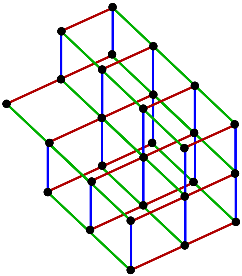
Orders and Lattices from Graphs

Spring School SGT 2018
Seté, June 11-15, 2018



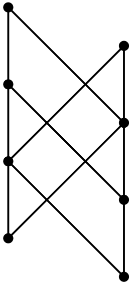
Stefan Felsner
Technische Universität Berlin

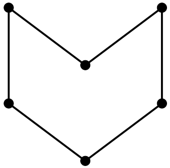
Outline

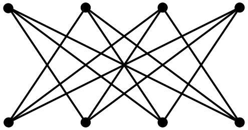
Finite Orders

$P = (X, <)$ is an order iff

- X finite set
- $<$ transitive and irreflexive relation on X.







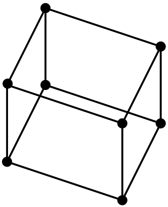
Lattices

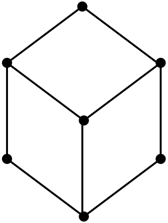
$P = (X, <)$ an order.

- Let $x \vee y$ be the least upper bound of x and y if it exists.
- Let $x \wedge y$ be the greatest lower bound of x and y if it exists.

$L = (X, <)$ is a finite lattice iff

- L is a finite order
- $x \vee y$ and $x \wedge y$ exist for all x and y.





Lattices - the algebraic view

$L = (X, \vee, \wedge)$ is a finite lattice iff

- X is finite and for all $a, b, c \in X$ and $\diamond \in \{\vee, \wedge\}$
- $a \diamond (b \diamond c) = (a \diamond b) \diamond c$ (associativity)
- $a \diamond b = b \diamond a$ (commutativity)
- $a \diamond a = a$ (idempotency)
- $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ (absorption)

Proposition. The two definitions of finite lattices are equivalent:

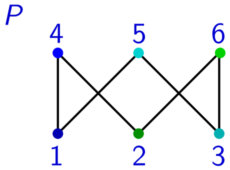
$$x \leq y \iff x = x \wedge y \quad \text{and} \quad x \leq y \iff y = x \vee y.$$

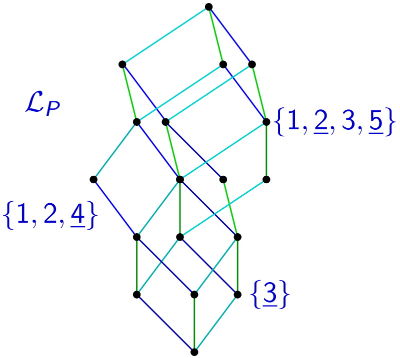
Distributive Lattice

A lattice $L = (X, \vee, \wedge)$ is a distributive lattice iff

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

FTFDL. L is a finite distributive lattice $\iff$ there is a poset P such that that L is isomorphic to the inclusion order on downsets of P.

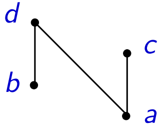




Linear Extensions

A linear extension of $P = (X, <_P)$ is a linear order $L = (X, <_L)$, such that

- $x <_P y \implies x <_L y$



d	c	d	d	c
c	d	b	c	d
b	b	c	a	a
a	a	a	b	b

Dimension of Orders I

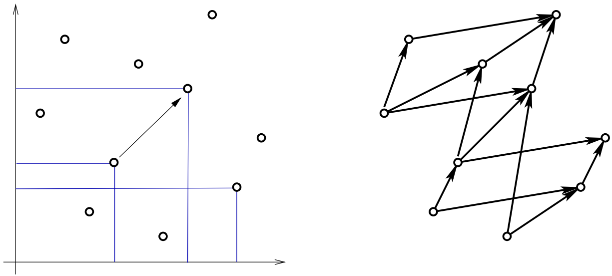
A family $\mathcal{L}$ of linear extensions is a realizer for $P = (X, <)$ provided that

* for every incomparable pair (x, y) there is an $L \in \mathcal{L}$ such that $x < y$ in L.

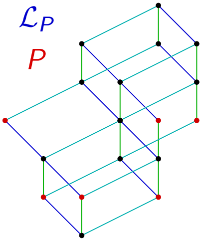
The dimension, $\dim(P)$, of P is the minimum t, such that there is a realizer $\mathcal{L} = \{L_1, L_2 \ldots, L_t\}$ for P of size t.

Dimension of Orders II

The dimension of an order $P = (X, <)$ is the least t, such that P is isomorphic to a suborder of $\mathbb{R}^t$ with the product ordering.



Dilworth's Imbedding Theorem (1950)



Theorem. $\dim(\mathcal{L}_P) = \text{width}(P)$.

- Let $C_1, \ldots, C_w$ be a chain partition of P.

  Imbed $\mathcal{L}_P$ in $\mathbb{R}^w$ by $I \to (|I \cap C_1|, \ldots, |I \cap C_w|)$.

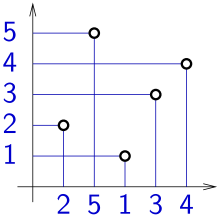
- If P contains an antichain A of size w,

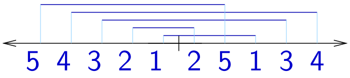
  then there is a Boolean lattice $\mathcal{B}_w$ in $\mathcal{L}_P$.

  Hence $\dim(\mathcal{L}_P) \geq \dim(\mathcal{B}_w) = w$.

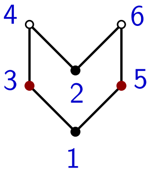
Small Dimension

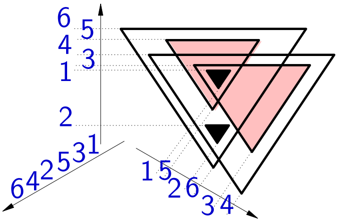
- Dimension 2: Containment orders of intervals.





- Dimension 3: Containment orders of triangles.

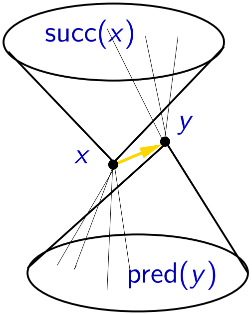




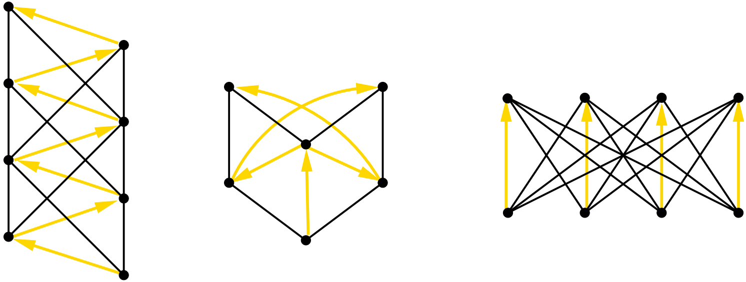
Critical Pairs

Definition. An incomparable pair (x, y) is critical if

- $a < x$ implies $a < y$.
- $y < b$ implies $x < b$.



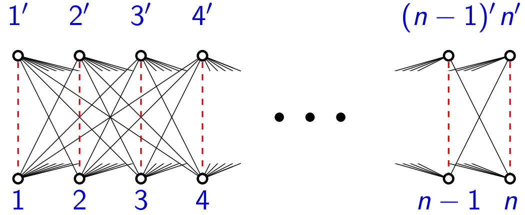
Critical Pairs



Proposition. A family $\mathcal{R}$ of linear extensions of P is a realizer of P $\iff$ $\mathcal{R}$ reverses all critical pairs.

Standard Examples

- Standard example of an n dimensional order:

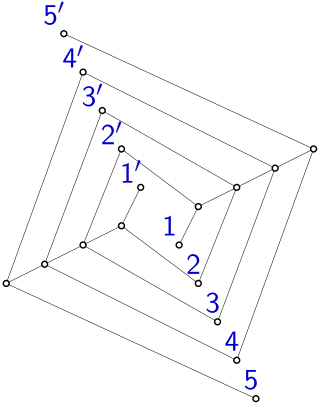


Dimension and Planarity

Theorem [Baker 1971]. If an order P has $\mathbf{0}$ and $\mathbf{1}$ and a planar diagram, then $\dim(P) \leq 2$.

Theorem [and Trotter and Moore 1977]. If an order P has $\mathbf{0}$ and a planar diagram, then $\dim(P) \leq 3$.

The dimension of an order P with a planar diagram can be unbounded (Kelly 1981).



Dimension beyond Planarity

Theorem [F., Li, and Trotter 2010 **].** The dimension of an order P of height ≤ 2 with a planar diagram is at most 4.

Theorem [Streib and Trotter 2014 **].** There is a function f such that $\dim(P) \leq f(h)$ for orders of height $\leq h$ with a planar cover graph.

Theorem [Joret, Micek, and Wiechert 2018 **].** There is a function $f_{\mathcal{C}}$ such that $\dim(P) \leq f_{\mathcal{C}}(h)$ for orders of height $\leq h$ whose cover graphs belong to a class $\mathcal{C}$ of graphs with bounded expansion. (This includes classes with a forbidden minor.)

Complexity

Theorem [Yannakakis 1982 **].** To test if a partial order has dimension $\leq k$ is NP-complete for all $k \geq 3$.
To test if a partial order of height 2 has dimension $\leq k$ is NP-complete for all $k \geq 4$.

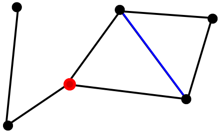
Theorem [F., Mustață, and Pergel 2014 **].** To test if a partial order of height 2 has dimension 3 is NP-complete.

Theorem [Chalermsook et al. 2013 **].** Unless NP $=$ ZPP there is no polynomial algorithm to approximate the dimension of a partial order with a factor of $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$

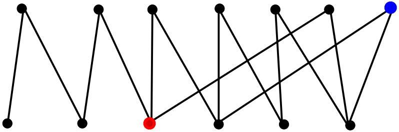
Incidence Orders and Dimension

The incidence order P_G of G

G



P_G



Theorem [Spencer '72 / Trotter '80 / Hoşten und Morris '98 **].**

$$\dim(K_n) = \dim(\mathbf{B}_n[1,2]) = \log\log n + (\tfrac{1}{2} + o(1)) \log\log\log(n)$$

$\dim(K_n) \leq$	2	3	4	5	6	7	8
$n \leq$	2	4	12	81	2646	1422564	229809982112

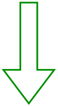
$a(9) = 423295099074735261880$ Andries E. Brouwer, 2012.

A Planarity Criterion

Theorem [Schnyder 1989].
A Graph G is planar $\iff$ $\dim(P_G) \leq 3$.

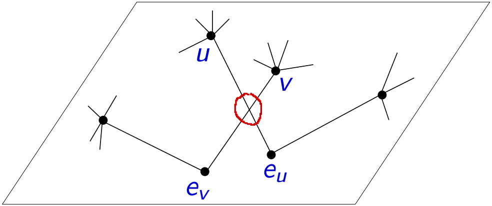
- $\dim(G) \leq 3 \implies G$ planar.

P_G in $\mathbb{R}^3$



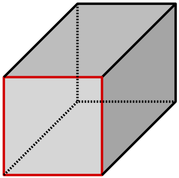
projection

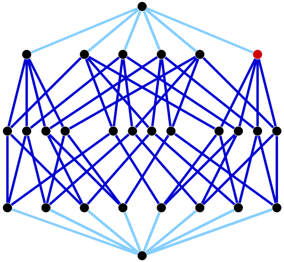
$x + y + z = 1$



Dimension of Polytopes

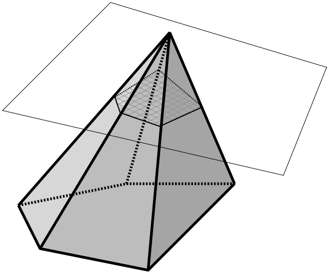
Let $\mathcal{F}_P$ be the face lattice of polytope P.



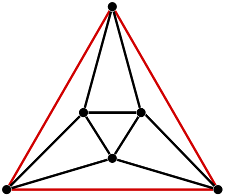


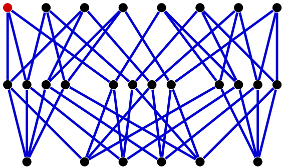
Dimension of Polytopes: Lower Bound

Theorem [Reuter 1990].
If P is a d-polytope, then $\dim(\mathcal{F}_P) \geq d + 1$.



Dimension of 3-Polytopes





Theorem [Schnyder 1989].
If G is a plane triangulation with a face F, then

- $\dim(P_{VEF}(G \setminus F)) = 3$
- $\dim(P_{VEF}(G)) = 4$

Theorem [Brightwell+Trotter 1993].
If G is a 3-connected plane graph with a face F, then

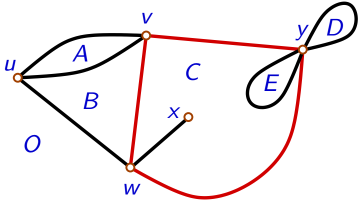
- $\dim(P_{VEF}(G \setminus F)) = 3$
- $\dim(P_{VEF}(G)) = 4$

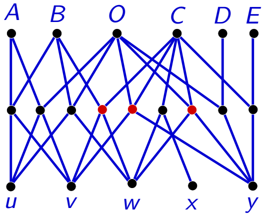
Dimension and Planar Graphs

Theorem [Schnyder 1989].
A Graph G is planar $\iff$ $\dim(P_G) \leq 3$.

Theorem [Brightwell+Trotter 1997].
If G is a plane multi-graph with loops, then

$$\dim(P_{VEF}(G)) \leq 4.$$





Outline

Orders and Lattices

Definitions
The Fundamental Theorem
Dimension and Planarity

Lattices and Graphs

α-orientations
Δ-Bonds and Further Examples
The ULD-Theorem

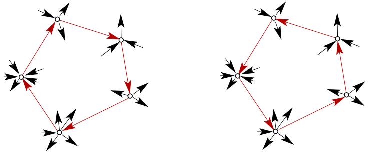
Distributive Lattices and Markov Chains

Coupling from the Past
Mixing time on α-orientations

α-Orientations

Definition. Given $G = (V, E)$ and $\alpha : V \to \mathbb{N}$.
An α-orientation of G is an orientation with $\text{outdeg}(v) = \alpha(v)$ for all v.

- Reverting directed cycles preserves α-orientations.



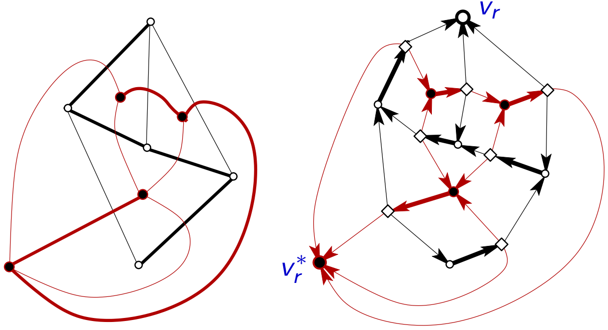
Theorem. The set of α-orientations of a planar graph G has the structure of a distributive lattice.

- Diagram edge $\sim$ revert a directed essential/facial cycle.

Example 1: Spanning Trees

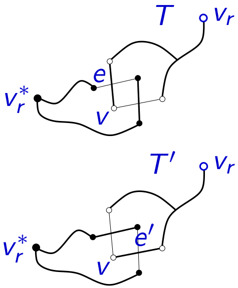
Spanning trees are in bijection with α_T orientations of a rooted primal-dual completion $\widetilde{G}$ of G

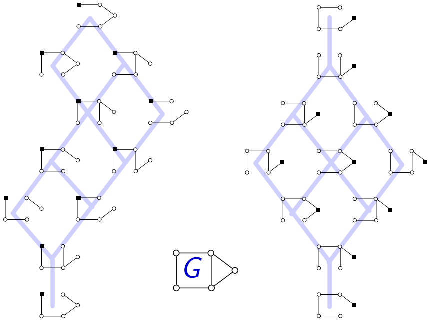
- $\alpha_T(v) = 1$ for a non-root vertex v and $\alpha_T(v_e) = 3$ for an edge-vertex v_e and $\alpha_T(v_r) = 0$ and $\alpha_T(v_r^*) = 0$.



Lattice of Spanning Trees

Gilmer and Litheland 1986, Propp 1993



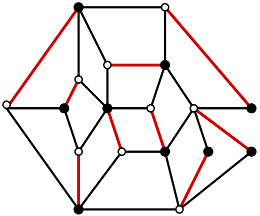


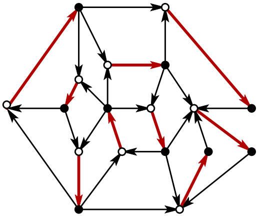
Example2: Matchings and f-Factors

Let G be planar and bipartite with parts (U, W). There is bijection between f-factors of G and α_f orientations:

- Define α_f such that $\text{indeg}(u) = f(u)$ for all $u \in U$ and $\text{outdeg}(w) = f(w)$ for all $w \in W$.

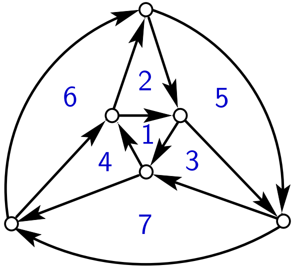
Example. A matching and the corresponding orientation.

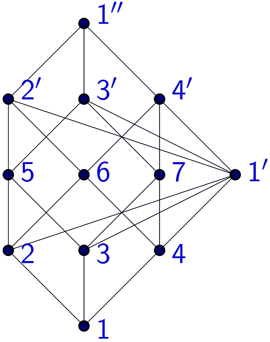




Example 3: Eulerian Orientations

- Orientations with $\mathrm{outdeg}(v) = \mathrm{indeg}(v)$ for all v, i.e. $\alpha(v) = \frac{d(v)}{2}$

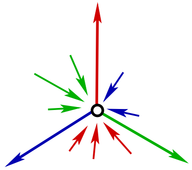




Example 4: Schnyder Woods

G a plane triangulation with outer triangle $F = \{a_1, a_2, a_3\}$.
A coloring and orientation of the interior edges of G with colors 1,2,3 is a Schnyder wood of G iff

- Inner vertex condition:



- Edges $\{v, a_i\}$ are oriented $v \to a_i$ in color i.

Schnyder Woods and 3-Orientations

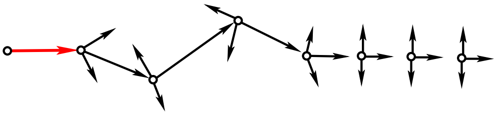
Theorem. Schnyder wood and 3-orientation are in bijection.

Proof.

- All edges incident to a_i are oriented $\to a_i$.

  G has $3n-9$ interior edges and $n-3$ interior vertices.

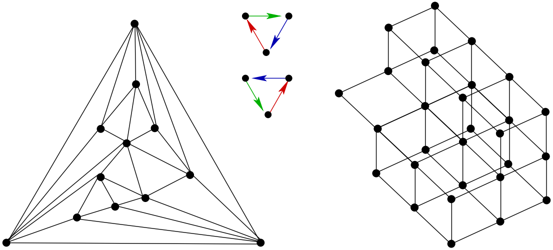
- Define the path of an edge:



- The path is simple (Euler), hence, ends at some a_i.
- Two path starting at a vertex do not cross (Euler).

The Lattice of Schnyder Woods

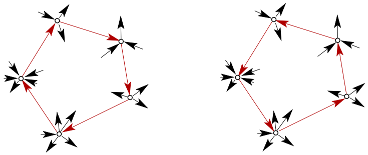
Theorem. The set of Schnyder woods of a plane triangulation G has the structure of a distributive lattice.



α-Orientations

Definition. Given $G = (V, E)$ and $\alpha : V \to \mathbb{N}$.
An α-orientation of G is an orientation with $\text{outdeg}(v) = \alpha(v)$ for all v.

- Reverting directed cycles preserves α-orientations.



Theorem. The set of α-orientations of a planar graph G has the structure of a distributive lattice.

Proof I: Essential Cycles

For the proof we assume that G is 2-connected.

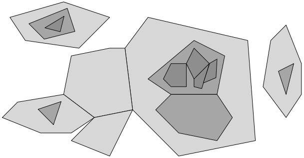
Definition.
A cycle C of G is an essential cycle if

- C is chord-free and simple,
- the interior cut of C is rigid,
- there is an α-orientation X such that C is directed in X.

Lemma.
C is non-essential $\iff$ C has a directed chordal path in every α-orientation.

Proof II

Lemma.
Essential cycles are interiorly disjoint or contained.



Lemma.
If C is a directed cycle of X, then X^C can be obtained by a sequence of reversals of essential cycles.

Lemma.
If $(C_1, .., C_k)$ is a flip sequence (ccw $\rightarrow$ cw) on X then for every edge e the essential cycles $C^{l(e)}$ and $C^{r(e)}$ alternate in the sequence.

Proof III: Flip Sequences

Lemma.
The length of any flip sequence (ccw $\to$ cw) is bounded and there is a unique α-orientation $X_{\min}$ with the property that all cycles in $X_{\min}$ are cw-cycles.

- $Y \prec X$ if a flip sequence $X \to Y$ exists.

Lemma.
Let $Y \prec X$ and C be an essential cycle. Every sequence $S = (C_1, \ldots, C_k)$ of flips that transforms X into Y contains the same number of flips at C.

Proof IV: Potentials

Definition. An α-potential for G is a mapping $\wp : \mathsf{Ess}_\alpha \to \mathbb{N}$ such that

- $|\wp(C) - \wp(C')| \leq 1$, if C and C' share an edge e.
- $\wp(C^{l(e)}) \leq \wp(C^{r(e)})$ for all e (orientation from $X_{\min}$)

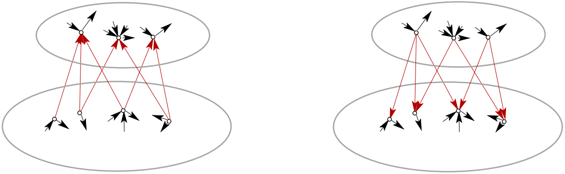
Lemma. There is a bijection between α-potentials and α-orientations.

Theorem. α-potentials are a distributive lattice with

- $(\wp_1 \vee \wp_2)(C) = \max\{\wp_1(C), \wp_2(C)\}$ and
- $(\wp_1 \wedge \wp_2)(C) = \min\{\wp_1(C), \wp_2(C)\}$ for all essential C.

A Dual Construction: c-Orientations

- Reorientations of directed cuts preserve flow-difference (#forward arcs − #backward arcs) along cycles.



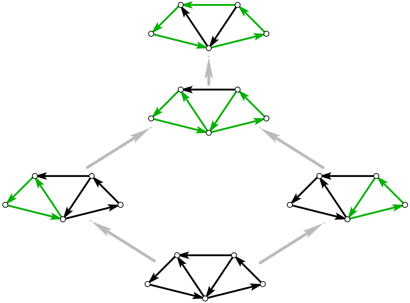
Theorem [Propp 1993]. The set of all orientations of a graph with prescribed flow-difference for all cycles has the structure of a distributive lattice.

- Diagram edge $\sim$ push a vertex ($\neq v_\dagger$).

Circulations in Planar Graphs

Theorem [Khuller, Naor and Klein 1993].
The set of all integral flows respecting capacity constraints ($\ell(e) \leq f(e) \leq u(e)$) of a planar graph has the structure of a distributive lattice.

$0 \leq f(e) \leq 1$



- Diagram edge $\sim$ add or subtract a unit of flow in ccw oriented facial cycle.

Δ-Bonds

$G = (V, E)$ a connected graph with a prescribed orientation.
With $x \in \mathbb{Z}^E$ and C cycle we define the circular flow difference

$$\Delta_x(C) := \sum_{e \in C^+} x(e) - \sum_{e \in C^-} x(e).$$

With $\Delta \in \mathbb{Z}^{\mathcal{C}}$ and $\ell, u \in \mathbb{Z}^E$ define

- $\mathcal{B}_G(\Delta, \ell, u) = \left\{ x \in \mathbb{Z}^E \;:\; \Delta_x = \Delta \text{ and } \ell \leq x \leq u \right\}$.

  $\Delta_x = \Delta$ (circular flow difference)

  $\ell \leq x \leq u$ (capacity constraints).

Δ-Bonds as Generalization

Special cases:

- c-orientations are $\mathcal{B}_G(\Delta, 0, 1)$
  ($\Delta(C) = \frac{1}{2}\big(|C^+| - |C^-| - c(C)\big)$).
- Circular flows on planar G are $\mathcal{B}_{G^*}(0, \ell, u)$
  (G^* the dual of G).
- α-orientations.

Theorem [Felsner & Knauer 2009 **].**
$\mathcal{B}_G(\Delta, \ell, u)$ has the structure of a distributive lattice.