Input-Sensitive Enumerations

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Plan of the lectures

- Introduction to branching enumeration algorithms and their analysis.
- Advanced analysis of branching algorithms; the "Measure and Conquer" technique.
- Lower bounds.
- Conclusions and open problems.

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- Could be efficient in practice.

Cons

- Difficult to analyze.
- Could be difficult to apply for some classes of problems.

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- *Reduction rules*
  - used to simplify instances,
  - typically reduce the size,
  - typically run in polynomial time.
- *Branching rules*
  - solve the problem for an instance by recursively solving $t \geq 2$ (smaller) instances,
  - typically run in polynomial time (without recursive calls).

Search trees

Search trees are used to illustrate, understand and analyze branching algorithms:

- *Root*: assign the input to the root.
- *Node*: assign to each node a problem instance.
- *Child*: each instance produced by a branching rule is assigned to a child.
- *Leaf*: outputs are assigned to leaves.

Analysis of branching algorithms

Running time analysis:

- upper bound of the maximum number of leaves $L(s)$ of a search tree for an input of given size s,
- if each reduction and branching rule can be done in polynomial time, then we have running time $O^*(L(s))$,
- $L(s)$ gives a combinatorial upper bound for the number of enumerating objects.

Bounding the number of leaves

Let $\mathcal{A}$ be a recursive branching algorithm that uses a single branching rule that generates $r \geq 2$ instances of the problem.

We associate a **measure** $\mu(I)$ with each instance I of the problem.

Typically, for simple branching algorithms, $\mu(I)$ is integer, and often $\mu(I)$ is the number of vertices for graph problems.

Let $I_1, \ldots, I_r$ be the instances of the problem generated by the branching rule from I.

Assume that there are positive (integers) $t_1, \ldots, t_r$ such that

$$\mu(I_i) \leq \mu(I) - t_i \text{ for } i \in \{1, \ldots, r\}.$$

It is said that

$$b = (t_1, \ldots, t_r)$$

is the *branching vector* for the rule.

Bounding the number of leaves

Let $L(s)$ be the maximum number of leaves of a search tree for the instances I with $\mu(I) = s$.

We have that

$$L(s) \leq L(s - t_1) + \ldots + L(s - t_r).$$

Let $t = \max\{t_1, \ldots, t_r\}$. We associate with $b = (t_1, \ldots, t_r)$ the *characteristic polynomial*:

$$p(x) = x^t - x^{t-t_1} - \ldots - x^{t-t_r}.$$

Claim: $p(x)$ has a unique positive real root λ and $L(s) = O^*(\lambda^s)$.

It is said that $\lambda = \lambda(t_1, \ldots, t_r)$ is the *branching number* of b.

Bounding the number of leaves

Given a *branching vector*

$$b = (t_1, \ldots, t_r),$$

we solve the equation

$$x^t - x^{t-t_1} - \ldots - x^{t-t_r} = 0$$

for $t = \max\{t_1, \ldots, t_r\}$ and find the *branching number*

$$\lambda = \lambda(t_1, \ldots, t_r).$$

Then we obtain the bound

$$L(s) = O^*(\lambda^s).$$

Often we can show that $L(s) \leq \lambda^s$.

Analyzing collections of recurrences

We consider all branching rules and construct the family of branching vectors

$$\mathcal{B} = \{b^{(i)} \mid i \in \mathbb{N}\}$$

$$b^{(i)} = (t_1^{(i)}, \ldots, t_{r(i)}^{(i)}).$$

We compute

$$\lambda = \sup\{\lambda(b^{(i)}) \mid b^{(i)} \in \mathcal{B}\}.$$

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Huge problem: The bound $O^*(\lambda^s)$ is limited by the worst branching number even if the corresponding branching occur in some few special cases.

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For $\lambda = \lambda(t_1, \ldots, t_r)$ and $s = \mu(I)$,

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Assume that we consider an enumeration problem for graphs where the aim is to list all vertex subsets that satisfy a property P.

Claim: If $\mu(I) \leq n$ for the inputs containing n-vertex graphs, then

$$L(n) = O^*(\lambda^n).$$

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This is what we need to obtain that for n-vertex graphs,

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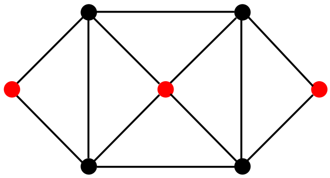

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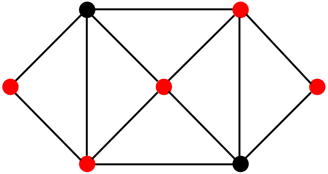

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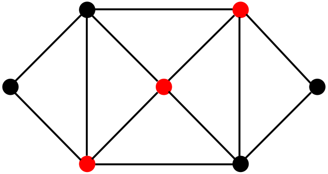

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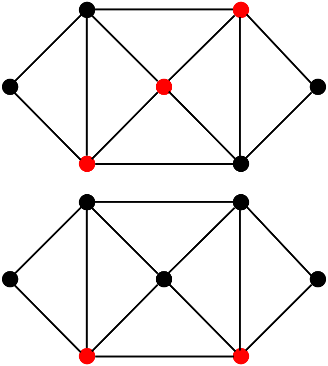
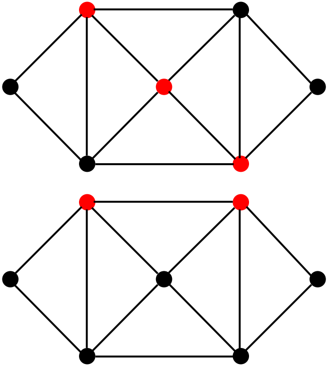

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Theorem

There is $\varepsilon > 0$ such that all minimal connected dominating sets of an n-vertex graph can be enumerated in time $2^{(1-\varepsilon)n}$.

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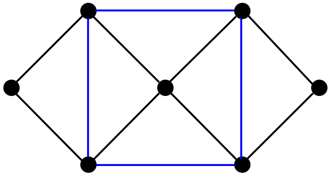

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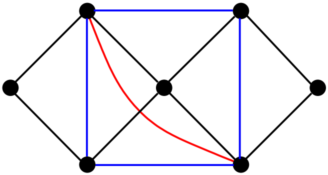

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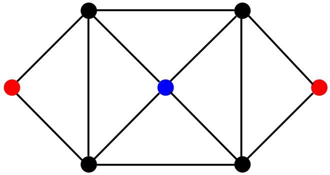

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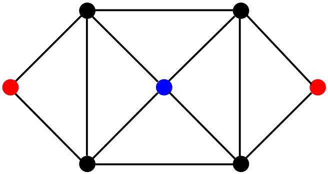

Lemma

Every chordal graph has a simplicial vertex.

Semi-simplicial vertices

For a vertex u of a graph G, denote by $S_G(u)$ the set of all simplicial vertices v of G such that $vu \in E(G)$.

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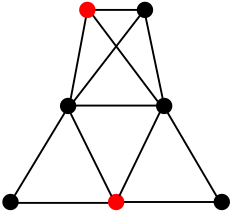

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- We have that u is a semi-simplicial vertex of G.

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- If x is not a cut-vertex of $G[D]$, then $D \setminus \{x\}$ is a connected dominating set of G.

Separators

Observation: Let (A, B) be a *separation* of a connected graph G, that is, $A, B \subseteq V(G)$, $A \cup B = V(G)$ and there is no edge $uv \in E(G)$ with $u \in A \setminus B$ and $v \in B \setminus A$.

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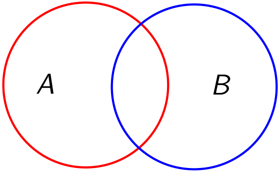

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The measure of the instance is $|V(H)|$.

Initial steps

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and the maximum branching number is $\lambda(3,3,3) = 3^{1/3} \approx 1.4423$ for $k = 3$.

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- If H has a cut-vertex $v \notin X$, then call **Emum MCDS**$(H, X \cup \{v\})$.

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- If H has a simplicial vertex v such that $N_H(v) \cap X \neq \emptyset$, then call **Emum MCDS**$(H - v, X)$.

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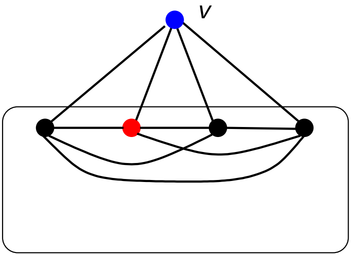

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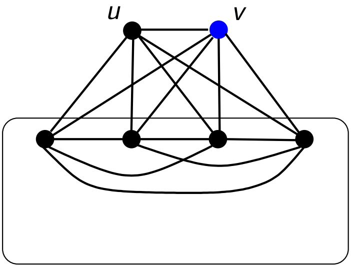

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We select a semi-simplicial vertex x and branch depending on $|S_H(x)|$:

- $|S_H(x)| = 1$,
- $|S_H(x)| = 2$,
- $|S_H(x)| \geq 3$.

Branching for $|S_H(x)| = 1$

Let y be the unique vertex of $S_H(x)$.

Branching for $|S_H(x)| = 1$

Let y be the unique vertex of $S_H(x)$.

We consider two cases:

- $d_H(y) = 2$,
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Branching for $|S_H(x)| = 1$ and $d_H(y) = 2$

Let z be the neighbor of y distinct from x.

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- If $y \in X$, then
  - **(i)** call **Emum MCDS**$(H/xy - z, (X \cup \{x\})/xy)$,
  - **(ii)** call **Emum MCDS**$(H/yz - x, (X \cup \{z\})/yz)$.

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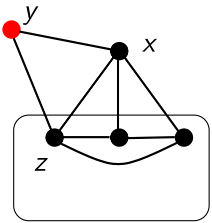

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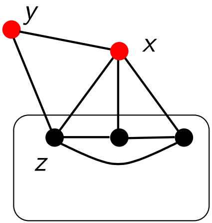

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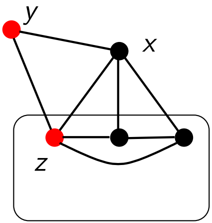

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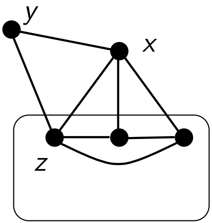

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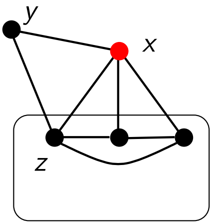

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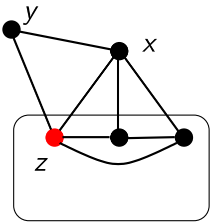

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The branching vector is $(2, 2)$ and $\lambda(2, 2) = 2^{1/2} \approx 1.4143$.

Branching for $|S_H(x)| = 1$ and $d_H(y) \geq 3$

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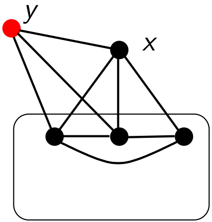

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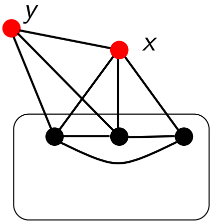

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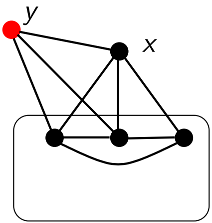

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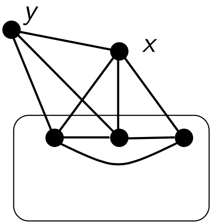

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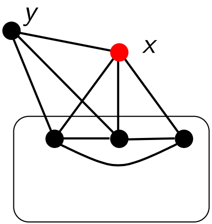

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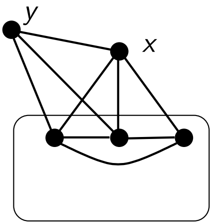

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The worst branching vector is $(3, 1)$ for $d_H(y) = 3$ and $\lambda(3, 1) \approx 1.4656$.

Branching for $|S_H(x)| = 2$

Let y and z be the neighbors of x, $d_H(y) \leq d_H(z)$.

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Observation: if for an X-minimal connected dominating set D, it holds that $x \in D \setminus X$, then

- either $(N_H(y) \setminus \{x\}) \cap D = \emptyset$
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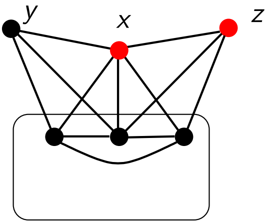

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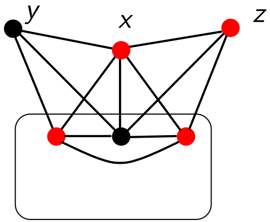

Branching for $|S_H(x)| = 2$

We consider the cases:

- $d_H(y) = d_H(z) = 2$,
- $d_H(y) = 2$ and $d_H(z) \geq 3$,
- $d_H(y), d_H(z) \geq 3$,

Branching for $|S_H(x)| = 2$

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and branch depending on the inclusion of y and z in X.

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Branching for $|S_H(x)| = 2$

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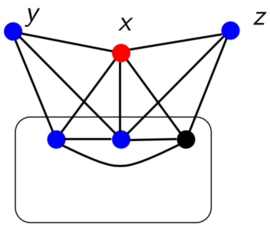

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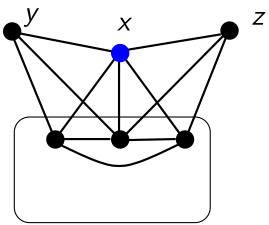

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The worst branching vector $(4, 4, 1)$ for $d_H(y) = d_H(z) = 3$ and $\lambda(4, 4, 1) \approx 1.5437$.

Branching for $|S_H(x)| \geq 3$

Assume that $S_H(x) \cap X = \emptyset$.

Branching for $|S_H(x)| \geq 3$

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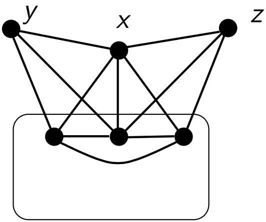
The worst branching vector is $(3, 1)$ for $|S_H(x)| = 3$ and $\lambda(3, 1) \approx 1.4656$.

Introducing measure

We have that the worst case is $|S_H(x)| = 2$ and $d_H(y) = d_H(z) = 3$ for $\{y, z\} = S_H(x)$.



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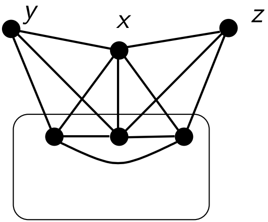

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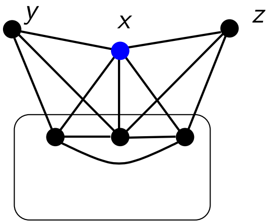

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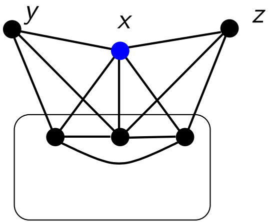

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Simplicial verices of degree 2 are good!

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We set

$$\omega(v) = \begin{cases} 1 - \varepsilon & \text{if } v \text{ is a simplicial vertex of degree 2} \\ 1 & \text{otherwise} \end{cases}$$

Improving branching numbers

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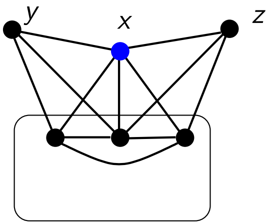

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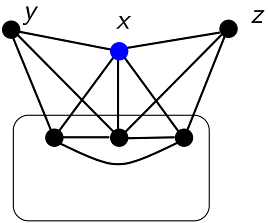

The branching vector is $(4, 4, 1 + 2\varepsilon)$ and $\lambda(4, 4, 1 + 2\varepsilon) < \lambda(4, 4, 1)$.

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Let $|S_H(x)| = 1$ and $d_H(y) = 2$ for $\{y\} = S_H(x)$. Let $N_H(y) = \{y, z\}$ and assume that $y \notin X$.

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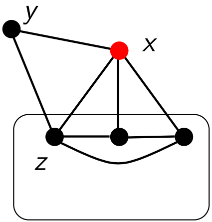

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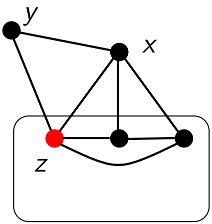

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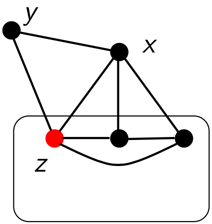

The branching vector is $(2 - \varepsilon, 2 - \varepsilon) > \lambda(2, 2)$.

Balancing branching numbers

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Eventually, we have to solve some equation

$$\lambda_i(\varepsilon) = \lambda_j(\varepsilon)$$

for $\lambda_i \in \mathcal{B}$ and $\lambda_j \in \mathcal{B}'$.

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$\varepsilon \approx 0.21199\ldots$ and $x \approx 1.47353\ldots$

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Theorem

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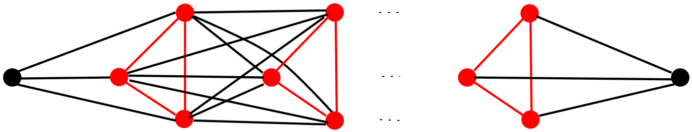

There are chordal graphs with at least $3^{(n-1)/3} = \Omega(1.4422^n)$ minimal CDS

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- The main idea: the removal of a vertex decreases the degrees of the neighbors.
- To determine k and $w(0), \ldots, w(k-1)$, we have to solve an auxiliary optimization problem.

Enumeration of minimal dominating sets

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Lower bound: there are graphs with $15^{n/6}$ (1.5704^n) minimal dominating sets.

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Observation: $D \subseteq V(G)$ is a minimal dominating set of G if and only if $\mathcal{S}^* = \{N_G[v] \mid v \in D\}$ is a minimal set cover for $(\mathcal{S}, \mathcal{U})$.

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The weights $\alpha_1, \ldots, \alpha_4$ and $\beta_2, \ldots, \beta_5$ are found by making use of the *quasiconvex programming*.