Input-Sensitive Enumerations

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- We use *exact exponential-time algorithms*, in particular *branching algorithms*.

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- Can be used to solve the decision, optimization and counting versions of the problem.
- The running time of the algorithm provides an *upper bound for the maximum number of enumerated objects*.
- Moreover, the algorithm can be used to obtain better bounds.

Upper and Lower bounds

Suppose that there is a family of instances $\mathcal{I}$ of an enumeration problem such that for every $n \in \mathbb{N}$, $\mathcal{I}$ contains an instance I with $|I| = n$ and the number of enumerated objects for $I \in \mathcal{I}$ is $f(|I|)$.

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- Construct an enumeration algorithm with the "best" running time.
- Construct the "best" lower bound.
- Ideally, we wish to get (asymptotically) tight upper and lower bounds for running time and combinatorial bounds for the number of enumerated objects.

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- Search tree algorithms.
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- Could be difficult to apply for some classes of problems.

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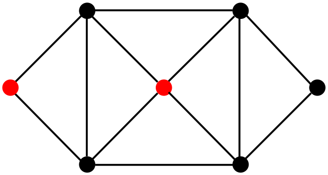

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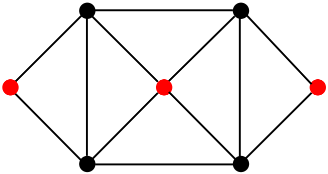

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Problem (Maximal Independent Set Enumeration)

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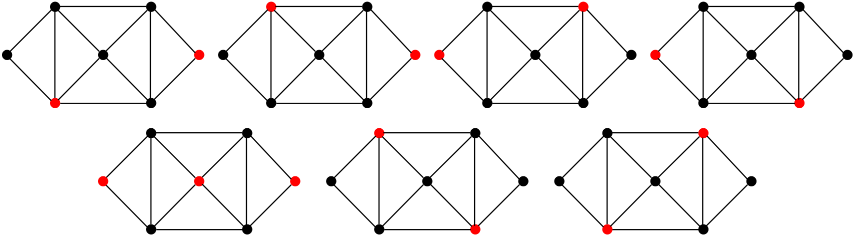

Enumeration of MIS

Enum MIS(G, S)

Input : A graph G, a set S of vertices already selected to be in a MIS, $S \cap V(G) = \emptyset$

Output: All sets $X \cup S$ for MIS S of G.

if $V(G) = \emptyset$ **then**
 output S
else
 find a vertex $v \in V(G)$ of minimum degree d;
 call **Enum MIS**$(G - N_G[v], S \cup \{v\})$;
 for $x \in N_G(v)$ **do**
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Call **Enum MIS**$(G, \emptyset)$ to enumerate all MIS of G.

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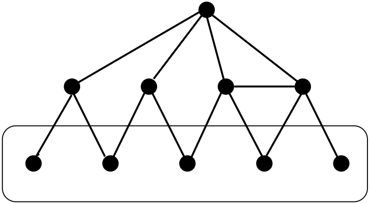

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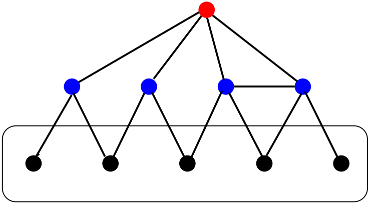

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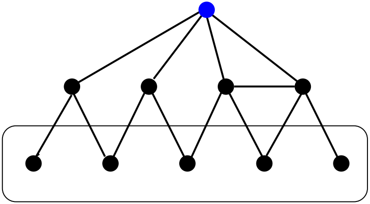

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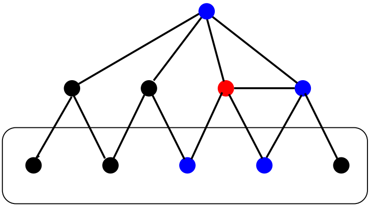

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Exercise: Give a formal inductive correctness proof.

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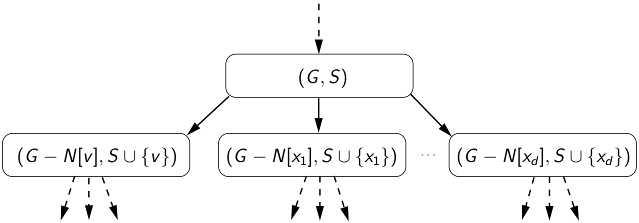

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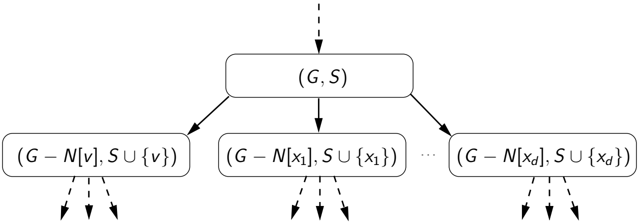

Observe that the algorithm produced maximal independent sets in the leaves of the search tree.

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Therefore,

$$L(n) \leq (d+1)L(n - (d+1)).$$

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$$L(n) \leq (d+1)L(n-(d+1)) \leq (d+1)3^{(n-(d+1))/3}$$
$$= (d+1)3^{(d+1)/3} \cdot 3^{n/3} \leq 3^{n/3}.$$

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We write $f(n) = O^*(g(n))$ to denote that there is a polynomial $p(n)$ such that $f(n) \leq g(n)p(n)$.

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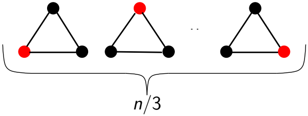

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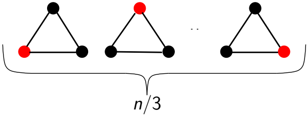

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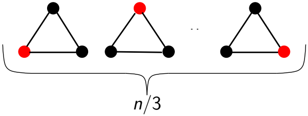

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We can enumerate MIS in time $O^*(3^{n/3})$ but cannot do it faster than $\Omega(3^{n/3})$.

Enumeration of MIS

Since the depth of the search tree is at most n, **Enum MIS** uses polynomial space.

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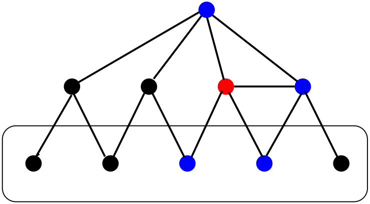

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Let $\mathcal{A}$ be an enumeration algorithm that lists all maximal independent sets (cliques) with polynomial delay. Then $\mathcal{A}$ runs in time $O^*(3^{n/3})$.

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Given a *branching vector*

$$b = (t_1, \ldots, t_r),$$

we solve the equation

$$x^t - x^{t-t_1} - \ldots - x^{t-t_r} = 0$$

for $t = \max\{t_1, \ldots, t_r\}$ and find the *branching number*

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Then we obtain the bound

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Claim: If $\mu(I) \leq n$ for the inputs containing n-vertex graphs, then

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Exercise: Show that $L(s) \leq \lambda^s$.

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Often it could be shown that $L(s) \leq \lambda^s$.

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Nevertheless, it is usually sufficient to consider few "worst" branching vectors and find

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- $\lambda(c, c) \leq \lambda(a, b)$ if $c = (a + b)/2$,
- $\lambda(a + \varepsilon, b - \varepsilon) < \lambda(a, b)$ for $0 < a < b$ and $0 < \varepsilon < (b - a)/2$.

Some branching numbers

Branching numbers for $b = (t_1, t_2)$:

	1	2	3	4	5	6
1	2.0000	1.6181	1.4656	1.3803	1.3248	1.2852
2	1.6181	1.4143	1.3248	1.2721	1.2366	1.2107
3	1.4656	1.3248	1.2560	1.2208	1.1939	1.1740
4	1.3803	1.2721	1.2208	1.1893	1.1674	1.1510
5	1.3248	1.2366	1.1939	1.1674	1.1487	1.1348
6	1.2852	1.2107	1.1740	1.1510	1.1348	1.1225

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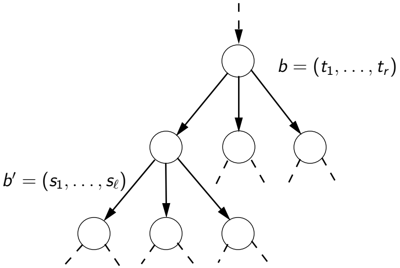

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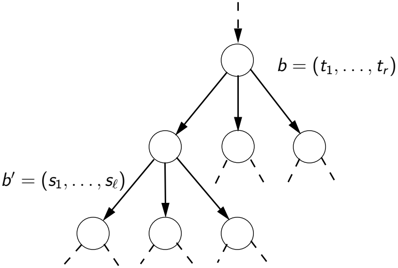

New branching vector: $c = (t_1 + s_1, \ldots, t_1 + s_\ell, t_2, \ldots, t_r)$.

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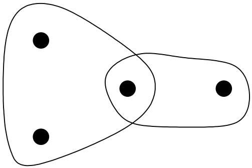

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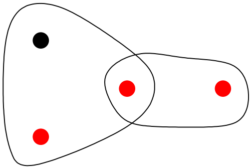

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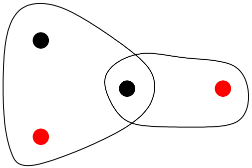

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Input: *A family $\mathcal{S}$ of subsets over an universe $\mathcal{U}$.*

Task: *Enumerate all maximal independent sets of G.*

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We consider the problem for $\mathcal{S}$ containing subsets of size at most 3.

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- **Emum Hitting Sets** enumerates minimal hitting sets in time $O(1.8394^n)$ where $n = |\mathcal{U}|$.
- there are at most 1.8394^n minimal hitting sets if $\mathcal{S}$ contains sets of size at most 3.

Exercises

1. Enumerate all maximal matchings in a graph.
   - Construct an algorithm that runs in time $O^*(c^m)$ for $c < 2$ where m is the number of edges.
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($*$) Improve the running time $O(1.8394^n)$ for the enumeration of minimal hitting sets.