4. Conservative weightings
Undirected shortest paths
T-joins
Paths in Graphs

Directed, nonnegative weights (Dijkstra)
-1 weights NP-hard (HAM)
Conservative (no circuit of neg total weight): $\in P$

Undirected shortest paths with nonnegative weights?
With -1 weights?
With a conservative weighting?

Exercise: Does the triangle inequality hold in the undirected case? Are subpaths of shortest paths shortest?
Can we solve undirected shortest path problems in the same way as directed ones? Or reduce one to the other?
Conservativeness

**Def:** \((G, w)\) where \(G\) is a graph, \(w : E(G) \to \mathbb{Z}\) is conservative, if for every circuit \(C\) of \(G\) : \(w(C) \geq 0\).

\[
\lambda(x, y) : = \lambda_w(x, y) : = \min \{w(P) : P \text{ path} \} = ?
\]

\[
\lambda(a, b) = \lambda(a, c) = -1 \quad ; \quad \lambda(b, c) = -2 \quad ;
\]

\[
\lambda(a, b) + \lambda(b, c) < \lambda(a, c)
\]

A shortest \((a, c)\)-path is not shortest between \(a\) and \(b\).

Exercise 3.4

Recursively with `Matrix Multiplication`?  
Bellman-Ford?  Floyd-Warshall?  

**NO!**
T-joins

Euler’s theorem: \(G = (V, E)\), \(E\) : streets
One can go through all the streets
Exactly once \(\Leftrightarrow G\) conn., \(\forall\) degree even

Easy facts about T-joins: \(G\) connected, \(|T|\) even \(\Rightarrow \exists T\)-join; Exercise 3.1

$\text{min weight «Eulerian replication»} = \text{duplication of a min weight } T_G\text{-join}$

\(G = (V, E), w : E \rightarrow IR, F\) is a minimum weight T-join \(\Leftrightarrow (G, w[F])\) is conservative, where 

\[w[F](e) := \begin{cases} 
-1 & \text{if } e \in F \\
1 & \text{if } e \notin F 
\end{cases}\]

Is it true: \(\lambda(x,y) := \lambda_w(x,y) := \min \{w(P) : P \{x,y\}-join\}\) ?
**A Quick Proof of Seymour’s theorem**

**Theorem:** $G$ bipartite, $w: E(G) \rightarrow \{-1,1\}$, $(G,w)$ conservative $\iff$ $E_-$ can be covered by disj cuts meeting it in exactly one edge each.

**Proof:**

$x_0 \in V(G)$  

S. : `Quick Proof', & ...  

Take $b \neq x_0$ such that  

Claim 1:  

\[ | \delta(b) \cap E_- | = 1 \]  

Claim 2: Swapping on a circuit $C$, $w(C)=0$:

\[ w[C] \text{ is conservative} \]

Claim 3: Contracting $\delta(b)$ deleting loops, cons. is kept!
**T-cuts**

Def: \( \delta(W) \subseteq E(G) \) \((W \subseteq V)\) is a **T-cut**, if \(|W \cap T|\) is odd

**Proposition:** \( F \) **T-join**, \( \delta(W) \) **T-cut** \( \Rightarrow \) \(| F \cap \delta(W) | \geq 1 \)

\[
\tau(G,T) := \min \{ |F| : F \subseteq E, F \text{ is a T-join} \}
\]

\[
\nu(G,T) := \max \{ |C| : C \text{ disjoint T-cuts} \}
\]

**Easy:** \( \tau(G,T) \geq \nu(G,T) \)

**Theorem (Seymour ‘81):** If \( G \) is bipartite, \( \tau(G,T) = \nu(G,T) \)

Exercise 5.1
Nonbipartite minmax

\[ \nu_2(G,T) := \operatorname{max} \{ |E| : E \text{ 2-packing of } T\text{-cuts} \} \], where a 2-packing is a family covering every element \( \leq \) twice

**Easy** : \( \tau(G,T) \geq \frac{\nu_2(G,T)}{2} \)

**Proof** : Let \( F \) be a T-join, and \( E \) a 2-packing of T-cuts.

Then

\[ 2 \tau(G,T) = 2 |F| \geq \sum_{C \text{ in } E} |F \cap C| \geq |E| = \nu_2(G,T) \]

On two minmax theorems in graph

**Theorem** (Lovász ‘76) If G is arbitrary : \( \tau(G,T) = \frac{\nu_2(G,T)}{2} \)

**Theorem** (Edmonds-Johnson ‘73) \( G=(V,E) \)

\[ \tau(G,T) = \nu^*(G,T) \]
Polynomial algorithm for the postman

Input: \( G=(V,E), \ w: E \rightarrow \text{IR} \)
Task: minimize the weight of a T-join

 Proposition (Edmonds): If the weights are nonnegative easy reduction:
\[ \min \text{ weight matching of the complete graph on } T \text{ where the weights are the } w\text{-shortest paths in } G \text{ between the vertices of } T. \]

Can we find a negative circuit and shortest paths in undirected graphs?

Can we reduce the augmenting paths for matchings to this?
6. Linear Programming (LP)
LP for bipartite matchings

MATCHING POLYTOPE for G=(V,E) bipartite
\[
x \in \mathbb{IR}^E : \\
x (\delta(v)) \leq 1 , \ \forall \ v \in V \\
x \geq 0
\]

Dual for the all 1 objective function:

VERTEX COVER for G=(V,E) bipartite
\[
x \in \mathbb{IR}^V : \\
x_i + x_j \geq 1 , \ \forall \ ij \in E \\
x \geq 0
\]

Proof: TDI, TU+Cramer, or comb. no odd circuit)
6.1 Fractional chromatic index

$\mathcal{M}$: set of all matchings

*Fractional chromatic index* := $\chi^* = \min \sum_{M \text{ in } \mathcal{M}} y_M$, $y_M \geq 0$

$\sum_{M \text{ in } \mathcal{M} \text{ contains } e} y_M \geq 1$ (or $\geq w(e)$ where $w$ is non-neg edge-weights)

$\chi^* = \chi^' (G,w) = \min \lambda : w/\lambda \in \text{ matching polytope}$

$\chi^' : \text{ in addition } \lambda \text{ integer and } w = \text{ integer comb of } \mathcal{M}$

What is $\chi^*$ for bipartite matchings?
Minmax and computation of \( \chi' \)

Fractional Chromatic Index for **bipartite graphs**?

At least \( \triangle \) for all graphs so \( \text{so} = \text{for bip;1/\triangle} \) on all edges \( \in \text{polytope} \)

For **general graphs**? Min \( \lambda : \ w / \lambda \in \text{matching polytope} \)

Edmonds (1965) \( x \in \mathbb{IR}^E : \ x (\delta(v)) \leq 1, \ x \geq 0 \)

\[ x (E(U)) \leq \frac{|U| - 1}{2} \quad \text{U} \subseteq V , |U| \text{ odd} \]

\( \lambda \geq \Delta , \ \lambda \geq \frac{2w(E(U))}{|U| - 1} , \quad \text{“} \quad \text{“} \)

**Polynomial algorithm**! Compare with average degree \( \frac{2w(E(U))}{|U|} \)

How does it compare if all weights are 1 (simple graphs)?
Nonbipartite matching polytope

The Perfect Matching Polytope: Kőnig (1916), Jacobi (1890)
Egerváry (1931), Birkhoff (1946), von Neuman (1952): easier to prove

If G is bipartite:
\[
\text{conv } (\chi_M : M \text{ p.m.}) = \{x \in \mathbb{R}^E : x(\delta(v)) = 1, x \geq 0 \}
\]

If G is arbitrary:
Edmonds (1965), add: if \( U \subseteq V \), \(|U|\) is odd \( x(\delta(U)) \geq 1 \)

The linear inequalities of the Matching Polytope of \( G=(V,E) \):
Edmonds (1965) \( x \in \mathbb{R}^E : x(\delta(v)) \leq 1, x \geq 0 \)

\[ x(E(U)) \leq \frac{|U|-1}{2} \quad U \subseteq V, \quad |U| \text{ odd} \]
Conjectures about additive gap 0 or 1

$P(G)$ matching polytope, k integer, $w \in k M(G)$ integer.

**Conjecture (Lovász):** $G$ without Petersen minor $\chi' = \chi'*$ i.e.

$$w = M_1 + \ldots + M_k$$

**Conjectures (Schrijver):** t-perfect graphs ...

**Conjecture (Goldberg, Seymour):** $\text{MID} = \text{ID} + 1$

$x \in \lambda$ matching$(G) \implies x$ is $\left\lceil \lambda \right\rceil + 1$–colorable; tight: Petersen

**Conjecture (Aharoni):** matroid indep set are MID

**Conjecture (Scheithauer and Terno):** cutting stock (bin packing patterns) are MID.
6.2 How are LP, polyhedra useful for insight?

Lower bound because relaxation

Can be part of the solution algorithm

Example of another use ...:

A generalization of Petersen’s theorem
Petersen’s theorem (1891)

A graph is *cubic* if all of its degrees are 3.

**Theorem**: $G$ is a cubic graph

$G$ has no bridge $\Rightarrow G$ has a p.m.
Weighted generalization

**Exercise**: Let $G=(V,E)$ be cubic, $w: E \rightarrow \mathbb{R}$ on the edges. Then

a. If $G$ is **bipartite**, or

b. If $G$ is arbitrary **bridgeless**

There exists a p.m. of weight $\geq 1/3 \ w(E)$

Bridgeless, but cannot be partitioned to 3 p.m.
Through the polyhedral lens

If $G=(V,E)$ cubic, bipartite
The constant $1/3$ function on the edges is in the convex hull of matchings.

If $G=(V,E)$ cubic, bridgeless
The constant $1/3$ function on the edges is in the convex hull of matchings.

If $G$ is cubic, bridgeless (or bipartite),
$\exists$ matching valued random variable $M$
so that $E(M) = \text{constant } 1/3$ on $E$.

**Theorem:** $G=(V,E)$ cubic, bridgeless (or bipartite),
$w : E \to \mathbb{R}$. $\exists$ matching $M$, $w(M) \geq w(E)/3$
Method: the inverse of the duality theorem

**Theorem** Edmonds, Johnson (1973): $Q_+(G,T) := \text{conv} (\text{T-joins}) + \mathbb{R}_+^n = \\
\{x \in \mathbb{R}_+^E \mid x(\delta(W)) \geq 1, \delta(W) \text{ is a T-cut, i.e. } |W \cap T| \text{ is odd}\}$

**Proof:** $\subseteq$: Clear!

For $=\,$ show $\forall c \in \mathbb{R}^S \min c^Tx \text{ for } x \text{ on the left } = \\
\min c^Tx \text{ for } x \text{ on the right}$

This suffices, since if not $\neq$, then $\subseteq$ and the hyperplane $c^Tx=b$ separating some $x$ on the right from all on the left ($\Rightarrow c \geq 0$ maybe changing the sign), shows that the min of $c^Tx$ is smaller on the right.

But min of $c^Tx$ on the right is equal, by the duality theorem to the max of its dual so the latter is smaller then the min of $c^Tx$ on the left, contradicting Edmonds and Johnson’s minimax theorem (Corollary of Seymour’s theorem):
Proving the T-join polyhedron Thm

Metatheorem: weighted minmax theorem $\iff$ polyhedron
( $\rho$-approximation for all weights $\iff$ $\rho$-polyhedron containment )

Q.E.D.

Edmonds-Johnson: $\tau(G,T,c) = \nu^*(G,T,c) := \text{fractional opt}$

Lovász (76): If $G$ arbitrary, $\tau(G,T) = \nu_2(G,T) / 2$

Seymour (81): If $G$ is bipartite, $\tau(G,T) = \nu(G,T)$
End of Part A: MATCHINGS

To come: TSP + a bit of submodularity, matroids

Exercises for the Courses 3-4: series 6