'WARM UP' HANDOUT PRIOR TO A COURSE OF COMBINATORIAL OPTIMIZATION IN SÈTE, JUNE 2018

Ideas for approximation algorithms: Matchings, edge-colorings and the Travelling Salesman

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Abstract

Some of the most important classical tools of combinatorial optimization, such as matching or matroid theory, have been recently used for approximating solutions to NP-hard problems. The course wishes to introduce the tools themselves, and to show how they act in recent approximation algorithms.

The purpose of this handout is to provide "warming up" exercises about the title-subject, at the same time introducing some preliminaries of the course. I presuppose basic definitions about graphs that can be easily looked up in any introductory course or book, for instance in the text-books suggested below. It is usually a good hint for the solution to use one of the immediately preceding exercises, or else we sometimes explicit other hints as well.

Some of these exercises will be restated and used during the course, and the full solution will be given if the result of an exercise is used.

Advised preliminaries: Shortest paths in digraphs, network flows (Ford and Fulkerson's algorithm, max flow min cut theorem, min cost flows), bipartite matchings, introduction to complexity theory (P, NP, coNP, PTAS, APX-complete, approximation ratio ...) Kruskal's algorithm for minimum weight spanning trees, definitions and basic facts about matroids, linear programming (simplex method, duality theorem), similar basic knowledge of first courses of graph theory or operations research.

We will fully include the necessary knowledge about non-bipartite matchings (see "1.") and the generalization we use (T-joins, see "5."), but if you know something about them beforehand, you will digest these more easily.

Have fun with the exercises !

Key-words: matchings, matroids, packing and covering, linear programming, polyhedral combinatorics, minmax theorems, good characterization, algorithmic proofs, edge-coloring, TSP (travelling salesman problem), T-joins, complexity theory, connectivity, ear-theorems, matroid intersection, P, NP, RP, coNP, NP-complete, approximation algorithms, APX-hard.

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Textbooks:

Korte, Vygen: Combinatorial Optimization, New Edition, (Springer 2018).

Lovász: Combinatorial Problems and Exercises (Akadémiai Kiadó)

Lovász, Plummer: Matching Theory (Akadémiai Kiadó)

Schrijver: Combinatorial Optimization (Springer)

Shmoys, Williamson : The Design of Approximation algorithms (2011)

Articles related to the last part of the course:

Rico Zenklusen, A 1.5-Approximation for path TSP (May 2018) https://arxiv.org/abs/1805.04131

Sebő, Vygen: Shorter Tours by Nicer Ears : 7/5-approximation for graphic TSP, 3/2 for the path version, and 4/3 for two-edge-connected subgraphs
http://pagesperso.g-scop.grenoble-inp.fr/∼seboa/sebo_files/papers/ccanicears.pdf

Sebő, Undirected distances and the postman-structure of graphs
http://pagesperso.g-scop.grenoble-inp.fr/∼seboa/sebo_files/papers/jctb90undirdist.pdf

Basic notations: Graphs are meant to be undirected unless we say otherwise (eg. end of series 3); $V(G)$ is the set of vertices of graph G, $E(G)$ is its edge-set; $\nu(G)$ denotes the matching number of G, that is, the maximum size of a set of disjoint edges in G; $\tau(G)$ is the minimum vertex cover of G, that is, a set of vertices of minimum cardinality that meet every edge of G. I use the common notations mainly inspired by the above references.

If you don't understand an exercise, or you have any question or comment concerning the course or this hand-out, please, don't hesitate contacting me by e-mail; I will try to answer if I am online.

If you cannot solve an exercise, don't worry: to have understood it, to have thought about it and to have realized the difficulty will already be helpful enough for the course.

1 Matchings

Exercise 1 Let G be a graph, and $uv \in E(G)$. Then either $\nu(G-u) < \nu(G)$, or $\nu(G-v) < \nu(G)$, or else for any maximum matching M_u of $G-u$, and M_v of $G-v$: $M_u \cup M_v$ contains an (u,v)-path P alternating between M_u and M_v.

Exercise 2 Let G be a bipartite graph, and $uv \in E(G)$. Then either $\nu(G-u) < \nu(G)$, or $\nu(G-v) < \nu(G)$. Deduce from this a simple inductive proof of Kőnig's theorem $\nu(G) = \tau(G)$ for every bipartite graph G.

If G is a graph, and $X \subseteq E(G)$, then G/X denotes the graph we get from G by identifying the endpoints of the edges in X (and deleting the edges induced by X). We say that a vertex is *covered* by a matching if it is the endpoint of one of the edges of the matching, otherwise it is *uncovered*.

Exercise 3 Let G be a graph, $uv \in E(G)$, $\nu(G-u) = \nu(G)$, $\nu(G-v) = \nu(G)$. Then for the alternating path P of Exercise 1.1 the minimum number of uncovered vertices in G/P is the same as in G.

Exercise 4 Deduce, by induction, using exercises 1 and 3, the theorem of Tutte-Berge: the minimum number of vertices not covered by a matching is equal to the maximum of $q(X) - |X|$, where $q(X)$ denotes the number of odd components of $G - X$. If you know Edmonds' algorithm deduce also a proof of its correctness.

If for a set X the value of $q(X) - |X|$ is maximum, then it is called a Tutte-set.

Exercise 5 If $v \in V(G)$ is contained in some Tutte-set then it is covered by every maximum matching of G.

2 Edge-Coloring

Exercise 1 Let $G = (V, E)$ be an undirected multigraph and $k \in \mathbb{Z}$. Suppose we are given a partial edge-k-coloring of G, and the set of colored edges is inclusionwise maximal. Suppose there is an uncolored edge between $u, v \in V$, where u misses red, and v misses blue (where red and blue are two of the k colors). Then the union of blue and red edges has a component which is a path between u and v.

Exercise 2 (Kőnig's edge-coloring theorem) Prove that a bipartite graph can be edge-colored to as many colors as the maximum degree. Is this true to every graph ?

Exercise 3 (Particular Tashkinov trees) Build a "breadth first tree" starting with tree $\{\{u, v\}, uv\}$, from root u, using only edges whose color has already been missed by some vertex of the already constructed part of the tree. Prove that the already colored edges plus edge uv can all be correctly colored under either of the following conditions:

a. There is a common missing color in u and in one of its neighbors in the tree.
b. There are two neighbors of u in the tree, missing the same color.

Exercise 4 (Vizing's theorem) Prove that a graph without parallel edges can be edge-colored with $\Delta + 1$ colors where Δ is the maximum degree.

3 Postman tours

A *postman tour* is a closed walk which uses every edge of the graph at least once. Let us call a set of edges a *postman set* if its deletion leads to a graph with all degrees even (but not necessarily connected).

Exercise 1 Let G be a connected graph. The minimum length of a postman tour is equal to $|E(G)| + \tau$ where τ is the minimum cardinality of a postman set.

Exercise 2 Let G be a graph. A postman set P has minimum cardinality if and only if there is no circuit of negative weight according to the weight function which is -1 on the edges of P and 1 on the other edges.

Exercise 3 Prove that the maximum matching problem can be solved by a simple (linear time) reduction to an algorithm that finds a negative circuit in a ± 1-weighted graph, or concludes correctly that such a circuit does not exist.

Hint: Add a new vertex to the graph, and join it to all uncovered vertices by edges of weight -1, and put weight -1 on the matching edges as well.

Exercise 4 Can you find a minimum postman set via Exercise 3.2, or a maximum matching via Exercise 3.3 by finding a negative circuit in some well-known way for directed graphs (eg. Floyd-Warshall's algorithm)? Hint: I cannot. Why ?

A digraph is said to be *weakly connected* if the underlying undirected graph is connected, and it is *strongly connected* if for any ordered pair of vertices u, v there exists a directed path from u to v. A *postman tour* of a digraph G is a weakly connected multigraph, where the multiplicity of every arc of G is at least 1, and the indegree of every vertex is equal to its outdegree.

Exercise 5 Show that a postman tour of a digraph is strongly connected, and a graph has a postman-set if and only if it is strongly connected. What is the complexity of finding a minimum weight postman tour in an edge-weighted digraph?

Hint: Apply network flows ("Hoffman's circulation theorem") with the appropriate lower capacities.

4 Conservative weightings

A weighting of the edges of a graph is called *conservative* if there is no circuit of negative weight. The *distance* between pairs of points is the minimum weight of paths.

Exercise 1 In a graph given with a conservative weighting, changing the sign of all edges of a 0-weight circuit, the weighting remains conservative and the distance between each pair of vertices, is unchanged.

Hint: Let C be a 0 weight circuit. Express the modified weight of an arbitrary path or circuit Q with the original weight of the symmetric difference of Q and C.

Exercise 2 Given a graph with two ± 1 conservative weightings, but where the parities of the number of negative edges adjacent to the vertices are the same, the distance between any pair of vertices is the same in the two weightings.

Exercise 3 Given a ± 1-weighted conservative graph $G = (V, E)$ and $a \in V$, let b be a vertex whose distance is minimum from a. If $b \neq a$, then b is incident to exactly one negative edge.

Exercise 4 Under the condition and notations of the preceding exercise, supposing in addition that G is bipartite, contracting the edges adjacent to b, the obtained graph is also conservative with the original weighting, and the distances of the vertices of $G - b$ from a do not change. What will be the distance of the new vertex from a ?

Exercise 5 In a ± 1-weighted conservative bipartite graph there exist edge-disjoint cuts covering all the negative edges so that each cut contains exactly one negative edge.

5 T-joins

Let $T \subseteq V(G)$, $|T|$ even. A *T-join* is a set of edges whose set of odd degree vertices is exactly T. Note that a postman set is a T_G-join, where T_G is the set of odd degree vertices of G. Let $\tau(G, T)$ be the minimum cardinality of a postman set in G.

Exercise 1 Prove that in a bipartite graph $\tau(G, T_G)$ is equal to the maximum number of pairwise edge-disjoint cuts defined by bipartitions $\{X, Y\}$ of $V(G)$, where X contains an odd number of vertices of odd degree.

Hint: Use Exercise 4.5.

Exercise 2 Is this a 'good characterization' (a theorem that puts the corresponding decision problem in NP intersection coNP) ?

Exercise 3 Is a similar theorem true for non-bipartite graphs ?

6 Matroid operations

If you never heard about matroids skip this series. If you try though, you may be rewarded by Exercise 6. If we are done with this somewhat boring introduction, we can better focus on using matroids for interesting theorems and algorithms.

A *minor* of the matroid $M = (S, \mathcal{F})$ is a matroid obtained from M by a succession of *deletions* and *contractions* of elements, that is:
$M \setminus e := M - e := (S \setminus e, \mathcal{F} \setminus e)$, $\quad M/e := (S \setminus e, \mathcal{F}/e)$,
where $\mathcal{F} \setminus e = \{F \in \mathcal{F} : F \subseteq S \setminus e\}$, $\quad \mathcal{F}/\{e\} = \{F \in \mathcal{F} : F \subseteq S \setminus e, F \cup \{e\} \in \mathcal{F}\}$.
The *dual* $M^* = (S, \mathcal{B}^*)$ of $M = (S, \mathcal{B})$ (matroids defined with the basis axioms) is defined as $\mathcal{B}^* := \{S \setminus B : B \in \mathcal{B}\}$.
The "cycle-matroid" associated to a graph G is denoted by $M(G)$.
The *sum* of two matroids $M = (S, \mathcal{F}_1)$, $M = (S, \mathcal{F}_2)$: $M = (S, \mathcal{F})$, where $\mathcal{F} := \{F = F_1 \cup F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$.

Exercise 1 Show that the result of all these operations is a matroid.

Exercise 2 Show $(M \setminus e)/f = (M \setminus f)/e$, that is, the result of a succession of deletions and contractions does not depend on the order of these operations.

Exercise 3 Show $(M \setminus e)^* = M^*/e$.

Exercise 4 Show that the rank function of the dual of a matroid with rank function r is: $r^*(X) = |X| - (r(S) - r(S \setminus X))$.

Exercise 5 Show that in the special case of graphic matroids these operations specialize to the well-known graph operations of the same name. In particular, if G is a planar graph, $M^*(G) = M(G^*)$; in addition, the circuits of G are the cuts of G^*.

Exercise 6 Prove Euler's formula: suppose $G = (V, E)$ is a connected planar graph, with f faces. Then $|V| - |E| + f = 2$.

Hint: Use the preceding exercise to show that deleting from $M(G)$ a spanning tree of G you get in $M^*(G)$ a spanning tree of G^*. Therefore $(|V| - 1) + (f - 1) = |E|$.

Exercise 7 Let $\mathcal{B}$ be the set of spanning trees of a graph $G = (V, E)$. Prove that the set $\{B \cup \{e\} : B \in \mathcal{B}, e \in E \setminus B\}$ also satisfies the basis axioms. Is this true for the set of bases $\mathcal{B}$ of any matroid ?