3. General {s,t} path TSP

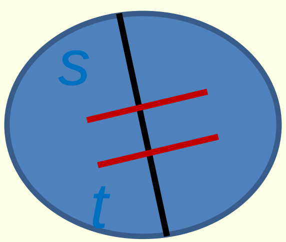
The {s,t}-path TSP

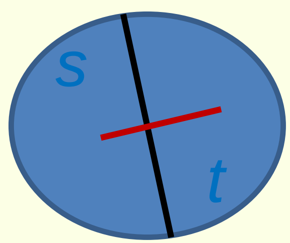
PATH TSP
INPUT : V cities, $s, t \in V$, $c: V \times V \rightarrow \mathbb{R}_+$ metric
OUTPUT: shortest s, t Hamiltonian path

The LP, i.e. the s-t-path TSP polytope:

$P(V,s,t) = \{ x \in \mathbb{R}_+^E : x(\delta(W)) \geq 2$, $\emptyset \neq W \subset V$, $s, t \in W$ or $\notin$

1, if s,t separated by W





= on vertices (1 for s, t , else 2) }

$= \{ x \in \mathbb{R}_+^E : x(E(W)) \leq n-1$, $\emptyset \neq W \subset V$,

$= n-1$,

and = 1 on stars of s, t and else 2 as before }

Remarks about the s-t path TSP polytope

Spanning tree polytope intersected with star inequalities !

Notation:

OPT := min Ham s-t path length.

OPT_{LP} (s,t) := min { $c^T x$: $x \in P(V,s,t)$ }

Integer points : Hamiltonian paths

The proof of 2 is as easy, and `the two proofs' of 5/3 are not difficult.

Of « Christofides » algorithm

Let us do 2 ! It looks easier, so why only 5/3 ?

Why 5/3 ? What is the difficulty ?

None of the proofs for 3/2 work,

Parity correction costs more than 1/2 :

Deleting an {s,t}-join, what remains is not an {s,t}-join

For $x \in P(V,s,t)$, $x(C)/2 < 1$ is possible,

$x/2$ is not in all T-join polyhedra.

Def : A cut C is called *narrow* if $x(C) < 2$

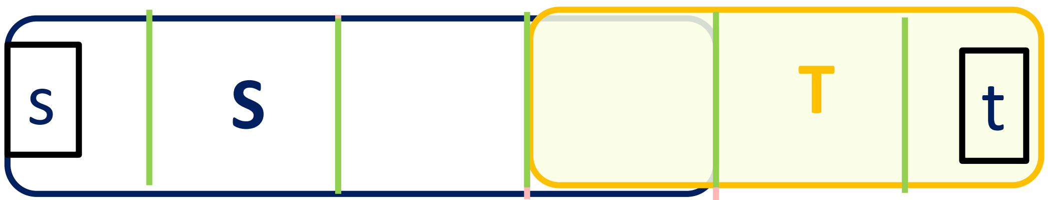
Narrow cuts are the guilty ones !

Narrow cuts form a chain

Lemma (An, Kleinberg, Shmoys, 2011) G=(V,E) graph, $x \in P(V,s,t)$.

The set of narrow cuts is a chain $\delta(S_i)$ (i=1, … , k), $s \in S_1 \subseteq \ldots \subseteq S_k$.

Proof : Suppose not :



$2+2 > d(S) + d(T) \geq d(S \cap T) + d(S \cup T) \geq 2 + 2$, a contradiction

The first results

cycle or path / cardinality or weights	cycle (s=t)	(s,t)-path
cardinality	Gamarnik,Lewenstein,Sviridenko (2005): 3/2 - ε for cubic 3-connected Boyd, Sitters,van der Ster,Stougie (2011): 4/3 for cubic Oveis, Gharan, Saberi, Singh (2011) : 3/2 - ε Mömke, Svensson (2011) : 1.461… Mucha (2011) : 13/9=1.444… **4**	Hoogeveen (1991) **5/3** An,Kleinberg,Shmoys(2011) **1.578 …**
general	**1** Christofides CHR, 1976 **1.5**	**3** Hoogeveen (1991) **5/3** An,Kleinberg,Shmoys(2011) “AKS” **1.619 …**

Last integrality gap (I) and approx ratio (A)

cycle or path / cardinality or weights	cycle	(s,t)-path
graph metric (cardinality tour)	Sebő, Vygen SV12, Jan 2012 **1.4 , conjectured: 4/3 (I)**	Gao: simpler proof, March, 2013 Sebő, Vygen SV12, Jan 2012 **1,5 best possible (I)** Traub, Vygen <3/2 (A) ? April 2018
General metric	Christofides CHR, 1976 **1,5 , conjectured: 4/3 (I)**	Sebő, van Zuylen, 2016 3/2 + 1/34 (I) **conjectured: 3/2 (I)** Traub, Vygen 3/2 + ε (A) 2017 **Zenklusen, April 2018: 3/2 (A)**

Zenklusen's 3/2 approximation
April 2018

Let $\mathcal{B}$ be a set of s-t cuts

This is a step `towards integrality'
And we show it is still tractable

y ∈ P(V,s,t) is $\mathcal{B}$ – *good* , if y(B) is 1 or ≥ 3 for all B ∈ $\mathcal{B}$

The incidence vector χ^H of a Hamiltonian path H :

χ^H ∈ P(V,s,t) , and χ^H *is* $\mathcal{B}$ – good for ∀ set $\mathcal{B}$

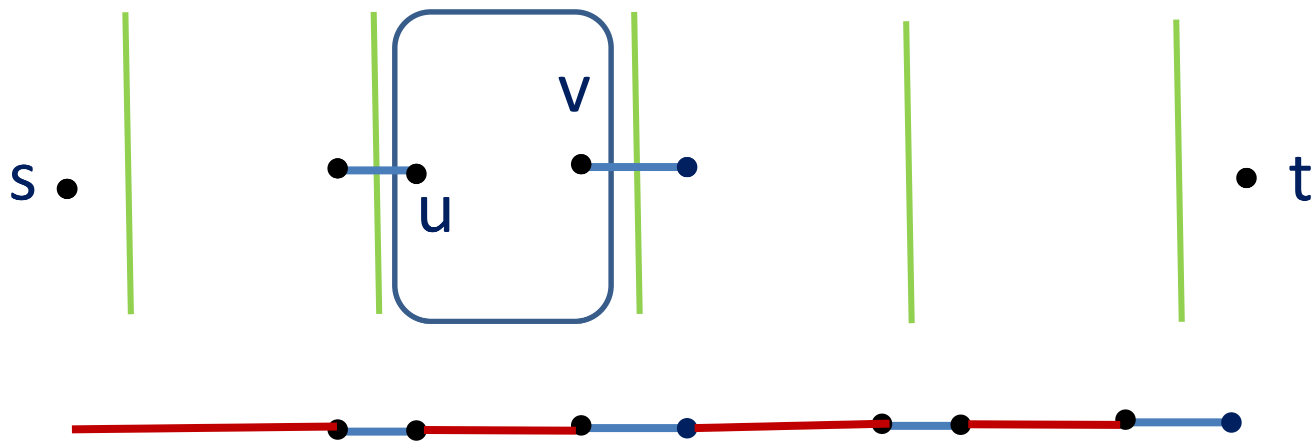
So min c(y), y $\mathcal{B}$ – good
still lower bounds OPT

Apply this to $\mathcal{B}$:= {C: x*(C))<3}

Karger: pol. Number
and can be listed

Temptative reduction

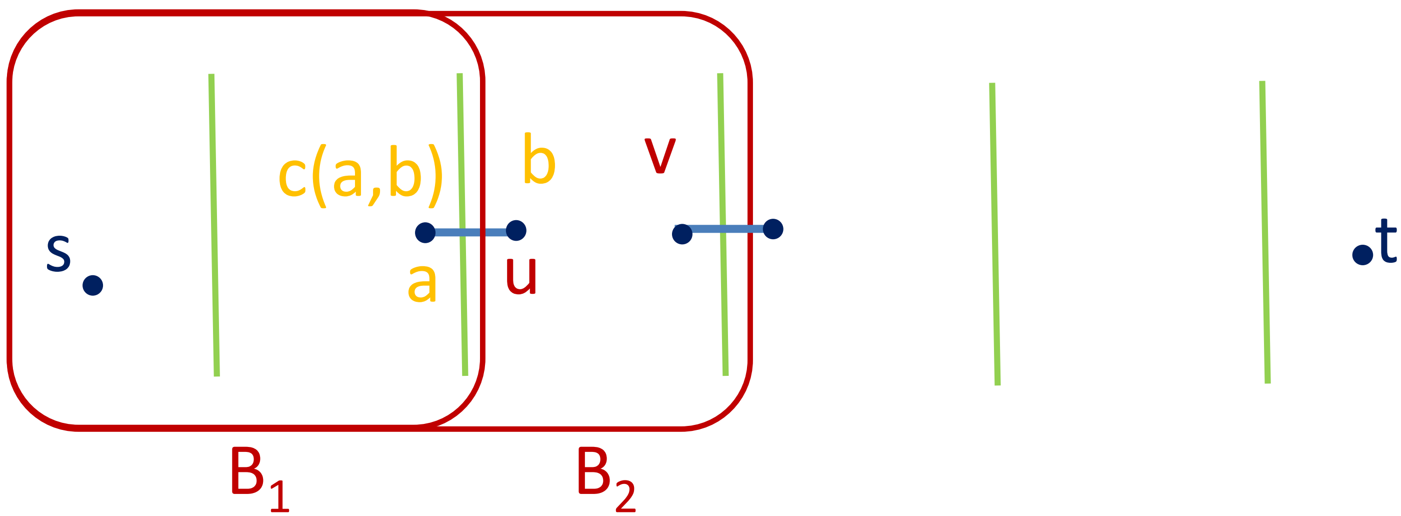
If we know about green cuts that we want to meet once, the problem can be decomposed to smaller ones :



Using Karger + Ellipsoids we can solve all small problems.

But : There may be new narrow cuts ...

Min $\mathcal{B}-good$ with shortest paths :



For all pairs $B_1 \subseteq B_2 \in \mathcal{B}$ and u, v

minimize on $P(B_2 \setminus B_1, u, v)$ requiring 3 on $\mathcal{B} \setminus \{B_1, B_2\}$

Solve shortest paths with $w(B_2 \setminus B_1, u, v)$, $c(a,b)$.

To be precise: the vertices of the auxiliary graph have to be triples …

How to find a TSP solution

Shortest paths with input w (u, v) , c (u,v)

found the min $\mathcal{B} - good$ solution y, c(y) $\leq$ OPT

If B $\in \mathcal{B}$ is in the chain of 1-c, y (B) ≥ 3 ;

if not, let B' in the chain not containing B and
not contained in B:

y(B) + y(B') ≥ y(B$\cap$B')+y(B$\cup$B) ≥ 2 + 2

= 1

$$\frac{x^{*} + y}{2}$$

is then in P(V,s,T) = the parity correction polyhedron !

Rico’s algorithm

THE END OF THIS COURSE

THE END OF THIS MEETING

MANY THANKS TO THE ORGANIZERS !

Many thanks to the participants !

Hopefully you know more than before !