# Input-Sensitive Enumerations

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# Plan of the lectures

- Introduction to branching enumeration algorithms and their analysis.
- Advanced analysis of branching algorithms; the "Measure and Conquer" technique.
- Lower bounds.
- Conclusions and open problems.

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# Upper and Lower bounds

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- Construct an enumeration algorithm with the "best" running time.
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- Ideally, we wish to get (asymptotically) tight upper and lower bounds for running time and combinatorial bounds for the number of enumerated objects.
- If we fails to produce a lower bound that is "sufficiently close" to our upper bound, then this usually means that the upper bound is too big.

# Maximal independent sets

An $n$-vertex graph has at most $3^{n/3}$ maximal independent sets that can be enumerated in time $O^*(3^{n/3})$ and there are $n = 3k$-vertex graphs that have $3^{n/3}$ maximal independent sets.

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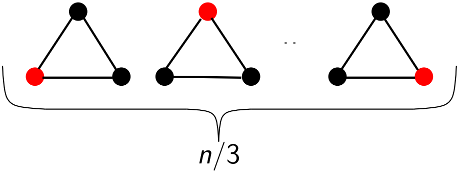

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An $n$-vertex chordal graph has at most $1.4736^n$ minimal connected dominating sets that can be enumerated in time $O(1.4736^n)$ and there are $n = 3k + 2$-vertex interval graphs that have $3^{(n-2)/3}$ $(1.44.22^n)$ minimal connected dominating sets.

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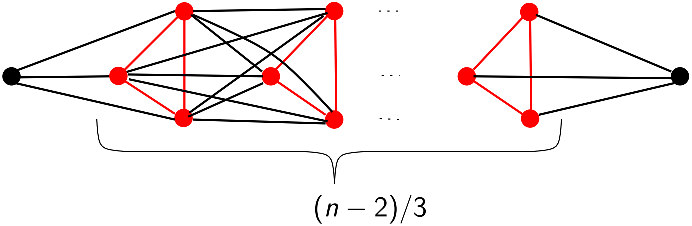

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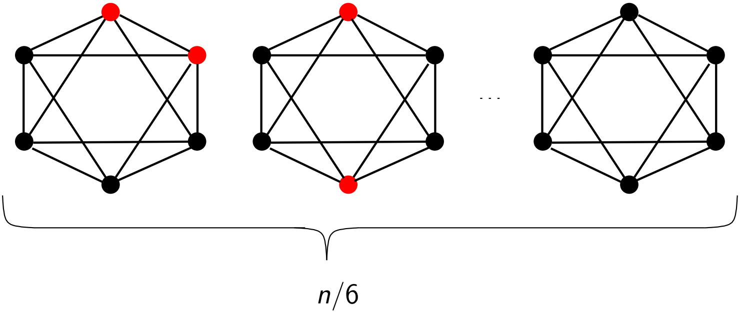

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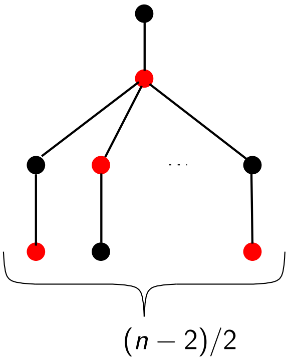

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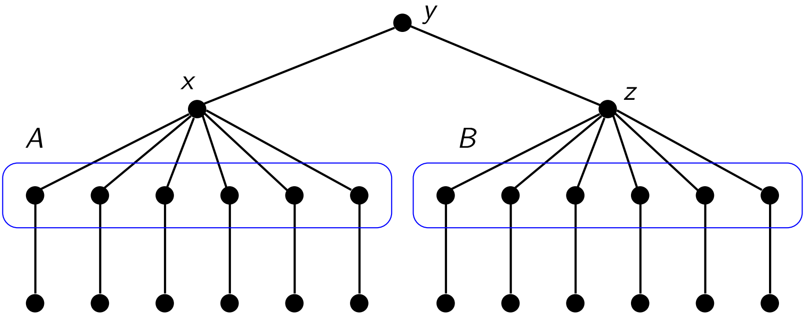
**Current upper bound:** $3^{n/3}$.

# Proof



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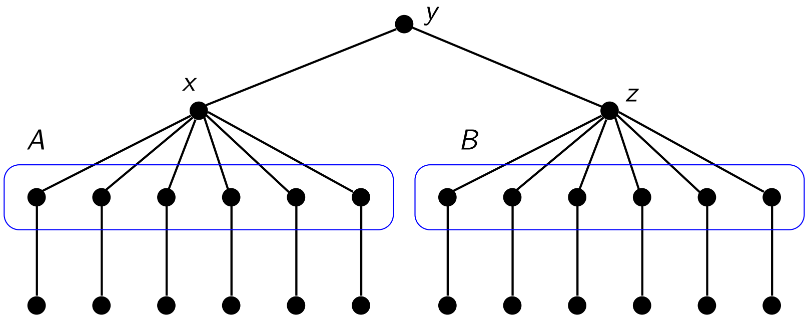
There are $3 \cdot (2^6 - 1)^2$ minimal dominating sets $D$ such that $D \cap A \neq \emptyset$ and $D \cap B \neq \emptyset$.



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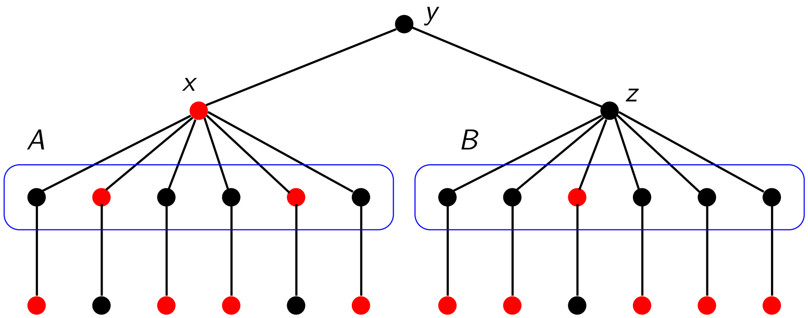

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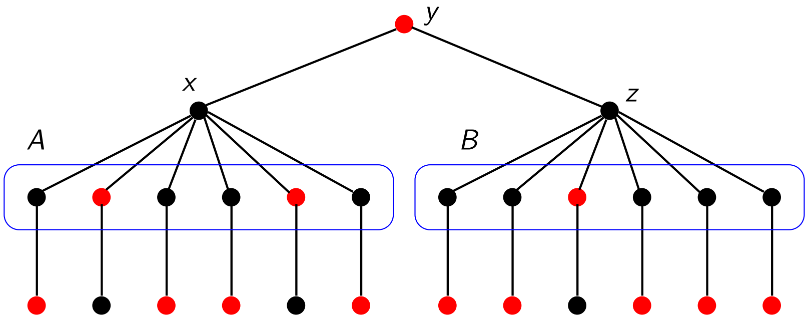

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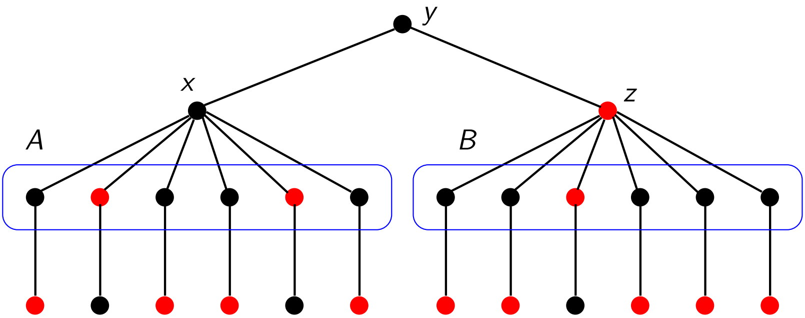

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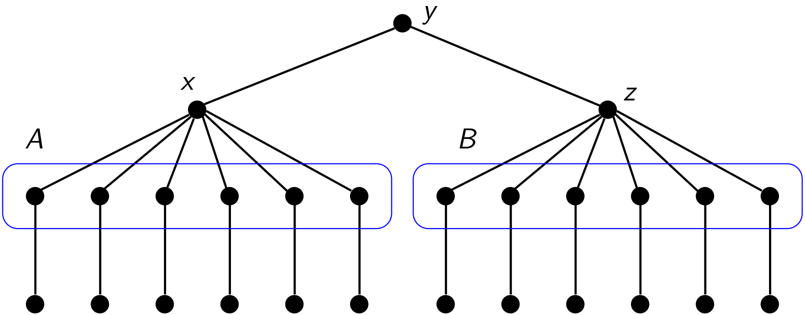

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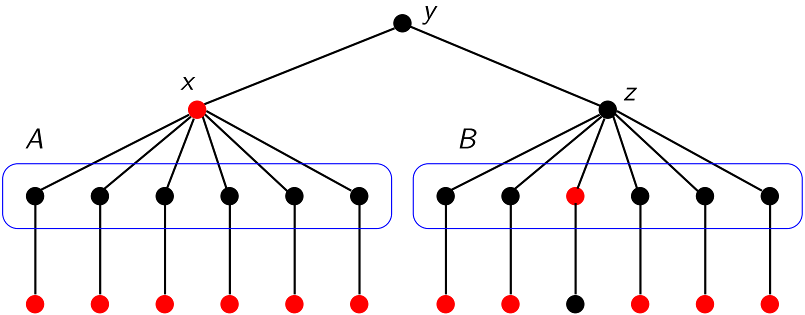

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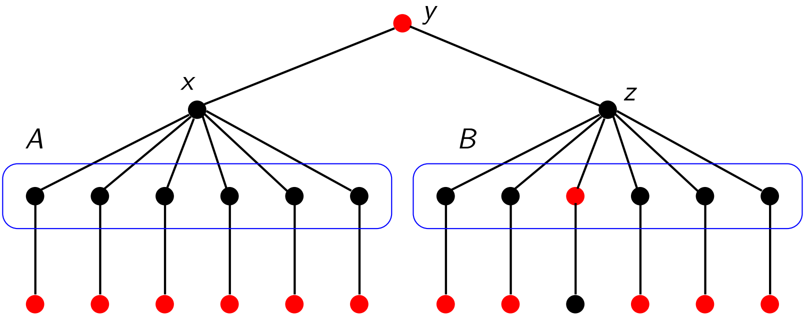

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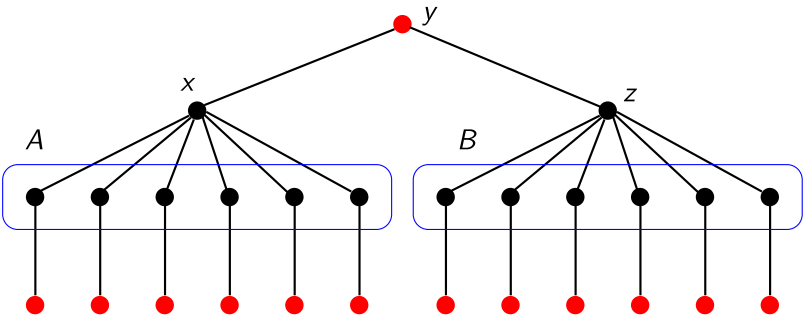

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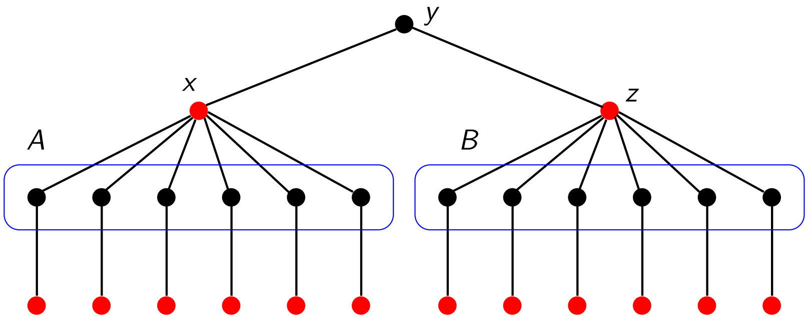

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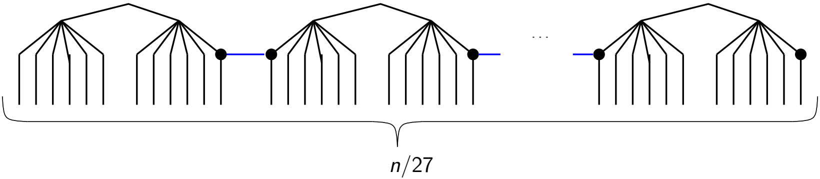

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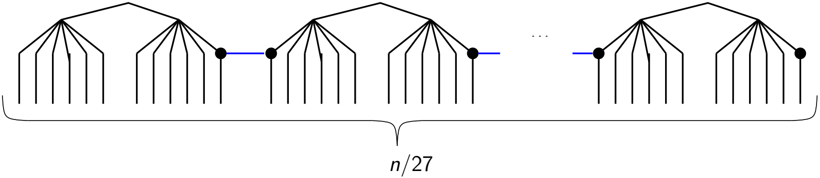


This tree has $12161^{n/27}$ minimal dominating sets.

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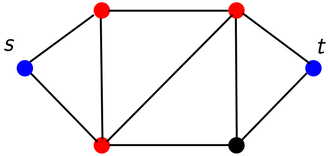

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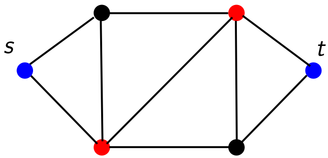

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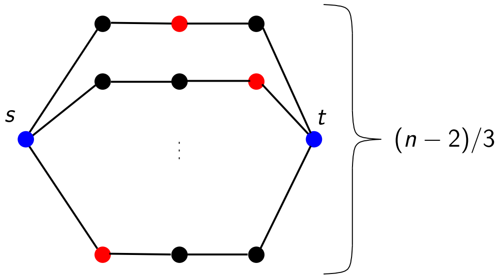

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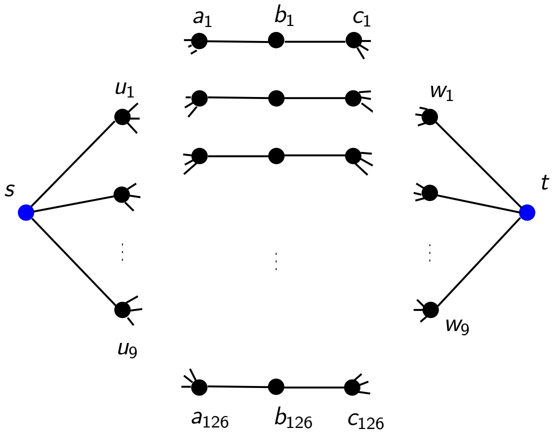
**Current upper bound:** $O(1.6180^n)$.

# Sketch of the proof



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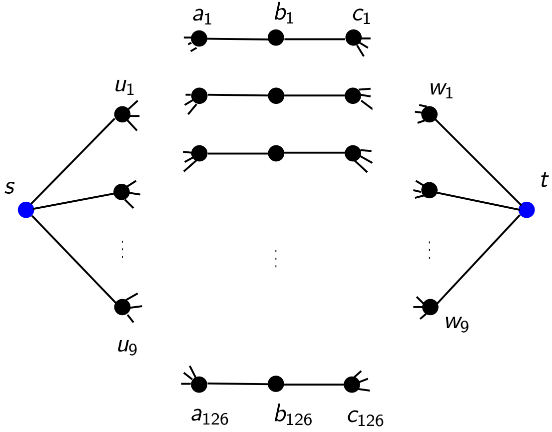


Note that $126 = \binom{9}{4}$; we make each $a_i$ adjacent to 4 vertices of $\{u_1, \ldots, u_9\}$ in such a way that $a_i$-s have distinct neighborhoods.

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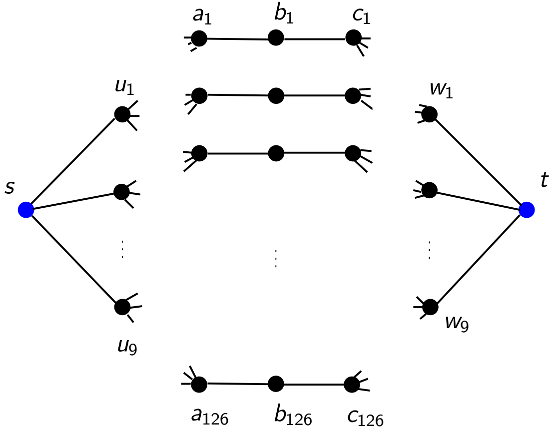


Symmetrically we make each $c_i$ adjacent to 4 vertices of $\{w_1, \ldots, w_9\}$ in such a way that $c_i$-s have distinct neighborhoods.

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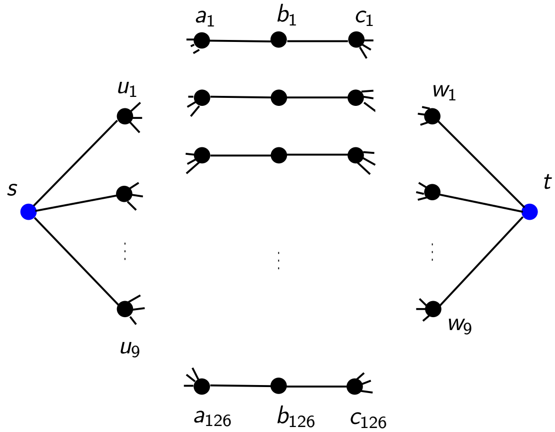


**Claim:** This graph has $> 2.4603 \cdot 10^{63}$ minimal $(s, t)$-separators.

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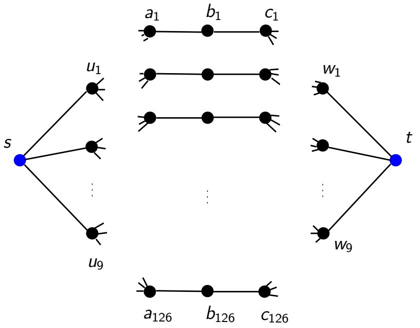


For $0 \leq p, q \leq 9$, let $N_{s,t}$ be the number of minimal $(s,t)$-separators that have $p$ vertices from $\{u_1, \ldots, u_9\}$ and $q$ vertices from $\{w_1, \ldots, w_9\}$.

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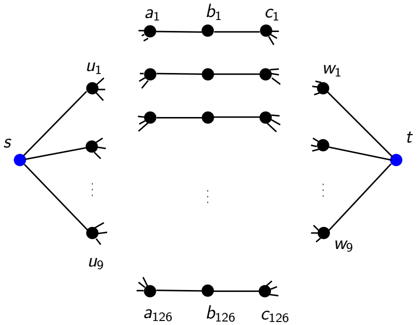


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Consider the cases (i) $p, q = 0$, (ii) $p = 0$ and $q \geq 4$, (iii) $p \geq 4$ and $q = 0$, (iv) $p, q \geq 4$ and lower bound $|S_{p,q}|$.

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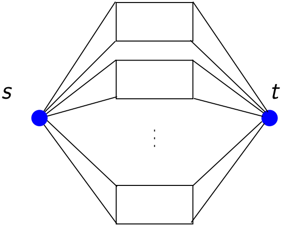

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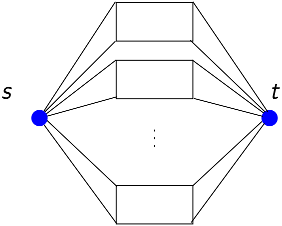


This graph has at least $1.4457^n$ minimal $(s, t)$-separators.

# Input-sensitive enumeration

- The running time depends on the length of the input only (e.g., the number of vertices of the input graph).
- We use the classical worst case running time analysis.
- If the number of objects to be enumerated is *exponential* (in the worst case), then an input-sensitive enumeration algorithm runs in *exponential* time.
- We use *exact exponential-time algorithms*, in particular *branching algorithms*.

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- Combinations of distinct techniques.

# Schöning's algorithm for 3-Satisfiability

**Problem (3-Satisfiability)**

**Input:** *A Boolean formula $\phi$ with $n$ variables in the conjunctive normal form such that each clause contain 3 literals.*

**Task:** *Decide whether $\phi$ has a satisfying assignment of variables.*

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$$\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3)$$

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The algorithm runs in time $O^*(1.5^n)$.

# Parameterized complexity

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A parameterized problem is *fixed-parameter tractable (FPT)* if it can be solved in time

$$f(k) \cdot n^{O(1)}.$$

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**Problem ($P$-Extension)**

**Input:** *A graph $G$, $U \subseteq V(G)$, and a non-negative integer $k$.*
**Parameter:** $k$
**Task:** *Decide whether there is a set $X \subseteq V(G) \setminus U$ of size at most $k$ such that $U \cup X$ satisfies $P$.*

# Exact algorithm via Local Search

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**Theorem**

*If there exists an algorithm for P-Extension with running time $c^k n^{O(1)}$, then there exists a randomized algorithm for P-Subset with running time $O^*((2 - \frac{1}{c})^n)$.*

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*If there exists an algorithm for P-Extension with running time $c^k n^{O(1)}$, then there exists a deterministic algorithm for P-Subset with running time $O^*((2 - \frac{1}{c})^{n+o(n)})$.*

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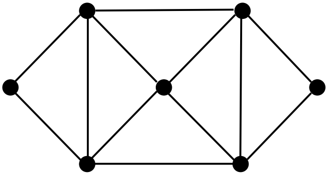

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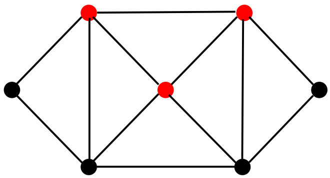

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Lower bound: There are $n = 10k$-vertex graphs with at least $105^{n/10}$ ($1.5926^n$) minimal feedback vertex sets.

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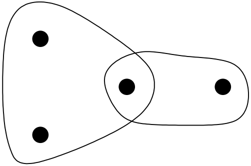

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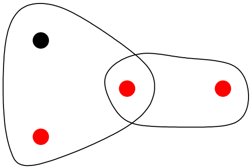

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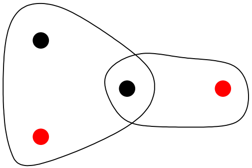

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We proved that if $\mathcal{S}$ contains sets of size at most 3, then $\mathcal{S}$ has at most $1.8394^n$ minimal hitting sets and these sets can be enumerated it time $O(1.8394^n)$ where $n = |\mathcal{U}|$.

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**Task:** Develop enumeration techniques for sets defined by *non-local* properties.

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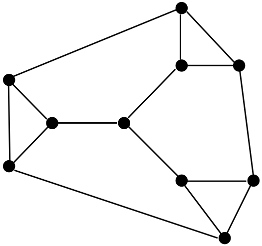

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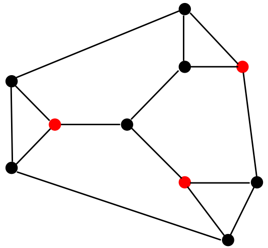

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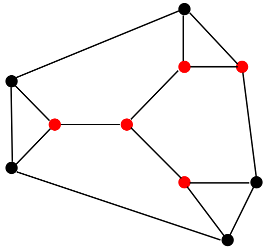

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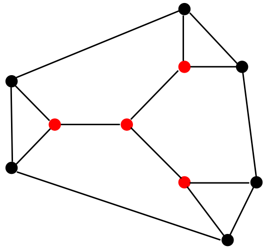

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## Problem (Enumeration of Minimal CDS)

**Input:** *A connected graph $G$.*

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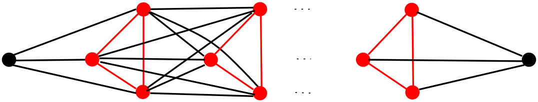
There are graphs with at least $3^{(n-2)/3}$ minimal CDS:



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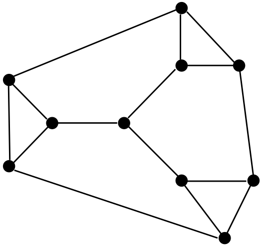

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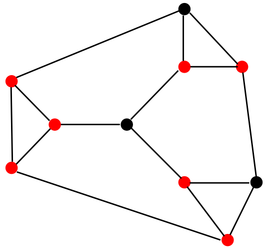

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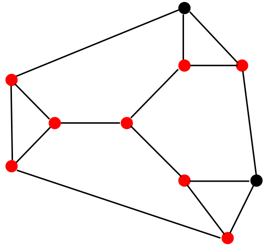

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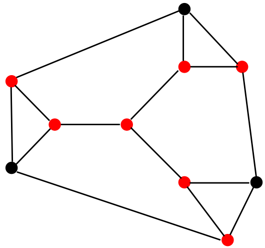

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**Problem (Enumeration of Minimal CVC)**

**Input:** *A connected graph $G$.*

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An $n$-vertex graph has at most $2 \cdot 1.7076^n$ connected vertex covers an these sets can be enumerated in time $O^*(1.7076^n)$ (Wingsternes, 2018).

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There are graphs with at least $1.5197^n$ minimal connected vertex covers (Ryland, 2018).

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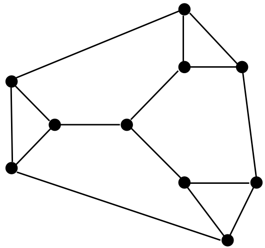

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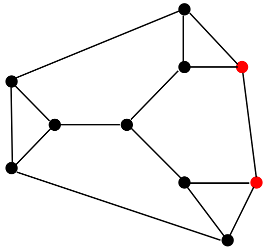

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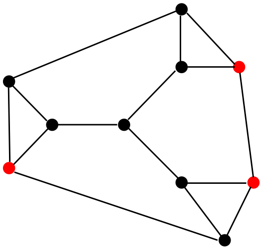

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Every minimal dominating set is a maximal irredundant set but not the other way around.

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**Problem (Enumeration of Maximal IS)**

**Input:** *A graph $G$.*

**Task:** *Enumerate all maximal irredundant sets.*

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There are graphs with at least $10^{n/5}$ maximal irredundant sets: