Ideas from exact and approximative classic and recent Combinatorial Optimization: Matchings, Edge-colorings and the TSP

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Part A : MATCHING

1. Basics

2. Edge-colorings Tashkinov (2000) for me: dec. 2017

3. Algorithms

Method of variables, class RP

Exact Matchings (Yuster, 2012), for me: June 2018

Edmonds' algorithm

4. Undirected shortest paths

Conservativeness, T-joins, Algorithms

5. Polyhedra, weights, Linear Programming

Approximation: **additive error of 1** for edge-coloring and exact matching

Randomized algorithms, LP lower bound

Exercises : series 1-5

Part B : TSP

1. Classical

s=t, General metric

2. Two-edge-connected spanning subgraph

ear theorems `graph TSP' , s=t (S., Vygen) 2014

Submodular functions, matroids

matroid intersection and approx. of submod max

3. General s,t path TSP

Zenklusen's 3/2 approx algorithm (April 2018)

Exercices series 6 Approximation : constant ratio

1. Basics

Matching

G=(V,E) graph.

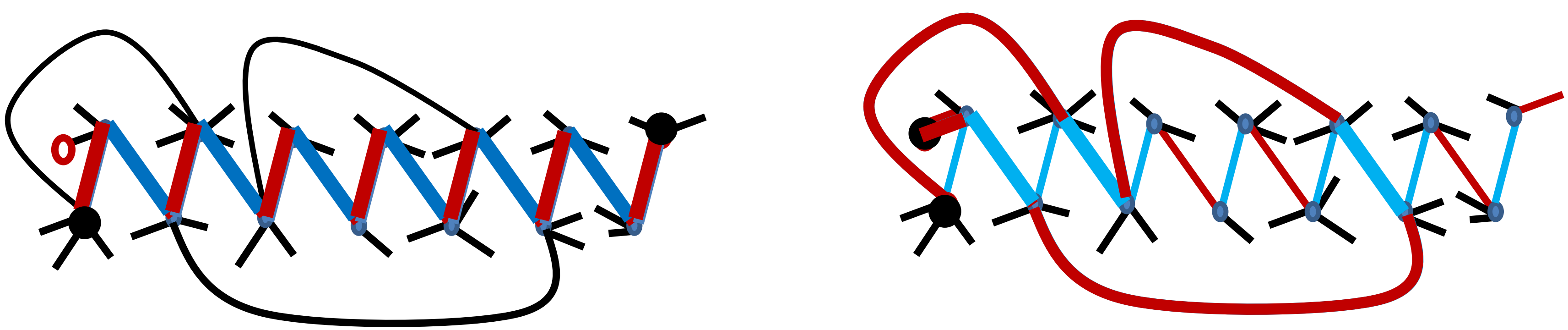
matching : a set M $\subseteq$ E of vertex-disjoint edges.

perfect matching : In addition M partitions V.

INPUT : G=(V,E) graph.

TASK : Find a matching of maximum size

Do the red edges form a maximum matching ?



Augmenting Paths

augmenting path with respect to matching M : path alternating between M and E \ M with the 2 endpoints uncovered by M.

Proposition (Berge) : G graph, M matching in G.
M is a maximum matching in G iff there is no augmenting path

Matching polytope: = conv (χ^M : M matching)

$$\chi_M(e) = \begin{cases} 1 \; if \; e \in M \\ 0 \; if \; e \notin M \end{cases}$$

Perfect matching polytope: = conv (χ^M : M perfect matching)

Interpretation with random sampling

$x = \sum_{M \in \mathcal{M}} \lambda_M \chi_M$, $(\lambda_M \geq 0, \sum_{M \in \mathcal{M}} \lambda_M = 1)$, $\mathcal{M}$ a set of p.m.

$\mathcal{M}$ can be viewed as a p.m. valued random variable

$$\Pr(\mathcal{M} = M) = \lambda_M$$

Then for $e \in E$: $\Pr(e \in \mathcal{M}) = x(e)$

$$E[\mathcal{M}] = x$$

Particular distributions (max entropy, or comb. restrictions)

Our use is notational, mainly: $E[\mathcal{F} + \mathcal{J}] = E[\mathcal{F}] + E[\mathcal{J}]$

Matching and vertex cover

matching : M set of vertex-disjoint edges

Max |M| : υ

vertex cover : T set of vertices so that G-T has no edges

Min |T| : τ

$\upsilon \leq \tau$

Min max

Theorem (Kőnig) : If $G=(V,E)$ is bipartite, then $\upsilon(G)= \tau(G)$

Exercise 1.2

Proof: ≤ is the proven 'easy part'; ≥ is to be proved:

If for some $v \in V$: $\nu(G - v) = \nu(G) - 1$, by induction :

$$\nu(G) = \nu(G - v) + 1 = \tau(G - v) + 1 \geq \tau(G) .$$

If $uv \in E$ then either u or v satisfy this condition !

Exercise 1.1

Q.E.D.

2. Edge-coloring

Def : G=(V,E). *edge-coloring* : each color is a matching

Edge-chromatic number = chromatic index = χ' :=min n. of colors

Bipartite edge-coloring

Theorem (Kőnig) : If G=(V,E) is bipartite, then $\chi'(G) = \Delta(G)$

Exercise 2.2

Proof: Now ≥ is now the 'easy part'; ≤ is to be proved:

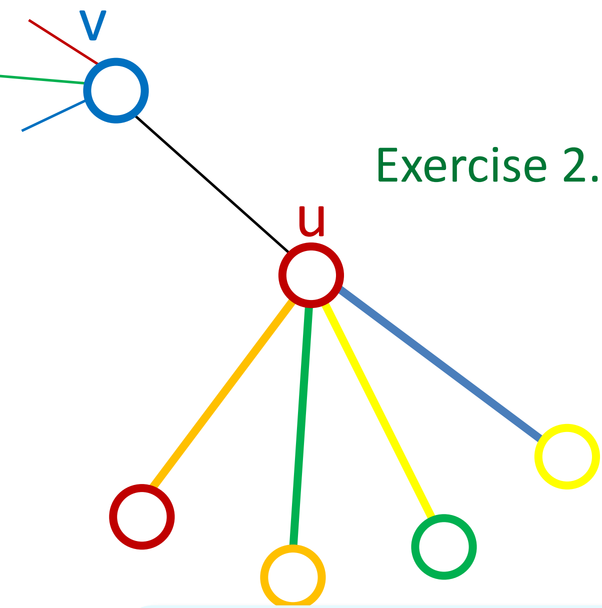
If uv $\in$ E is not yet colored then u , v both miss some color !

If it is the same color or can be recolored so : DONE

If not, they are joined by an even path:

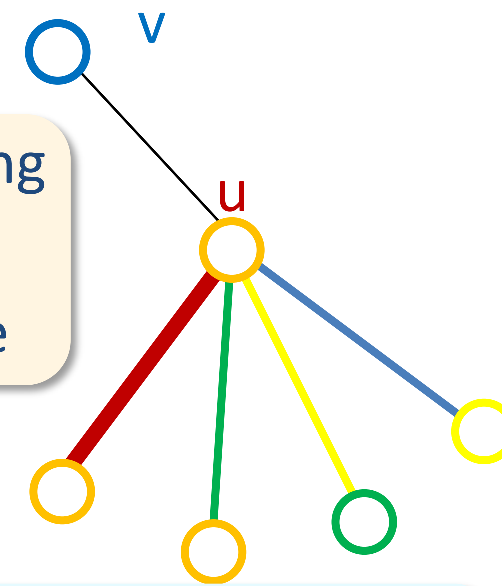
Q.E.D.

Tashkinov tree 1



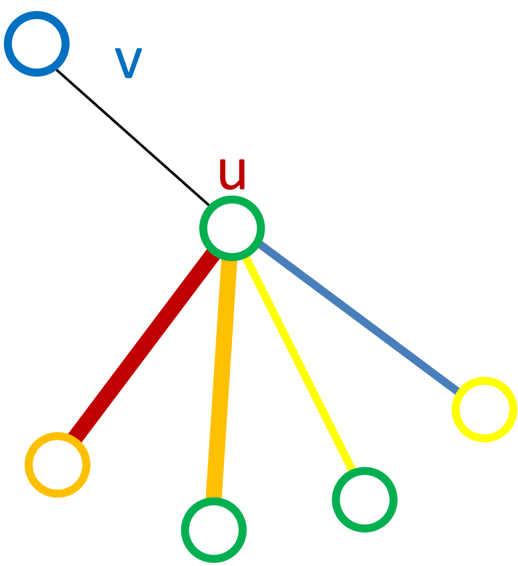
Exercise 2.3

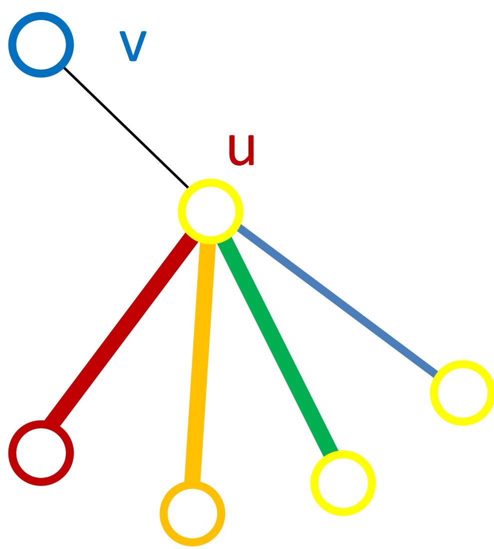
BFS tree F from {u,v,uv} using only edges whose color has already been missed before

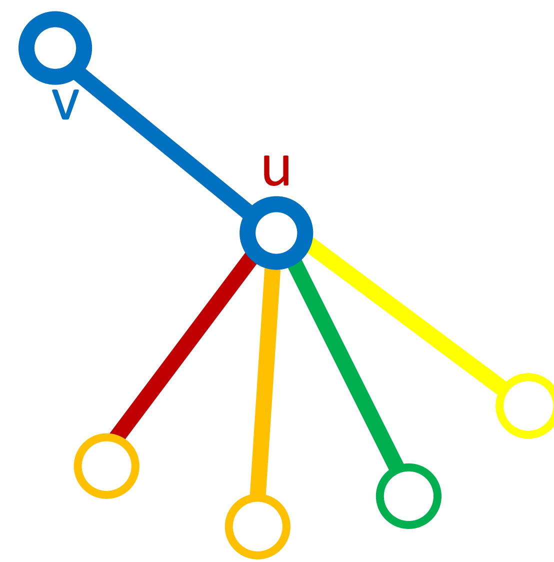


a. There is a common missing color in u and N(u) on F

$\Rightarrow$ uv can be colored





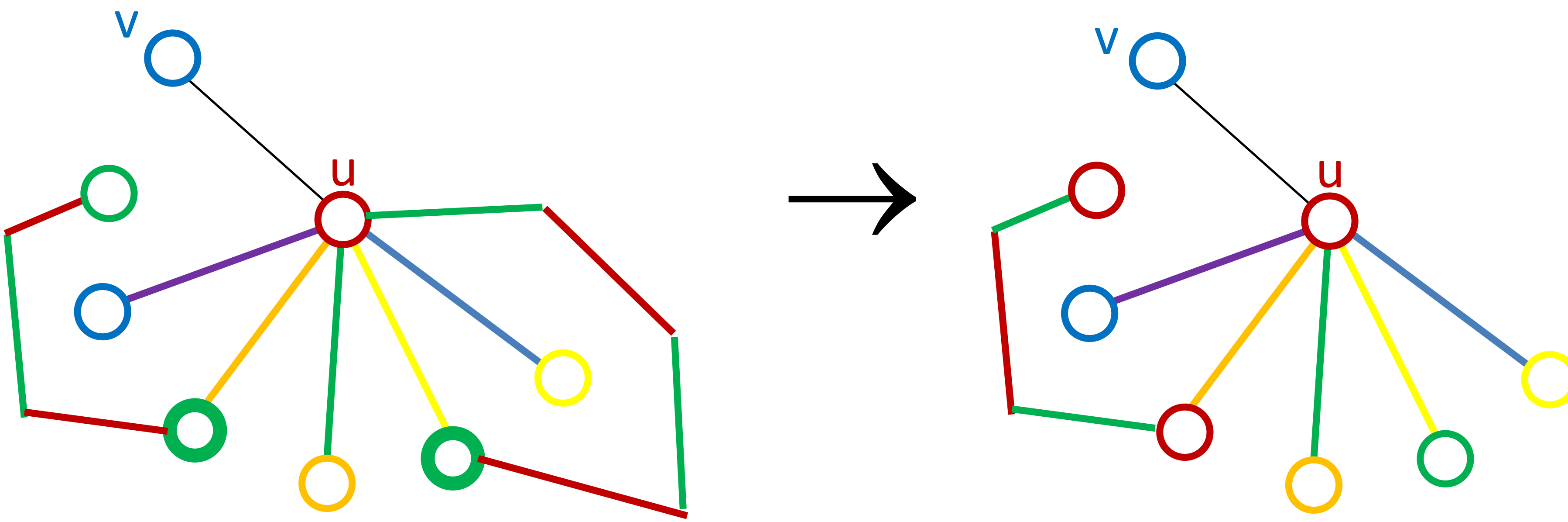


Tashkinov tree 2

b. There are two neighbors of u in F missing the same color

$\Rightarrow$ uv can be colored

swap a red-green component



Reduced to Case a.

Vizing's theorem

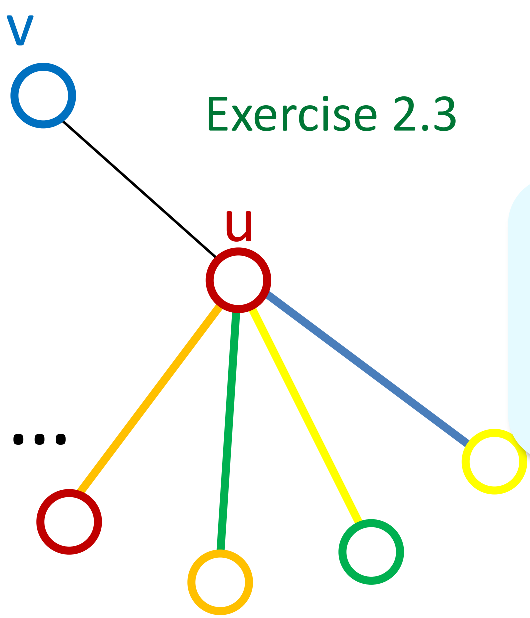
Theorem: If G=(V,E) is simple, then $\chi(G) \leq \Delta(G) + 1$

Exercise 2.4

Proof: Color as much as you can with $\Delta(G) + 1$ colors.

Then : every vertex has always a missing color !

Exercise 2.3



either a. or b. :

a. There is a common missing color in u and N(u) on F

b. There are two neighbors of u in F missing the same color

either $\Rightarrow$ uv can be colored

Tashkinov’s theorem

Generalizing the above proof :

Theorem: If such a BFS tree has two vertices missing the same color, then all colored edges + edge uv can be colored .

Corollaries: Better and better edge-coloring

3. Algorithms

Method of variables, class RP

Exact Matchings

Edmonds'algorithm

The method of variables (Tutte, Lovász, Geelen,…)

$G = (A, B, E)$ bipartite, $|A|=|B|$. $M := (x_{ij}$ if $ij \in E$, else $0)_{n\times n}$:

Proposition : det(M) is a nonzero polynomial ⇔ ∃ perfect matching

Proof : All terms of M are different, so there is no cancellation.

n! Terms, but determinants can be computed in polynomial time :
randomized algorithm: substitute values and then compute !

Questions : If then the det is nonzero can we conclude ?
If it is zero ?
What to do for nonbipartite graphs ?

The method of variables

The probability of error, precisely

Lemma: (Schwartz, Zippel) Let q be a nonzero polynomial of n variables $x_1,\dots, x_n$, and let it be of degree d ; $S \subseteq$ IN is finite, $s:=|S|$. Moreover, let $X_1,\dots, X_n$ be random variables taken independently and uniformly from S.

Then $\Pr(q(X_1,\dots, X_n)=0) \leq d/s$.

Proof: For n=1 obvious. Let $p \in Q[x_1,\dots, x_{n-1}]$ the coefficient of the highest power μ of x_n , and let π be the degree of p.

$$\Pr(q(X_1,\dots, X_n)=0) \leq \Pr(p(X_1,\dots, X_{n-1})=0) + \Pr(q(X_1,\dots, X_n)=0 \mid p(X_1,\dots, X_{n-1}) \neq 0)$$

$$\leq \quad \pi/s \quad + \quad \mu/s \quad \leq d/s$$

The method of variables

A Randomized Algorithm

Oracle Algorithm :

An oracle tells the substitution values of a polynomial in pol(deg) time.

1. Let $S = \{1,\ldots,2n\}$.
2. Make independent uniform choices in S for each variable.
3. Compute the polynomial (oracle call) for the chosen values.

   If $\neq 0$: the polynomial is nonzero ($\exists$ perfect matching)

   If $=0$? We decide: no perfect matching: Pr (error) $\leq$ ½

Why not bigger S ? Better to choose |S| = const x deg and repeat !

Proposition : After O(log $1/\varepsilon$) repetitions Pr (error) $\leq \varepsilon$

The complexity class P $\subseteq$ RP $\subseteq$ NP

Σ alphabet

$L \subseteq \Sigma^*$

$L \in NP \iff \exists R_L : \Sigma^* \times \Sigma^* \to \{0,1\}$

$x \notin L : \quad R(x,y) = 0 \quad \forall y \in \Sigma^*$

$x \in L : \exists y \in \Sigma^* : R(x,y) = 1$

Imagine : x= a graph, y the certificate (eg a substitution with ≠0 polynomial value)

$L \in RP \iff$ —"— $\quad x \to Y_x$

—"— —"—

$x \in L : \quad \{y \in \Sigma^* : R(x,y) = 1\} \geq \frac{|Y_x|}{2}$

The same def as NP but there are many certificates : constant proportion

Randomized algorithms for matching generalizations

RP thought of ≅ P

G = (A, B, E) **bipartite**, n= |A|=|B|. M := $(x_{ij}$ if ij ∈ E, else 0 $)_{n\times n}$

G = (V, E), n= |V| skew symm M := $(x_{ij} = -x_{ji}$ if ij ∈ E, else 0 $)_{n\times n}$

`Tutte matrix'** : square of the `Pfaffian'. **Good for testing !

Path matchings (Cunningham, Geelen)

*Exact matching***:** Given R⊆E, k∈IN, a max matching M, |M∩R|=k.

∃ Exact matching ⇔ multiplying x_{ij} for ij ∈R by y in the Tutte matrix, the coeff of y^k is not the 0 pol: ∀substitution this pol can be evaluated with n+1 x n+1 lin equ; ∈ RP (Lovász 1979) and not known to be in P !

Approximation for exact matchings

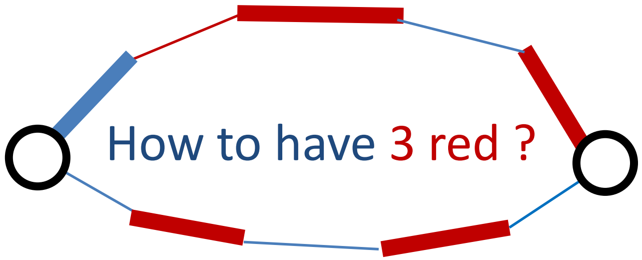
Theorem : (Yuster 2012) $G=(V,E)$ graph, $R \subseteq E$, $k \in \mathbb{N}$, then
Exactly one of the following possibilities holds :

(i) Each maximum matching of G meets R in $<$ k edges.
(ii) Each maximum matching of G meets R in $>$ k edges.

(iii) There exists a matching **of size at least $\upsilon(G) - 1$**
that meets R in $=$ k edges

Remark: (i), (ii) certified, checked in polytime (weighted match.)

Proof: If neither (i) nor (ii) holds,
Then M_1 : R-min max matching
$\mathbf{M_2}$: R-max max matching



Additive error of 1

Tutte-Berge theorem

Theorem : Let G=(V,E) be a graph. Then the minimum, over all matchings M of the number of uncovered vertices of V =

$$\max \{ q(X) - |X| \; : \; X \subseteq V \}$$

Def : q(X) is the n. of comps of G-X having an odd number of vertices

Proof : $\geq$: easy.

$\leq$: We can adapt the proof of Kőnig's theorem: Exercise 1.4

- If $\nu(G - v) = \nu(G) - 1$, induction is easy.

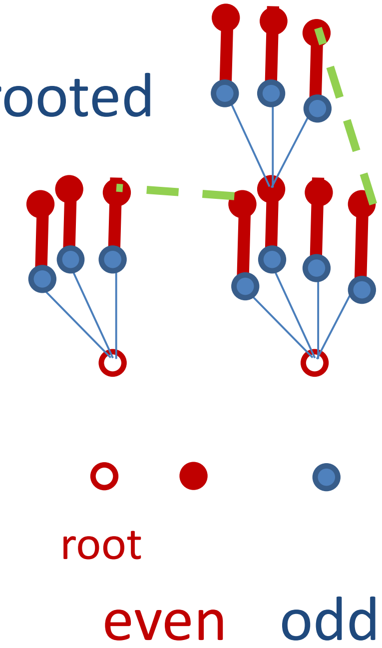
- If $\nu(G - u) = \nu(G - v) = \nu(G)$, apply Exercises 1.1, 1.3, 1.5 extended.

Hint : Observe that the new vertex is unconvered by a (actually two) maximum matching of the contracted graph. Can it be in X ? See Exercise 1.5.

Q.E.D.

Edmonds' algorithm

1. Grow an (inclusionwise max) alternating forest F rooted in uncovered vertices



2. If two even vertices are adjacent
   a.) between 2 different components : augment
   b.) in the same component :
       Adapt Exercise 1.3 to this case.
       Heureka you shrink ! (Edmonds)
   In both cases **GOTO 1** (possibly using the actual forest).

3. If there is no edge between the even vertices **STOP**
   X:= odd vertices

Theorem : X is a Tutte-set and F is a maximum matching

Summary of algorithms for matchings

Unweighted :

- Algorithms for bipartite graphs: paths in digraphs;
- Method of variables
- Edmonds' algorithm;
- Structural algorithms (for matchings by Lovász, S.:T-joins, b-match)

Weighted : Mainly two possibilities

- Primal-Dual framework with max cardinality subroutine
- Ellipsoid method