4. Conservative weightings
Undirected shortest paths
T-joins

Paths in Graphs

Directed, nonnegative weights (Dijkstra)

-1 weights NP-hard (HAM)

Conservative (no circuit of neg total weight): $\in$ P

Undirected shortest paths with nonnegative weights ?

With -1 weights ?

With a conservative weighting ?

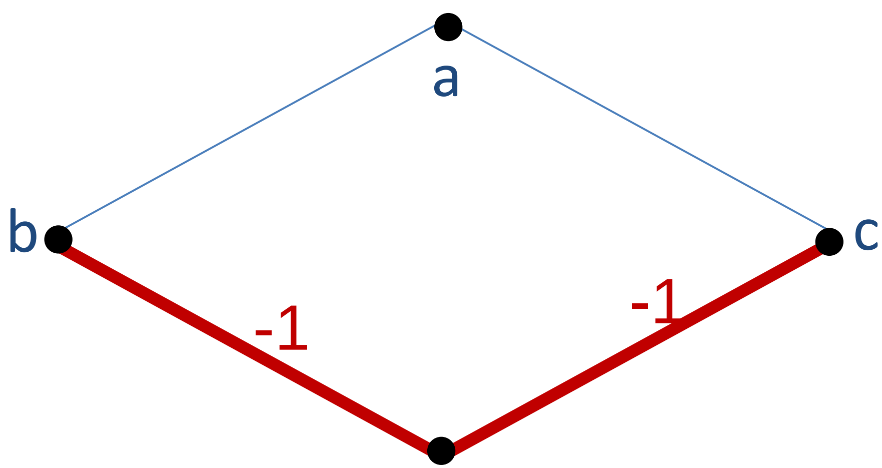
Exercise : Does the triangle inequality hold in the undirected case ? Are subpaths of shortest paths shortest ?

Can we solve undirected shortest path problems in the same way as directed ones ? Or reduce one to the other ?

Conservativeness

Def: (G,w) where G is a graph, w: E(G) $\rightarrow$ Z is *conservative*, if for every circuit C of G : w(C) $\geq$ 0 .

$\lambda(x,y) := \lambda_w(x,y) :=$ min {w(P) : P path } = ?



$\lambda(a,b) = \lambda(a,c) = -1$; $\lambda(b,c) = -2$;

$\lambda(a,b) + \lambda(b,c) < \lambda(a,c)$

A shortest (a,c)-path is not shortest between a and b.

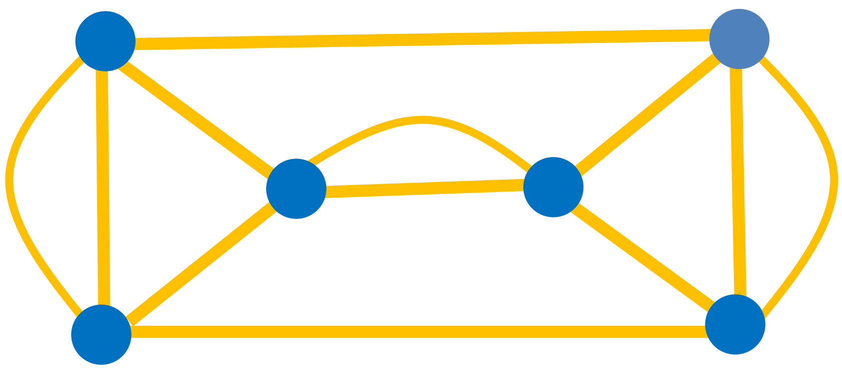
Exercise 3.4

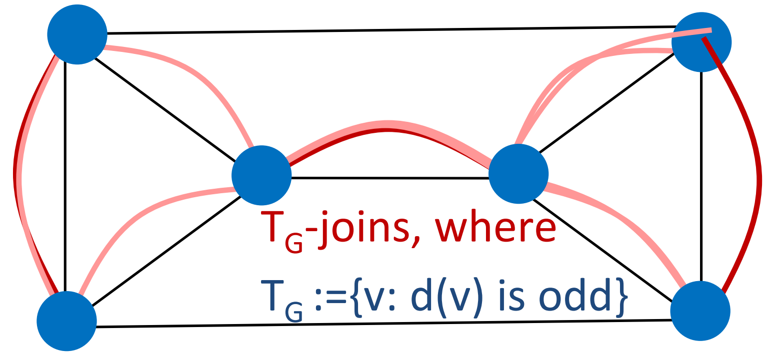
Recursively with `Matrix Multiplication' ?

Bellman-Ford ? Floyd-Warshall ?

NO !

T-joins





Euler's theorem : G= (V,E), E : streets
One can go through all the streets
Exactly once ⇔ G conn., ∀degree even

F ⊆ E(G) is a *T-join*, if

T = vertices of odd degree of F.

Easy facts about T-joins : G connected, |T| even ⇒ ∃ T-join ; Exercise 3.1

min weight «Eulerian replication» = duplication of a min weight T_G-join

G=(V,E), w: E → IR, F is a minimum weight T-join ⇔

(G, w[F]) is *conservative*, where $w[F](e) := \begin{cases} -1 \text{ if } e \in F \\ 1 \text{ if } e \notin F \end{cases}$

Is it true : $\lambda(x,y) := \lambda_w(x,y) :=$ min {w(P) : P {x,y}-join} ?

A Quick Proof of Seymour's theorem

Exercise 4.5

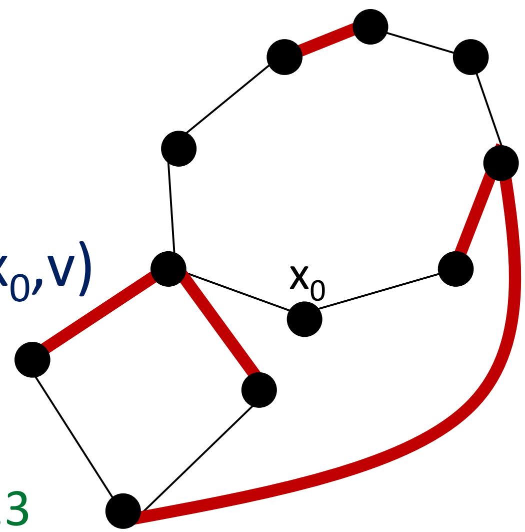
Theorem: G bipartite, $w:E(G) \to \{-1,1\}$, (G,w) conservative $\Leftrightarrow$ E_- can be covered by disj cuts meeting it in exactly one edge each.

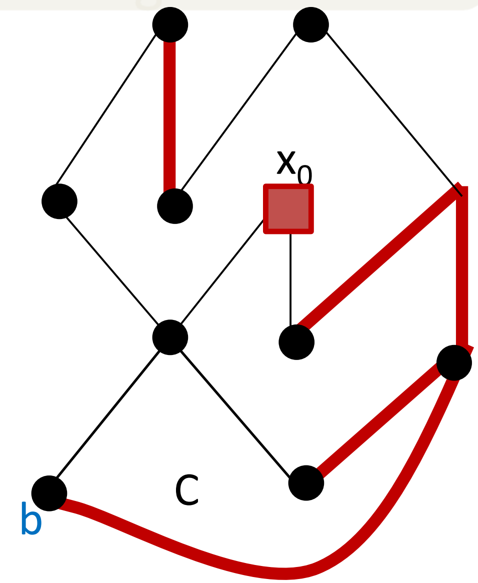
Proof : $x_0 \in V(G)$

S. : `Quick Proof', & ...

Take $b \neq x_0$ such that

$\lambda_w(x_0,b)=\min_{v\in V(G)} \lambda_w(x_0,v)$



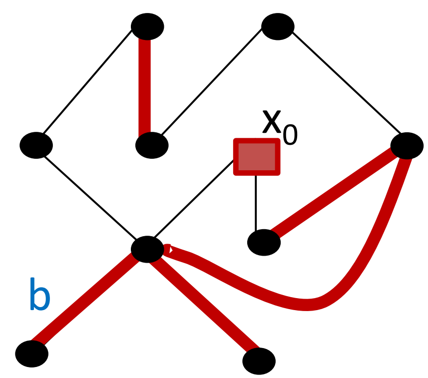


Claim 1 : $|\,\delta(b) \cap E_-\,| = 1$

Exercise 4.3

Claim 2 : Swapping on a circuit C, $w(C)=0$:

$w[C]$ is conservative



Claim 3 : Contracting $\delta(b)$ deleting loops, cons. is kept Exercise 4.4

!

T-cuts

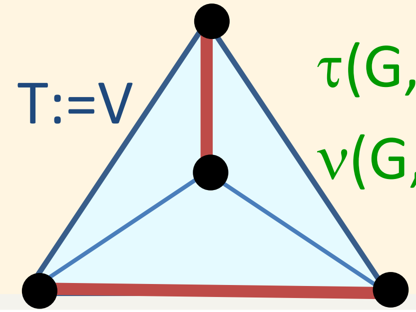
Def : $\delta(W) \subseteq E(G)$ $(W \subseteq V)$ is a *T-cut,* if $|W \cap T|$ is odd

Proposition : F T-join, $\delta(W)$ T-cut $\Rightarrow$ $|F \cap \delta(W)| \geq 1$

$\tau(G,T)$:= min { $|F|$: $F \subseteq E$, F is a T-join }

$\nu(G,T)$:= max{ $|\mathcal{C}|$: $\mathcal{C}$ disjoint T-cuts }

Easy : $\tau(G,T) \geq \nu(G,T)$



Exercise 5.1

Theorem (Seymour '81) If G is bipartite, $\tau(G,T) = \nu(G,T)$

Nonbipartite minmax

Exercise 5.3

$\nu_2(G,T)$:= max{ $|\mathcal{C}|$: $\mathcal{C}$ 2-packing of T-cuts }, where
a *2-packing* is a family covering every element ≤ twice

Easy : $\tau(G,T) \geq \nu_2(G,T)/2$

Proof : Let F be a T-join, and $\mathcal{C}$ a 2-packing of T-cuts.

Then $2\,\tau(G,T) = 2\,|F| \geq \sum_{C \text{ in } \mathcal{C}} |F \cap C| \geq |\mathcal{C}| = \nu_2(G,T)$

On two minmax theorems in graph

Theorem (Lovász '76) If G is arbitrary : $\tau(G,T) = \nu_2(G,T)/2$

Theorem (Edmonds-Johnson '73) G=(V,E)

$$\tau(G,T) = \nu^*(G,T)$$

Polynomial algorithm for the postman

Input : $G=(V,E)$, $w: E \rightarrow IR$

Task : minimize the weight of a T-join

Proposition (Edmonds) : If the weights are nonnegative easy reduction: min weight matching of the complete graph on T where the weights are the w-shortest paths in G between the vertices of T.

Can we find a negative circuit and shortest paths in undirected graphs ?

Can we reduce the augmenting paths for matchings to this ?

6. Linear Programming (LP)

LP for bipartite matchings

MATCHING POLYTOPE for G=(V,E) bipartite

$$x \in \mathbb{R}^E :$$
$$x(\delta(v)) \leq 1 , \forall v \in V$$
$$x \geq 0$$

Dual for the all 1 objective function:

VERTEX COVER for G=(V,E) bipartite

$$x \in \mathbb{R}^V :$$
$$x_i + x_j \geq 1 , \forall ij \in E$$
$$x \geq 0$$

Proof : **TDI** , TU+Cramer, or comb. no odd circuit)

6.1 Fractional chromatic index

$\mathcal{M}$: set of all matchings

fractional chromatic index := $\chi'^* = \text{Min} \sum_{M \in \mathcal{M}} y_M$, $y_M \geq 0$

$\sum_{M \in \mathcal{M} \text{ contains } e} y_M \geq 1$ (or $\geq w(e)$ where w is non-neg edge-weights)

$\chi'^* = \chi'(G,w) = \text{Min } \lambda \; : \; w / \lambda \in$ matching polytope

χ' : in addition λ integer and w = integer comb of $\mathcal{M}$

What is χ'^* for bipartite matchings ?

Minmax and computation of χ'^*

Fractional Chromatic Index for **bipartite graphs** ? Δ

At least Δ for all graphs so = for bip;$1/\Delta$ on all edges $\in$ polytope

For **general graphs** ? Min λ : $w/\lambda \in$ matching polytope

Edmonds (1965) $x \in \mathbb{R}^E$: $x(\delta(v)) \leq 1$, $x \geq 0$

$$x(E(U)) \leq \frac{|U|-1}{2} \qquad U \subseteq V,\ |U| \text{ odd}$$

$$\lambda \geq \Delta, \quad \lambda \geq \frac{2w(E(U))}{|U|-1}, \quad \text{“} \qquad \text{“}$$

Polynomial algorithm ! Compare with average degree $\frac{2w(E(U))}{|U|}$!

How does it compare if all weights are 1 (simple graphs) ?

Nonbipartite matching polytope

The Perfect Matching Polytope: Kőnig (1916), Jacobi (1890)

Egerváry (1931), Birkhoff (1946), von Neuman (1952): easier to prove

If G is bipartite :

conv (χ_M : M p.m.) = $\{x \in IR^E : x(\delta(v))=1, x \geq 0\}$

If G is arbitrary :

Edmonds (1965), add : if $U \subseteq V$, $|U|$ is odd $x(\delta(U)) \geq 1$

The linear inequalities of the Matching Polytope of G=(V,E):

Edmonds (1965) $x \in IR^E$: $x(\delta(v)) \leq 1$, $x \geq 0$

$$x(E(U)) \leq \frac{|U|-1}{2} \qquad U \subseteq V, |U| \text{ odd}$$

Conjectures about additive gap 0 or 1

P(G) matching polytope, k integer, $w \in k\, M(G)$ integer.

Conjecture (Lovász) : G without Petersen minor $\chi' = \chi'^*$ i.e.

$$w = M_1 + \ldots + M_k$$

Conjectures (Schrijver) : t-perfect graphs …

Conjecture (Goldberg, Seymour) : MID = ID +1

$x \in \lambda$ matching(G) $\Rightarrow$ x is $\lceil \lambda \rceil$+1–colorable; tight: Petersen

Conjecture (Aharoni): matroid indep set are MID

Conjecture (Scheithauer and Terno): cutting stock (bin packing patterns) are MID.

6.2 How are LP, polyhedra useful for insight ?

Lower bound because relaxation

Can be part of the solution algorithm

Example of another use … :

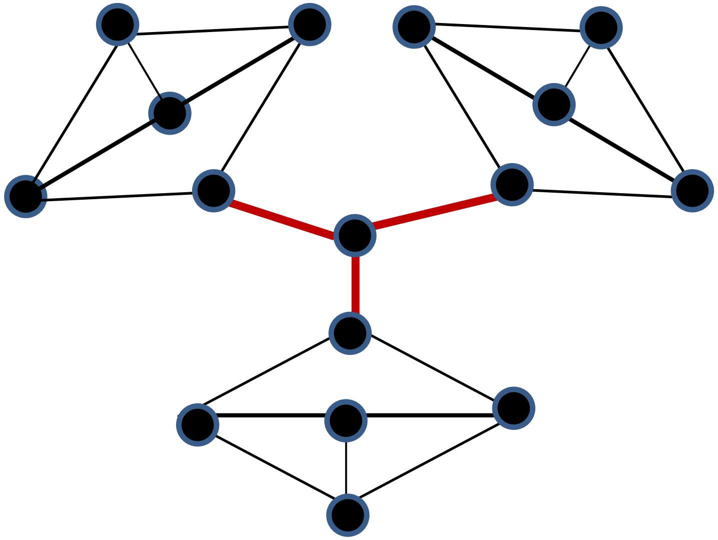
A generalization of Petersen's theorem

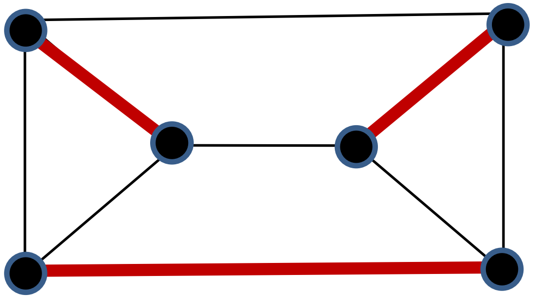
Petersen's theorem (1891)

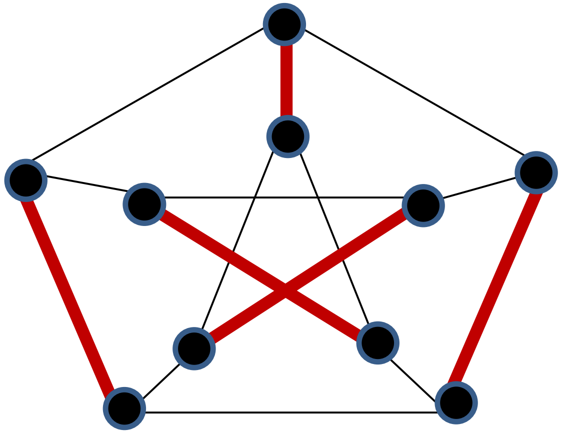
A graph is *cubic* if all of its degrees are 3.

Theorem: G is a cubic graph

G *has no bridge* $\Rightarrow$ G has a p.m.







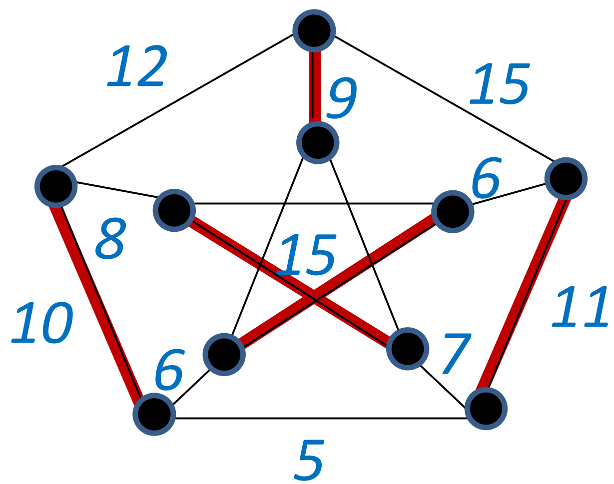
Weighted generalization

Exercise : Let G=(V,E) be cubic, w: E → IR on the edges. Then

a. If G is bipartite, or

b. If G is arbitrary bridgeless

There exists a p.m. of weight $\geq 1/3\ w(E)$



10 + 9 + 11 + 2x15

= **60** $\geq$ 1/3 w(E)

(w (E) = 179)

Bridgeless, but cannot be partitioned to 3 p.m.

Through the polyhedral lens

If G=(V,E) cubic, bipartite
The constant 1/3 function
on the edges is in the
convex hull of matchings.

If G=(V,E) cubic, bridgeless
The constant 1/3 function
on the edges is in the
convex hull of matchings.

If G is cubic, bridgeless (or bipartite),
$\exists$ matching valued random variable $\mathcal{M}$
so that $E(\mathcal{M})$ = constant 1/3 on E .

Theorem: G=(V,E) cubic, bridgeless (or bipartite),
w: E $\to$ IR. $\exists$ matching M, $w(M) \geq w(E)/3$

6.3 The T-join polyhedron

Method: the inverse of the duality theorem

Theorem Edmonds,Johnson (1973) : $Q_+(G,T) :=$ conv (T-joins) + IR_+^n =

$\{x \in IR_+^E \ \ x(\delta(W)) \geq 1, \delta(W)$ is a T-cut, i.e. $|W \cap T|$ is odd$\}$

Proof : $\subseteq$: Clear !

For = show $\forall\, c \in IR^S$ min c^Tx for x on the left =
min c^Tx for x on the right

This suffices , since if not =, then $\subset$ and the hyperplane $c^Tx=b$ separating **some** x on the right from **all** on the left (=> $c \geq 0$ maybe changing the sign), shows that the min of c^Tx is smaller on the right .

But min of c^Tx on the right is equal, by the duality theorem to the max of its dual so the latter is smaller then the min of c^Tx on the left, contradicting Edmonds and Johnson’s minimax theorem (Corollary of Seymour’s theorem):

Proving the T-join polyhedron Thm

Metatheorem : weighted minmax theorem ⇔ polyhedron

(ρ-approximation for all weights ⇔ ρ- polyhedron containment)

Q.E.D. ⇔

Edmonds-Johnson : $\tau(G,T,c)= \nu^*(G,T,c)$:= fractional opt

⇐

Lovász (76): If G arbitrary, $\tau(G,T) = \nu_2(G,T)/2$

⇐

Seymour (81): If G is bipartite, $\tau(G,T) = \nu(G,T)$

End of Part A: MATCHINGS

To come : TSP +

a bit of submodularity, matroids

Exercises for the Courses 3-4 : series 6