Part B : TSP

1. Classical TSP
   s=t, General metric

2. Graph metric
   ear theorems `graph TSP’, s=t (S., Vygen) 2014
   Submodular functions, matroids
   matroid intersection and approx. of submod max

3. General s,t path TSP
   Zenklusen’s 3/2 approx algorithm (April 2018)

Exercises series 6   Approximation : constant ratio
Optimal orders

s-t-Path Travelling Salesman Problem

INPUT : V «cities», s , t ∈ V, c: V×V → IR+ metric

OUTPUT: shortest s-t -Hamiltonian path

\[ P(V,s,t) = \{ x\in \mathbb{R}^E_+: x(\delta(W)) \geq 2, \emptyset \neq W \subseteq V, s, t \in W \text{ or } \in \}, \]  
\[ \text{min } c^Tx = \text{ on vertices (1 for } s, t \text{ ; else 2 )} \]

\[ \text{OPT}(c) \]

\[ \text{OPT}_{LP}(c) \]

Metric: triangle inequality, satisfied by reasonable applications, without it: even approx is hard
Approximation and Integrality ratio

For a minimization problem

- the approximation ratio is at most $\rho$ if for any input a solution of value at most $\rho \cdot \text{OPT}$ can be found in polynomial time.

- the integrality gap is at most $\rho$ if for any input $\text{OPT} / \text{OPT}_{LP} \leq \rho$.

Lower bound for approximation ratio: 123/122 NP-hard (Karpinski Lampis, Schmied)

Lower bound for integrality gap: graph metrics
1. Classical TSP
TSP

**INPUT**: $V$ cities, $c: V \times V \rightarrow \mathbb{IR}_+$ metric

**OUTPUT**: shortest Hamiltonian circuit

Without it no constant ratio (easy from HAM)

Christofides (1976)

**Determine**: a minimum weight spanning tree

**Add**: Add a minimum $T_F$ - join $J_F$ to make it Eulerian

**Shortcut** the Eulerian tour

NP-hard (Karp, 1972)
A proof of ratio 2 and two proofs of $\frac{3}{2}$

Approximation ratio 2: **Double** a min cost spanning tree $F$ and shortcut $J_F$.

Approximation ratio $\frac{3}{2}$: $F + J_F$, where $c(F) \leq \text{OPT}$, $c(J_F) \leq \frac{1}{2} \text{OPT}$, since connected, Eulerian $\Rightarrow$ has two disjoint $T$-joins for all $T$.

$\text{OPT}_{LP} := \{ \min c(x) : x \in \mathbb{R}^+_E, x(\delta(W)) \geq 2, \text{ for all } \emptyset \neq W \subset V, = \text{ for vertices} \}$

**Theorem** (Wolsey '80, Cunningham 1984) G = (V, E) graph.

We find at most $\frac{3}{2} \text{OPT}_{LP}$ since $c(F) \leq \text{OPT}_{LP}$, $c(J_F) \leq \frac{1}{2} \text{OPT}_{LP}$

2. Classical TSP with graph metric, and min size Two-edge-connected spanning subgraph
‘Network reliability’

2-Edge Connected Spanning Subgraph, 2ECSS

graph-TSP, graph-TSP paths

Def: A graph $G=(V,E)$ is 2-edge-connected, if $(V, E \setminus e)$ is connected for all $e \in E$.
Ears

The longer the ears, the smaller the quotient $n.\text{ of edges} / \text{vertices}$

$$G = P_0 + P_1 + P_2 + \ldots + P_k$$

2-approx for 2ECSS: delete 1-ears!

Exploited by Cheriyan, S., Szigeti (1998) for a $17/12$-approx
C = (S, ℱ) , ℱ ⊆ ℙ(S) is a matroid if
(i) ∅ ∈ ℱ
(ii) F ∈ ℱ , F' ⊆ F ⇒ F' ∈ ℱ
(iii) F₁, F₂ ∈ ℱ , |F₁| < |F₂| ⇒ ∃ e ∈ F₂ \ F₁ : F₁ ∪ {e} ∈ ℱ

F ∈ ℱ is called an independent set.

The rank function of M is
\( r : 2^S \rightarrow \mathbb{IN} \) defined as
\( r(X) := \max \{ |F| : F \subseteq X, F \in ℱ \} \)

Examples: Forests in graphs, Linearly independent sets, partition matr.
Matroid Intersection Theorem

\[ M = (S, F) \text{ matroid} \]

\[ \text{conv} \left( \chi_F : F \in \mathcal{F} \right) = \left\{ x \in \mathbb{R}^S : x(A) \leq r(A) \text{ for all } A \subseteq S \right\} \quad \text{(Edmonds)} \]

**maximize** \{ |F| : F \in F_1 \cap F_2 \} = ?

\[ \max \left\{ 1^T x : x(A) \leq r_i(A) \quad (i=1, 2) \text{ for all } A \subseteq S \right\} \]

**Theorem** (Edmonds 1979):

\[ \max |F| = \min_{F \in F_1 \cap F_2} r_1(X) + r_2(S \setminus X) \quad X \subseteq S \]

Polynomial algorithm for both and also if weights are given.
Matroid Intersection Algorithm
Generalization of bipartite matching
(of the alternating paths in the « Hungarian method »)

**Proof of** $\geq$: that is, there is $F$ and $X$ with $|F| = r_1(X) + r_2(S \setminus X)$.

We prove that the following algorithm terminates with such an $F$ and $X$.

What is the INPUT? $\Rightarrow$ **ORACLE** - rank, independence, etc

0.) Let: $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ maximal by inclusion (greedily)

1.) Define arcs from unique cycles:

- $C_1 \in \mathcal{C}_1$
- $C_2 \in \mathcal{C}_2$
Approx for submod max \( \text{mon, size } k, f(0)=0, \)

**Algorithm (for sets of size k):** (Nemhauser, Wolsey) Having \( X \) already,

\[
\text{WHILE } |X| < k \quad \text{choose } x \text{ that maximizes } f(X \cup \{x\}) - f(X)
\]

**Lemma:** \( f(X \cup \{x\}) - f(X) \geq \frac{f(OPT) - f(X)}{k} \)

**Proof:** Since \( \text{mon: } f(OPT) \leq f(OPT \cup X) \leq f(X) + k \left( f(X \cup \{x\}) - f(X) \right) \)

Let \( X^i \) be what we found until step \( i \). Then
\[
f(X^k) - f(X^{k-1}) \geq \frac{f(OPT)}{k} - \frac{f(X^{k-1})}{k}, \text{ so }
\]
\[
f(X^k) \geq \frac{f(OPT)}{k} + (1 - \frac{1}{k}) f(X^{k-1})
\]
\[
f(X^k) \geq f(OPT) \left( 1 - \left(1 - \frac{1}{k}\right)^k \right) \geq \left( 1 - \frac{1}{e} \right) f(OPT)
\]