\mathcal{M}_1$ and the edges in $\{d_x a_y\} \cup \{d_x e_i \mid i \in [1, t]\}$(See Figure 31). Note that $F(T_{d_x a_y}) = F(T_{d_x a_w}) \cup F(T_1) \cup \{a_y e_t d_x, d_x a_w e_1\} \cup \{d_x e_i e_{i+1} \mid i \in [1, t-1]\}$ (See Figure 29). Note that the faces represented in **F** in $\mathcal{M}$ are the faces represented in **F** in $\mathcal{M}_1$ and that no face is represented in **A** (resp. **D**) in $\mathcal{M}', \mathcal{M}_1$ or $\mathcal{M}$ (resp. $\mathcal{M}$). One can check that no face is represented in **E** in $\mathcal{M}_1$ and that the faces represented in **E** in $\mathcal{M}$ are exactly the faces represented in **E** in $\mathcal{M}'$, and the faces in $\{d_x a_w e_1\} \cup \{d_x e_i e_{i+1} \mid i \in [1, t-1]\}$ (See Figure 31). Since we have added a face segment $\underline{\mathbf{a_y e_t}}\mathbf{d_x}$, the edges represented in $\mathcal{M}$ are exactly the edges of $T_{d_x a_y}$.

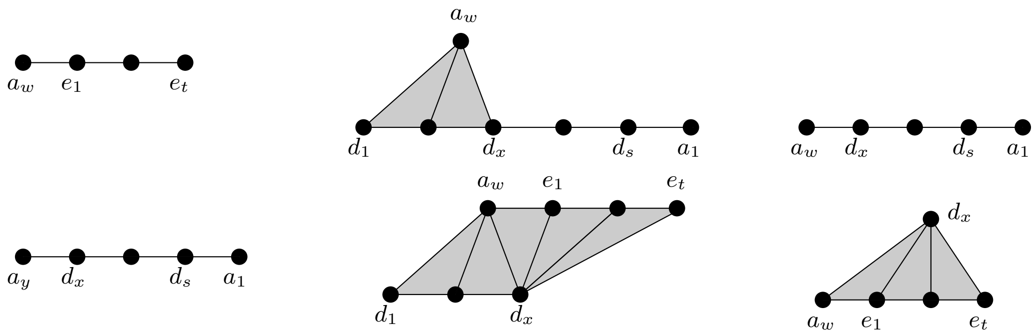

Fig. 31. Case 2.2.3: the graphs represented in **E** in $\mathcal{M}_1$ (top left), in **E** in $\mathcal{M}'$ when $w = p$ (top middle), in **E** in $\mathcal{M}'$ when $w < p$ (top right), in **D** in $\mathcal{M}$ (bottom left), in **E** in $\mathcal{M}$ when $w = p$ (bottom middle) and in **E** in $\mathcal{M}$ when $w < p$ (bottom right).

We know that $Const_{S_1 \cup F_1}$ is acyclic. Let $Const'_1$ be the digraph obtained from $Const_{S_1 \cup F_1}$, where the arc from **E** to $\mathbf{a}_w$ and the arc from **F** to $\mathbf{a}_y$ have been respectively replaced by an arc from $\mathbf{a}_w$ to **E** and an arc from $\mathbf{a}_y$ to **F**. Since **E** and **F** are free in $\mathcal{M}_1$, $Const'_1$ is acyclic.

We know that $Const_{S' \cup F'}$ is acyclic. Let $Const'_2$ be the digraph obtained from $Const_{S' \cup F'}$, where there are two new vertices $\mathbf{a}_y$ and **D**, where the arcs from **E** to $\mathbf{d}_i$, $i \in [x+1, s]$ and from **E** to $\mathbf{a}_1$ have been respectively replaced by some arcs from **D** to $\mathbf{d}_i$, $i \in [x+1, s]$ and from **D** to $\mathbf{a}_1$ and where there is an arc from $\mathbf{d}_x$ (resp. $\mathbf{a}_y$) to **D**. We also add a new vertex **I** representing the end of $\mathbf{a}_y$ and an arc from **I** to $\mathbf{a}_y$. Since **E** is free in $\mathcal{M}'$ and since $\mathbf{a}_y$ has only one predecessor (**I**) that has no predecessor in $Const'_2$, **D** is free in $Const'_2$ and thus, $Const'_2$ is acyclic.

Note that $Const_{S \cup F}$ is the union of $Const'_1$ and $Const'_2$ where the two vertices corresponding to $\mathbf{a}_y$ (resp. $\mathbf{a}_w$, **E**, **A**) have been identified. Since $Const'_1$ and $Const'_2$ are acyclic, any cycle of $Const_{S \cup F}$ must contains two vertices among $\mathbf{a}_w, \mathbf{a}_y, \mathbf{A}, \mathbf{E}$. Since **A** has no predecessor, since **A** is the only predecessor of $\mathbf{a}_w$ (resp. $\mathbf{a}_y$) in $Const'_1$ and since the only predecessor of **E** in $Const'_1$ is $\mathbf{a}_w$, there is no cycle going from $Const'_1$ to $Const'_2$ through any of these points and thus $Const_{S \cup F}$ is acyclic. For the same reasons as in the proof of Case 1.1, the special points belonging to $\mathbf{a}_w$ when $w < p$ remain free in $Const_{S \cup F}$.

If $w < p$, we realize all the special points appearing on $\mathbf{a}_w$ (they are on [**AE**]), except **A** (but we realize **E**). Then, in both cases, we have constructed a premodel $\mathcal{M}$ of $T_{d_x a_y}$ that satisfies Property 2.

Case 2.2.4: $z = x - 1$ and $w > y$ (see Figure 32).

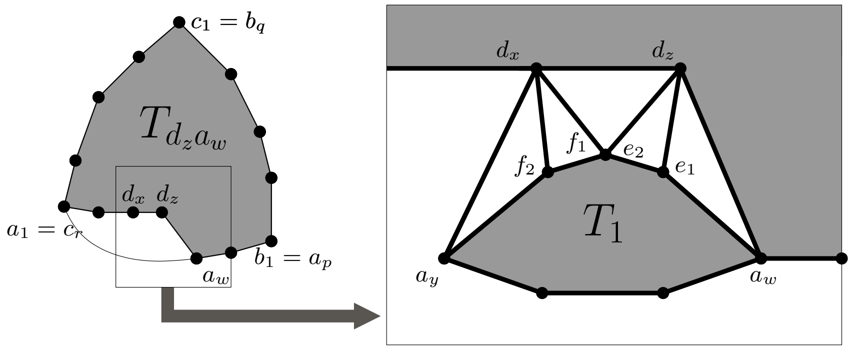

Fig. 32. Case 2.2.4: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x - 1$ and $w > y$.

Let us denote $e_1, e_2, \dots, e_t$ the neighbors of d_z strictly inside the cycle $(d_z, d_x, a_y, \dots, a_w, d_z)$, going "from right to left" (see Figure 32). Since z is maximal there is no edge $d_x a_w$, so $t \geq 1$. Let us denote $f_1, \dots, f_u$ the neighbors of d_x strictly inside the cycle $(d_x, a_y, \ldots, a_w, d_z)$, going "from right to left" (see Figure 32). Note that $f_1 = e_t$ and that w being minimal, there is no edge $d_z a_y$, so $u \geq 1$.

Since w is minimal (resp. z is maximal) we have $e_i \neq a_j$ (resp. $f_i \neq a_j$), for all $1 \leq i \leq t$ (resp. $1 \leq i \leq u$) and $y \leq j \leq w$. Let T_1 be the subgraph of $T_{d_x a_y}$ that lies inside the cycle $(a_y, \ldots, a_w, e_1, \ldots, e_t, f_2, \ldots, f_u, a_y)$. By Lemma 3.2, T_1 is a W-triangulation. Since the W-triangulation $T_{d_x a_y}$ has no separating 3-cycle (d_z, a_w, e_i), (d_z, e_i, e_j), (d_x, f_i, f_j), or (d_x, f_i, a_y), there exists no chord $a_w e_i$, $e_i e_j$, $f_i f_j$, or $f_i a_y$ in T_1. With the fact that $t \geq 1$ and $u \geq 1$, we know that $(f_1, f_2, \ldots, f_u, a_y)$-$(a_y, \ldots, a_w)$-$(a_w, e_1, \ldots, e_t)$ is a 3-boundary of T_1. Finally, since T_1 has less edges than $T_{d_x a_y}$ ($d_x a_y \notin E(T_1)$), Property 1 holds for T_1 with respect to the mentioned 3-boundary.

We want to construct a premodel $\mathcal{M} = (S, F, \tau)$ of $T_{d_x a_y}$ contained in some concave polygon **ABCD**. Consider three non-collinear points $\mathbf{B}, \mathbf{C}, \mathbf{E}$.

If $w = p$ (See Figure 33, top), consider a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_z a_w}$ satisfying Property 2 that is contained in **BCE** and where the points $\mathbf{B}, \mathbf{C}, \mathbf{E}$ are respectively a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point and a fan-path–$a_w \triangleleft (d_1, \ldots, d_z) \cdot (d_z, d_x, \ldots, d_s, a_1)$–point. We then prolong $\mathbf{a}_w$ after $\mathbf{E}$ to a new point $\mathbf{A}$ (since $\mathbf{E}$ is free, it keeps the constraints digraph acyclic).

If $w < p$ (See Figure 33, bottom), consider a premodel $\mathcal{M}' = (S', F', \tau')$ of $T_{d_z a_w}$ satisfying Property 2 that is contained in a concave polygon **ABCE** for some point $\mathbf{A}$ and where the points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{E}$ are respectively a path–$(a_w, \ldots, a_p)$–point, a path–$(b_1, \ldots, b_q)$–point, a path–$(c_1, \ldots, c_r)$–point and a path–$(a_w, d_z, d_x, \ldots, d_s, a_1)$–point.

In both cases, as in Cases 2.2.1 and 2.2.3, we do a gliding of $(\mathbf{d}_{x+1}, \ldots, \mathbf{d}_s, \mathbf{a}_1)$ on $\mathbf{d}_x$. Let $\mathbf{D}$ be the new intersection point of $\mathbf{d}_x$ and $\mathbf{d}_{x+1}, \ldots, \mathbf{d}_s, \mathbf{a}_1$. Since we have done exactly the same moves as in previous cases, for the same reasons as before, the constraints digraph is still acyclic after these modifications.

If $u = 1$ (See Figure 33, left), let $\mathbf{F}$ be an inner point of $[\mathbf{AD}]$ and consider a premodel $\mathcal{M}_1 = (S_1, F_1, \tau_1)$ of T_1 satisfying Property 1 that is contained in **AEF** and where the points $\mathbf{A}, \mathbf{E}, \mathbf{F}$ are respectively a path–$(a_y, \ldots, a_w)$–point, a path–$(a_w, e_1, \ldots, e_t)$–point and a fan–$a_y \triangleleft (f_1, \ldots)$–point. Then, we prolong $\mathbf{a}_y$ after $\mathbf{F}$ in such a way that $\mathbf{D}$ is an inner point of $\mathbf{a}_y$. We now add a face segment $\underline{\mathbf{f_1 a_y d_x}}$ from $\mathbf{F}$ to an inner point of $[\mathbf{DE}]$ (that is contained in $\mathbf{d}_x$).

If $u > 1$ (See Figure 33, right), consider a premodel $\mathcal{M}_1 = (S_1, F_1, \tau_1)$ of T_1 satisfying Property 1 that is contained in a concave polygon **DAEF** for some point $\mathbf{F}$ and where the points $\mathbf{A}, \mathbf{D}, \mathbf{E}, \mathbf{F}$ are respectively a path–$(a_y, \ldots, a_w)$–point, a path–$(f_2, \ldots, f_u, a_y)$, a path–$(a_w, e_1, \ldots, e_t)$–point and the crossing point of $\mathbf{e}_t = \mathbf{f}_1$ and $\mathbf{f}_2$. We prolong $\mathbf{a}_y$ after $\mathbf{D}$. We now add a face segment $\underline{\mathbf{f_1 f_2 d_x}}$ from $\mathbf{F}$ to an inner point of $[\mathbf{DE}]$ (that is contained in $\mathbf{d}_x$).

By using Lemma 2.12, we can ensure that when $w < p$, there are no representative points $\mathbf{p}_1$ of $\mathcal{M}_1$ and $\mathbf{p}_2$ of $\mathcal{M}'$ exactly at the same position on $\mathbf{a}_w$, except $\mathbf{A}$ and $\mathbf{E}$.

Note that the two segments $\mathbf{a}_w$ of S_1 and S' form now a single segment $\mathbf{a}_w$. If $u = 1$ (resp. $u > 1$), consider now $\mathcal{M} = (S, F, \tau)$ where $S = S' \cup S_1$ (up to the identification of the $\mathbf{a}_w$s), $F = F' \cup F_1 \cup \{\underline{\mathbf{a_y e_t d_x}}\}$ (resp. $F = F' \cup F_1 \cup \{\underline{\mathbf{f_1 f_2 d_x}}\}$) and where τ is defined as follows. For any $\mathbf{p} \in Rep_{S' \cup F'} \setminus \{\mathbf{A}, \mathbf{E}\}$ (resp. $\mathbf{p} \in Rep_{S_1 \cup F_1} \setminus \{\mathbf{A}, \mathbf{D}, \mathbf{E}, \mathbf{F}\}$), $\tau(\mathbf{p}) = \tau'(\mathbf{p})$ (resp. $\tau(\mathbf{p}) = \tau_1(\mathbf{p})$) and $\tau(\mathbf{A}), \tau(\mathbf{D}), \tau(\mathbf{E})$ and $\tau(\mathbf{F})$ are defined as follows. $\mathbf{A}$ is now a path–$(a_y, \ldots, a_p)$–point; this is possible, since its incidence sequence is $(\mathbf{a}_y, \ldots, \mathbf{a}_w, \ldots, \mathbf{a}_p)$. $\mathbf{D}$ is now a path-fan–$(d_x, \ldots, d_s, a_1) \cdot d_x \triangleleft (a_y, f_u, \ldots, f_2)$–point; this is possible since its incidence sequence is $(\mathbf{d}_x, \ldots, \mathbf{d}_s, \mathbf{a}_1, \mathbf{a}_y, \mathbf{f}_u, \ldots, \mathbf{f}_2, \mathbf{d}_x, \mathbf{a}_y)$.

If $w < p$, $\mathbf{E}$ is a fan–$d_z \triangleleft (a_w, e_1, \ldots, e_t, d_x)$–point; this is possible since its incidence sequence is $(\mathbf{d}_z, \mathbf{a}_w, \mathbf{e}_1, \ldots, \mathbf{e}_t, \mathbf{d}_x, \mathbf{d}_z, \mathbf{a}_w)$. If $w = p$, $\mathbf{E}$ is a double-fan–$a_w \triangleleft (d_1, \ldots, d_z) \cdot d_z \triangleleft (d_x, e_t, \ldots, e_1, a_w)$–point; this is possible since its incidence sequence is $(\mathbf{a}_w, \mathbf{d}_1, \ldots, \mathbf{d}_z, \mathbf{d}_x, \mathbf{e}_t, \ldots, \mathbf{e}_1, \mathbf{a}_w)$.

If $\mathbf{F}$ is the crossing of $\mathbf{f}_1$ and $\mathbf{f}_2$ in $\mathcal{M}_1$, then $\mathbf{F}$ remains the crossing of $\mathbf{f}_1$ and $\mathbf{f}_2$ in $\mathcal{M}$; this is possible, since if there was a face segment incident to $\mathbf{F}$ in $\mathcal{M}_1$, then either $\mathbf{f}_1$ or $\mathbf{f}_2$ separates it from $\underline{\mathbf{f_1 f_2 d_x}}$ in $\mathcal{M}$. If $\mathbf{F}$ is a fan–$a_y \triangleleft (f_1)$–point in $\mathcal{M}_1$, then $\mathbf{F}$ is the crossing point of $\mathbf{a}_y$ and $\mathbf{f}_1$ in $\mathcal{M}$; this is possible since if there was a face segment incident to $\mathbf{F}$ in $\mathcal{M}_1$, then $\mathbf{f}_1$ separates it from $\underline{\mathbf{a_y e_t d_x}}$ in $\mathcal{M}$. Otherwise, there is no face segment incident to $\mathbf{F}$ and $\mathbf{F}$ is a fan–$a_y \triangleleft (f_1, \ldots)$–point in $\mathcal{M}_1$; it remains a fan–$a_y \triangleleft (f_1, \ldots)$ in $\mathcal{M}$; this is possible since its incidence sequence is $(\mathbf{a}_y, \underline{\mathbf{a_y f_1 d_x}}, \mathbf{f}_1, \ldots, \mathbf{a}_y)$.

Since $V(T_{d_x a_y}) = V(T_{d_z a_w}) \cup V(T_1)$, every vertex $v \in V(T_{d_x a_y})$ corresponds to exactly one segment $\mathbf{v}$ in S.

Note that $E(T_{d_x a_y}) = E(T_{d_z a_w}) \cup E(T_1) \cup \{d_x a_y, e_t d_x\} \cup \{d_x f_i \mid i \in [2, u]\} \cup \{d_z f_i \mid i \in [1, t]\}$ (See Figure 32). Note that the edges represented in $\mathbf{F}$ in $\mathcal{M}$ are the edges represented in $\mathbf{F}$ in $\mathcal{M}_1$ and that an edge uv is represented in $\mathbf{A}$ in $\mathcal{M}$ if and only if uv is represented in $\mathbf{A}$ in $\mathcal{M}'$ or $\mathcal{M}_1$. One can check that

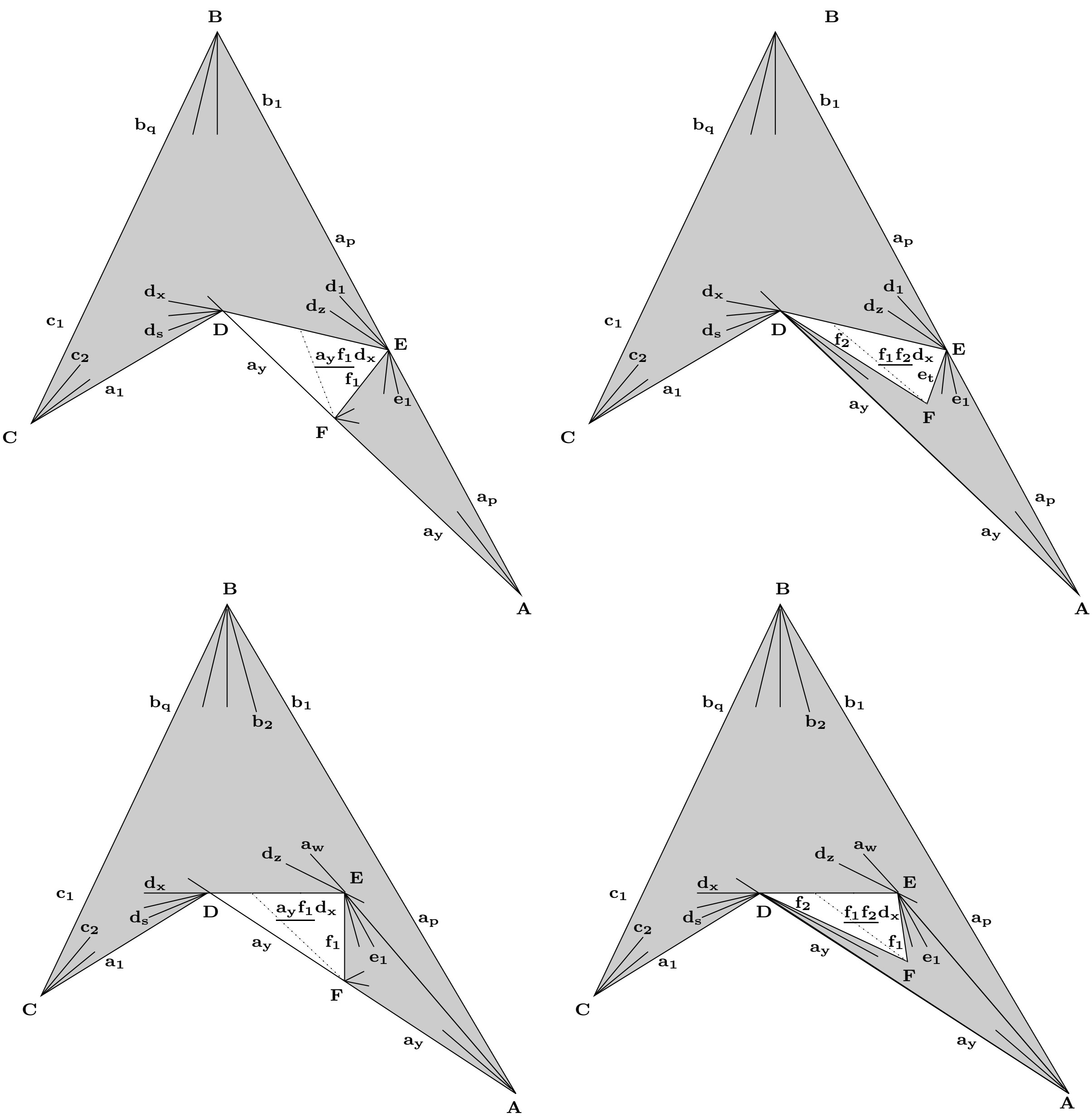

Fig. 33. Case 2.2.4: $\mathcal{M} = (S, F, \phi)$.

in any case, the edges represented in **D** and **E** in $\mathcal{M}$ are exactly the edges represented in **E** in $\mathcal{M}'$, in **D** in $\mathcal{M}_1$, in **E** in $\mathcal{M}_1$ and the edges in $\{d_x a_y, d_x e_t\} \cup \{d_x f_i \mid i \in [2, u]\} \cup \{d_z f_i \mid i \in [1, t]\}$ (See Figure 34 when $w = p$ and Figure 35 when $w < p$).

Note that $F(T_{d_x a_y}) = F(T_{d_x a_w}) \cup F(T_1) \cup \{a_w d_z e_1, d_x d_z e_t, d_x a_y f_u\} \cup \{d_z e_i e_{i+1} \mid i \in [1, t-1]\} \cup \{d_x f_i f_{i+1} \mid i \in [1, u-1]\}$ (See Figure 32). Note that the faces represented in **F** in $\mathcal{M}$ are the faces represented in **F** in $\mathcal{M}_1$ and that no face is represented in **A** in $\mathcal{M}', \mathcal{M}_1$ or $\mathcal{M}$. One can check that no face is represented in **D** or **E** in $\mathcal{M}_1$ and that the faces represented in **D** and **E** in $\mathcal{M}$ are exactly the faces represented in **E** in $\mathcal{M}'$, and the missing faces except $a_y f_1 d_x$ if $u = 1$ and $f_1 f_2 d_x$ if $u > 1$ (See Figure 34 when $w = p$ and Figure 34 when $w < p$). Since we have added a face segment $\underline{\mathbf{a_y f_1}}\mathbf{d_x}$ if $u = 1$ and a face segment $\underline{\mathbf{f_1 f_2}}\mathbf{d_x}$, the edges represented in $\mathcal{M}$ are exactly the edges of $T_{d_x a_y}$.

We know that $Const_{S_1 \cup F_1}$ is acyclic. Let $Const'_1$ be the digraph obtained from $Const_{S_1 \cup F_1}$, where the arc from **E** to $\mathbf{a}_w$ has been replaced by an arc from $\mathbf{a}_w$ to **E** and where the arc from **F** to $\mathbf{a}_y$ (resp. from **D**

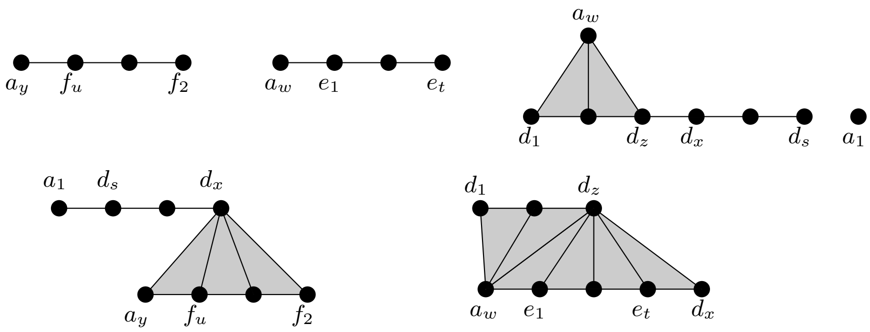

Fig. 34. Case 2.2.4 when $w = p$: the graphs represented in **D** in $\mathcal{M}_1$ (top left), in **E** in $\mathcal{M}_1$ (top middle), in **E** in $\mathcal{M}'$ (top right), in **D** in $\mathcal{M}$ (bottom left) and in **E** in $\mathcal{M}$ (bottom right).

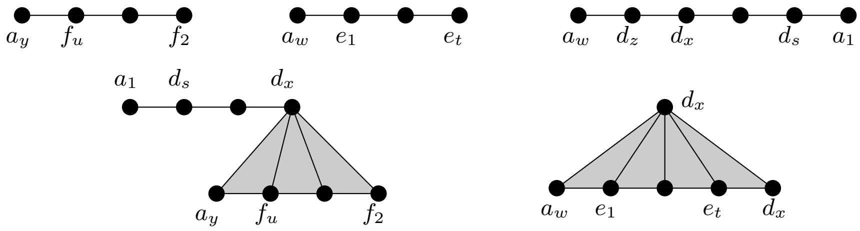

Fig. 35. Case 2.2.4 when $w < p$: the graphs represented in **D** in $\mathcal{M}_1$ (top left), in **E** in $\mathcal{M}_1$ (top middle), in **E** in $\mathcal{M}'$ (top right), in **D** in $\mathcal{M}$ (bottom left) and in **E** in $\mathcal{M}$ (bottom right).

to $\mathbf{a}_y$) has been replaced by an arc from $\mathbf{a}_y$ to **F** (resp. $\mathbf{a}_y$ to **D**) when $u = 1$ (resp. $u > 1$). Since **E** and **F** (resp. **D**) are free in $\mathcal{M}_1$, $Const'_1$ is acyclic.

We know that $Const_{S'\cup F'}$ is acyclic. Let $Const'_2$ be the digraph obtained from $Const_{S'\cup F'}$, where there are two new vertices $\mathbf{a}_y$ and **D**, where the arcs from **E** to d_i, $i \in [x+1, s]$ and from **E** to $\mathbf{a}_1$ have been respectively replaced by some arcs from **D** to d_i, $i \in [x+1, s]$ and from **D** to $\mathbf{a}_1$ and where there is an arc from $\mathbf{d}_x$ (resp. $\mathbf{a}_y$) to **D**. We also add a new vertex **I** representing the end of $\mathbf{a}_y$ and an arc from **I** to $\mathbf{a}_y$. Since **E** is free in $\mathcal{M}'$ and since $\mathbf{a}_y$ has no predecessor in $Const'_2$, **D** is free in $Const'_2$ and thus, $Const'_2$ is acyclic.

Note that $Const_{S\cup F}$ is the union of $Const'_1$ and $Const'_2$ where the two vertices corresponding to $\mathbf{a}_y$ (resp. $\mathbf{a}_w$, **E**, **A**, **D**) have been identified. Since $Const'_1$ and $Const'_2$ are acyclic, any cycle of $Const_{S\cup F}$ must contains two vertices among $\mathbf{a}_w, \mathbf{a}_y, \mathbf{A}, \mathbf{D}, \mathbf{E}$. Since **A** has no predecessor in $Const'_1$, since **A** is the only predecessor of $\mathbf{a}_w$ (resp. $\mathbf{a}_y$) in $Const'_1$ and since the only predecessor of **E** (resp. **D**) in $Const'_1$ is $\mathbf{a}_w$ (resp. $\mathbf{a}_y$), there is no cycle going from $Const'_1$ to $Const'_2$ through any of these points and thus $Const_{S\cup F}$ is acyclic. For the same reasons as in the proof of Case 1.1, the special points belonging to $\mathbf{a}_w$ if $w < p$ remain free in $\mathcal{M}$.

If $w < p$, we realize all the special points appearing on $\mathbf{a}_w$ (they are on [**AE**]), except **A** (but we realize **E**). Then, we have to partially realize **D** in order to obtain a path–$(a_y, d_x, \ldots, d_s, a_1)$–point. If $u = 1$, we are done. Otherwise, by using Lemma 2.13, we do a traversing of $\mathbf{d}_x$ by $(f_u, \ldots, f_2)$ along $\mathbf{a}_y$, we add the face segments corresponding to $d_x f_u a_y$ and $d_x f_i f_{i+1}$ for $i \in [2, u-1]$, as explained in the proof of Proposition 2.14 and then we realize the path–$(a_y, f_u, \ldots, f_2)$–point.

Once these realizations have been done, we have constructed a premodel $\mathcal{M}$ of $T_{d_x a_y}$ that satisfies Property 2.

This completes the study of Case 2 and ends the proof of Lemma 3.9. □

4 Proof of Theorem 2.5

We prove that every triangulation T has a full model (S, F) by induction on the number k of separating 3-cycles in T. If $k = 0$ the triangulation T is a W-triangulation 3-bounded by (a, b)-(b, c)-(c, a), where a, b and c are the vertices on its outer-boundary. Then Property 1 provides us a premodel $\mathcal{M} = (S, F, \tau)$ of T and by Corollary 2.18 we obtain a full model (S', F') of T.

If $k \geq 1$, let $C = (a, b, c)$ be a 3-cycle such that the triangulation T' induced by the vertices on and inside C does not contain any separating 3-cycle. Let T_1 be the triangulation obtained by removing all the vertices that lie strictly inside the cycle C. Let T_2 be the subgraph of T induced by all the vertices of T that lie strictly inside the cycle C. By definition of C, T_2 is either (A) a single vertex v or (B) a W-triangulation (see Figure 36). In T_1, the cycle C delimits a face and is no more a separating 3-cycle. Since T_1 has one

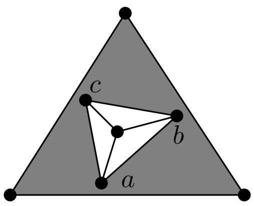

Fig. 36. The cases (A) and (B).

separating 3-cycle less than T, the induction hypothesis implies that T_1 admits a full model $\mathcal{M} = (S, F)$. Since abc is an inner face of T_1 there is a corresponding face segment, say $\mathbf{\underline{ac}b}$, in F and let respectively $\mathbf{B}$ and $\mathbf{C}$ be its flat end and its cross end. Note that there might be an other face segment incident to $\mathbf{C}$. If it exists we denote it $\mathbf{\underline{ac}d}$ since it would correspond to a face acd adjacent to the edge ac in T_1. Since F is non-interfering we know that ($\mathbf{a}$) or ($\mathbf{c}$) separate $\mathbf{\underline{ac}b}$ and $\mathbf{\underline{ac}d}$ in distinct half-planes. Here we assume, without loss of generality that the line ($\mathbf{a}$) separates them. Now let $\epsilon > 0$ be a real such that for every representative point $\mathbf{p} \in Rep_{S \cup F} \setminus \{\mathbf{B}, \mathbf{C}\}$ we have $dist(\mathbf{p}, \mathbf{\underline{ac}b}) > \epsilon$, and let the region $\mathcal{R}_\epsilon$ be the set of points at distance at most ϵ from $\mathbf{\underline{ac}b}$. The definition of ϵ implies that (1) the only segments intersecting $\mathcal{R}_\epsilon$ are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{\underline{ac}b}$ and eventually $\mathbf{\underline{ac}d}$ if it exists; and that (2) the endpoints of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (resp. the flat end of $\mathbf{\underline{ac}d}$) are not in $\mathcal{R}_\epsilon$. Since there is no inner face abc in T we remove $\mathbf{\underline{ac}b}$ from F and we add some segments and face segments in $\mathcal{R}_\epsilon$ to obtain a full model of the whole T.

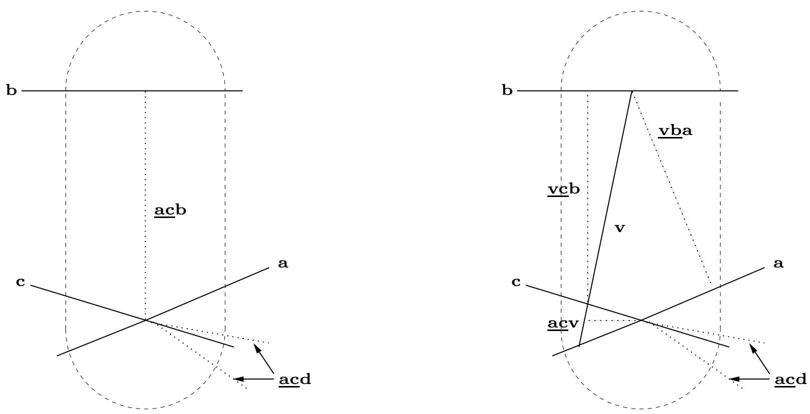

Fig. 37. Case (A): Modifications inside $\mathcal{R}_\epsilon$.

Case (A): T_2 is a single vertex v. Since $\mathbf{\underline{ac}b}$ and $\mathbf{\underline{ac}d}$ (if it exists) are non-interfering, it is easy to draw in the region $\mathcal{R}_\epsilon$ a segment $\mathbf{v}$ that only intersect $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$; and three face segments $\mathbf{\underline{vb}a}$, $\mathbf{\underline{vc}b}$, and $\mathbf{\underline{ac}v}$ such that the set $\{\underline{\mathbf{vb}}\mathbf{a}, \underline{\mathbf{vc}}\mathbf{b}, \underline{\mathbf{ac}}\mathbf{v}, \underline{\mathbf{ac}}\mathbf{d}\}$ is non-interfering (see Figure 37). Now it is clear that from the model $\mathcal{M}$ of T_1 we have added a segment for v, three crossings for va, vb and vc, removed the face segment of acb, and added the face segments of vba, acv and vcb; thus we have a full model of T.

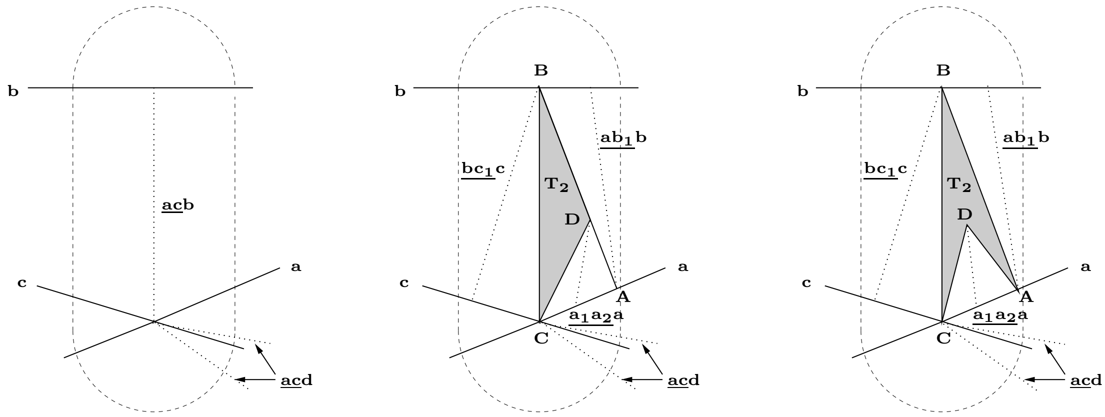

Fig. 38. Case (B): Modifications inside $\mathcal{R}_\epsilon$.

Case (B): T_2 is a W-triangulation. Let $a_1, a_2, \ldots, a_p$ be the neighbors of a inside the cycle (a, b, c) going from c to b excluded. Similarly let $b_1, b_2, \ldots, b_q$ (resp. $c_1, c_2, \ldots, c_r$) be the neighbors of b (resp. c) inside the cycle (a, b, c) going from a to c (resp. from b to a) excluded. It is clear that $a_1 = c_r$, $b_1 = a_p$, and $c_1 = b_q$. Furthermore, since there is no separating 3-cycle inside C, we have that:

– p, q, and $r \geq 2$.
– $(a_1, a_2, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_r)$ is a cycle, thus T_2 is a W-triangulation.
– T_2 has no chord $a_x a_y$, $b_x b_y$, or $c_x c_y$ with $y > x + 1$.

Thus T_2 is a W-triangulation 3-bounded by $(a_1, a_2, \ldots, a_p)$-$(b_1, b_2, \ldots, b_q)$-$(c_1, c_2, \ldots, c_r)$. Here we choose this particular 3-boundary because of the assumption that $(\mathbf{a})$ separates $\underline{\mathbf{ac}}\mathbf{b}$ and $\underline{\mathbf{ac}}\mathbf{d}$ (if it exists). We now apply Property 1 with respect to this 3-boundary and this implies that if $p = 2$ (resp. $p > 2$) then T_2 has a premodel $\mathcal{M}' = (S', F', \tau')$ inside the triangle $\mathbf{BCD}$ (resp. the polygon $\mathbf{ABCD}$), where $\mathbf{A}$ is a point of $\mathbf{a} \cap \mathcal{R}_\epsilon$ (See Figure 38) and $\mathbf{D}$ is an internal point of $[\mathbf{A}, \mathbf{B}]$ (resp. a point strictly inside $\mathbf{ABC}$). If $p = 2$ we prolong $\mathbf{b}_1 = [\mathbf{BD}]$ across $\mathbf{D}$ until reaching $\mathbf{A}$ and note that since $\mathbf{D}$ is free, then the constraints digraph of $\mathcal{M}'$ remains acyclic (*cf.* Lemma 2.9). Note also that according to the definition of $\mathcal{R}_\epsilon$, the full model $\mathcal{M}$ and the premodel $\mathcal{M}'$ only intersect at $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Now we are going to merge $\mathcal{M}$ and $\mathcal{M}'$ in order to construct a premodel $\mathcal{M}^* = (S^*, F^*, \tau^*)$ of the whole T. To do this, let $S^* = S \cup S'$ and $F^* = (F \setminus \underline{\mathbf{ac}}\mathbf{b}) \cup F' \cup \{\underline{\mathbf{a_1 a_2}}\mathbf{a}, \underline{\mathbf{ab_1}}\mathbf{b}, \underline{\mathbf{bc_1}}\mathbf{c}\}$; where $\underline{\mathbf{a_1 a_2}}\mathbf{a}$ goes from $\mathbf{D}$ to a point of $[\mathbf{A}, \mathbf{C}]$, $\underline{\mathbf{ab_1}}\mathbf{b}$ goes from $\mathbf{A}$ to a point of $\mathbf{b} \cap \mathcal{R}_\epsilon$, and $\underline{\mathbf{bc_1}}\mathbf{c}$ goes from $\mathbf{B}$ to a point of $\mathbf{c} \cap \mathcal{R}_\epsilon$ (See Figure 38). Observe that F^* is non-interfering, in particular we see that $\underline{\mathbf{a_1 a_2}}\mathbf{a}$ does not interfere with another face segment $\mathbf{f}$ at $\mathbf{D}$, since $\mathbf{f}$ would be inside $\mathbf{ABCD}$. We now define τ^* as follows. Let $\mathbf{A}$ be a fan–$a \triangleleft (a_p, \ldots, a_2)$–point, let $\mathbf{B}$ be a fan–$b \triangleleft (b_q, \ldots, b_1)$–point, and let $\mathbf{C}$ be a fan–$c \triangleleft (a, c_r, \ldots, c_1)$–point. If $p > 2$ the point $\mathbf{D}$ remains the crossing point of a_1 and a_2, even with its new incident face segment. If $p = 2$ the point $\mathbf{D}$ was either a fan–$a_2 \triangleleft (d_1, \ldots, d_s, a_1)$–point (for some vertices $d_1, \ldots, d_s$) or a fan–$a_2 \triangleleft (a_1)$–point. In the first case let $\mathbf{D}$ be a fan–$a_2 \triangleleft (a_1, d_s, \ldots, d_1)$–point (possible since it has no incident face segment in $\mathcal{M}'$). In the second case let $\mathbf{D}$ be the crossing point of a_1 and a_2 with one or two incident face segments. Note that in both case the graph corresponding to $\mathbf{D}$ remains unchanged. For the other representative points of $\mathcal{M}^*$ let their type remain as in $\mathcal{M}$ or $\mathcal{M}'$.

We now verify that $\mathcal{M}^*$ is a premodel of T.

- It is clear that $S^* \cup F^*$ is unambiguous and we show here that $Const_{S^* \cup F^*}$ is acyclic. Indeed this digraph arises from the union of $Const_{S \cup F}$ and $Const_{S' \cup F'}$ (where S' has a segment a_2 prolonged until $\mathbf{A}$ when $p = 2$) by adding the vertices corresponding to the new face segments and their flat end point, and adding the arcs incident to these vertices. But since the face segments have out-degree zero in the constraints digraphs, there is no cycle in $Const_{S^*\cup F^*}$ passing through a face segment. Thus a cycle would be in the union of $Const_{S\cup F}$ and $Const_{S'\cup F'}$. These two digraph being acyclic, this cycle should successively pass through a segment of $Const_{S'\cup F'}$, through one of the points **A**, **B** and **C**, and through a segment of $Const_{S\cup F}$. But this is impossible since in $Const_{S'\cup F'}$ the only points that intersect $\mathcal{M}$, **A**, **B** and **C**, have in-degree zero.

- Since $V(T)$ is the disjoint union of $V(T_1)$ and $V(T_2)$ we have that a vertex $v \in V(T)$ if and only if $\mathbf{v} \in S^*$.
- Note that $E(T) = E(T_1) \cup E(T_2) \cup \{aa_1 = ac_r\} \cup \{aa_2, \dots, aa_p\} \cup \{bb_1, \dots, bb_q\} \cup \{cc_1, \dots, cc_r\}$, that **A** was not a representative point in $\mathcal{M}$ (resp. was either an end point or a path–$(a_2, \dots, a_p)$–point in $\mathcal{M}'$) and that now it is a fan–$a \lessdot (a_p, \dots, a_2)$–point, that **B** was a flat face segment end in $\mathcal{M}$ (resp. was a path–$(b_1, \dots, b_q)$–point in $\mathcal{M}'$) and that now it is a fan–$b \lessdot (b_q, \dots, b_1)$–point that **C** was the crossing point of **a** and **c** in $\mathcal{M}$ (resp. was a path–$(c_1, \dots, c_r)$–point in $\mathcal{M}'$) and that now it is a fan–$c \lessdot (a, c_r, \dots, c_1)$–point. Since the other representative points remain with the same corresponding graphs, one can easily check (see Figure 39) that $E(T)$ is exactly the set of edges induces by $\mathcal{M}^*$.
- Note that $F(T) = (F(T_1) \setminus acb) \cup F(T_2) \cup \{a_1a_2a, ab_1b, bc_1c\} \cup \{aa_ia_{i+1} \mid 2 \leq i < p\} \cup \{bb_ib_{i+1} \mid 1 \leq i < p\} \cup \{cc_ic_{i+1} \mid 2 \leq i < p\} \cup \{acc_r\}$. According to the face segments added in F^* (the ones in $F^* \setminus (F \cup F')$), the faces induced by **A**, **B** and **C**, and since the other representative points remain with the same corresponding graphs, one can easily check (see Figure 39) that $F(T)$ is exactly the set of faces induced by $\mathcal{M}^*$.

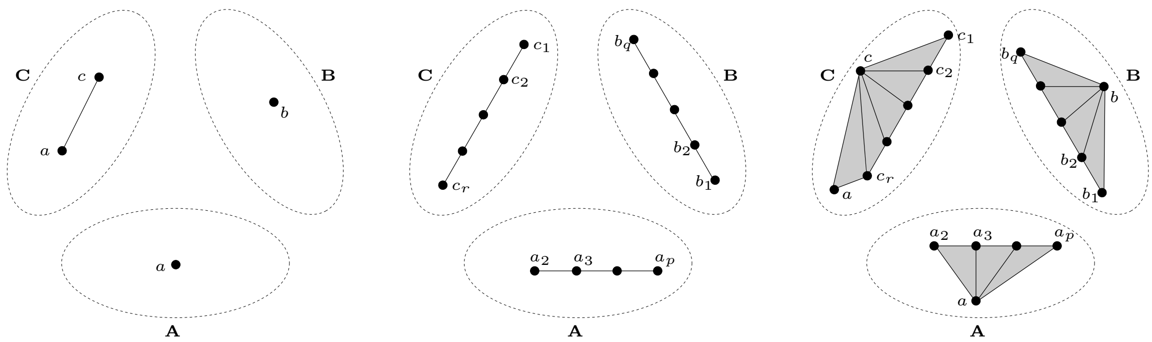

Fig. 39. The graphs corresponding to **A**, **B** and **C** in $\mathcal{M}$ (left), $\mathcal{M}'$ (center) and $\mathcal{M}^*$ (right).

Finally since T has a premodel $\mathcal{M}^*$, Corollary 2.18 implies that it has a full model, proving Theorem 2.5. □

5 Conclusion

West conjectures that every planar graph is the intersection graph of segments using only four directions [17]. Furthermore if the segment set is unambiguous, parallel segments induce a stable set, and the four directions would correspond to a four coloring of the planar graph. This conjecture is true for some families of planar graphs. Indeed, every bipartite planar graph has a representation with two directions [9,3,5] and every triangle free planar graph (that is 3-colorable by Grötzsch's theorem) has a representation with three directions [1].

De Fraysseix and Ossona de Mendez proposed [4] the following generalization of Scheinerman's Conjecture: "Every planar linear hypergraph is the intersection hypergraph of segments in the plane.", where a linear hypergraphs is an hypergraph such that two hyperedges intersect in at most one vertex. This generalization does not holds since the second author found a counterexample [8].

In our proof we need the constraints digraph to be acyclic in order to perform local perturbations on the segment set, like gliding or traversing. We wonder whether this condition is necessary: is it always possible to do local perturbations in any flexible segment set R (with possibly cycles in $Const_R$)? The flexibility of R is required since Pappus's construction gives us a segment set with only one point that is internal in 3 segments, and such that some glidings are impossible.

References

1. N. de Castro, F. Cobos, J.C. Dana, A. Márquez, and M. Noy. Triangle-free planar graphs as segment intersection graphs. *J. Graph Algorithms Appl.*, 6(1):7–26, 2002.
2. J. Chalopin, D. Gonçalves, and P. Ochem. Planar graphs are in 1-STRING. *Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*, 609–617, 2007.
3. J. Czyzowicz, E. Kranakis, and J. Urrutia. A simple proof of the representation of bipartite planar graphs as the contact graphs of orthogonal straight line segments. *Inform. Process. Lett.*, 66(3):125–126, 1998.
4. H. de Fraysseix and P. Ossona de Mendez. Representations by Contact and Intersection of Segments. *Algorithmica*, 47:453–463, 2007.
5. H. de Fraysseix, P. Ossona de Mendez, and J. Pach. Representation of planar graphs by segments. *Intuitive geometry (Szeged, 1991), Colloq. Math. Soc. János Bolyai*, 63:109–117, 1994.
6. G. Ehrlich, S. Even, and R.E. Tarjan. Intersection Graphs of Curves in the Plane. *J. Combin. Theory. Ser. B* 21:8–20, 1976.
7. D. Gonçalves. Edge-Partition of Planar Graphs into two Outerplanar Graphs. *Proceedings of the 37th Annual ACM Symposium on Theory of Computing*, 504–512, 2005.
8. D. Gonçalves. A Planar linear hypergraph whose edges cannot be represented as straight line segments. *European J. Combin.* 30 (2009), pp. 280-282
9. I.B.-A. Hartman, I. Newman, R. Ziv. On grid intersection graphs. *Discrete Math.*, 87(1):41–52, 1991.
10. P. Koebe. Kontaktprobleme der Konformen Abbildung. *Ber. Verh. Sachs. Akademie der Wissenschaften Leipzig, Math.-Phys. Klasse*, 88 (1936), 141–164.
11. J. Kratochvíl. String graphs. II. Recognizing String Graphs is NP-hard. *J. Combin. Theory. Ser. B*, 52:67–78, 1991.
12. J. Kratochvíl. Geometric representations of Graphs. Graduate Course, Barcelona, april 2005. retrieved Jan. 2006 at `http://kam.mff.cuni.cz/~honza/ig.ps`.
13. J. Kratochvíl and J. Matoušek. Intersection Graphs of Segments. *J. Combin. Theory. Ser. B*, 62:180–181, 1994.
14. P. Ossona de Mendez. retrieved Mar. 2006 at `http://www.ehess.fr/centres/cams/person/pom/langfr/research.html`.
15. J. Pach and J. Solymosi Crossing Patterns of Segments. *J. Combin. Theory. Ser. A*, 96:316–325, 2001.
16. E.R. Scheinerman. Intersection classes and multiple intersection parameters of graphs. *PhD Thesis, Princeton University*, 1984.
17. D. West. Open problems. *SIAM J. Discrete Math. Newslett.*, 2(1):10–12, 1991.
18. H. Whitney. A theorem on graphs. *Ann. of Math. (2)*, 32(2):378–390, 1931.