

Graph Representations: Rectangles and Squares

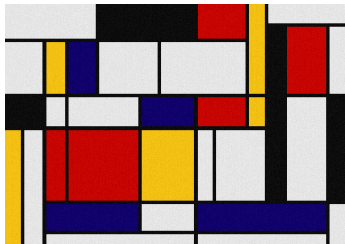
Spring School **SGT 2018**

Seté, June 11-15, 2018

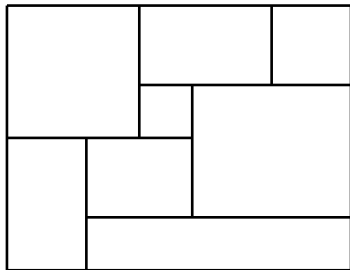
Stefan Felsner

Technische Universität Berlin

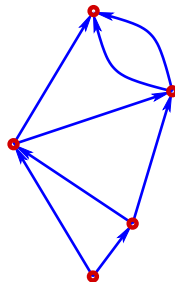
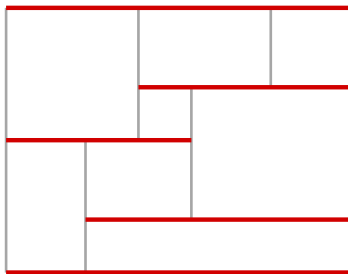
felsner@math.tu-berlin.de



A Rectangular Dissection

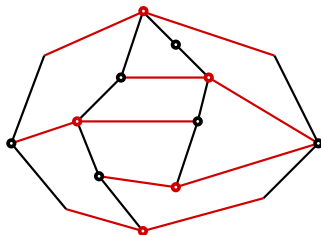
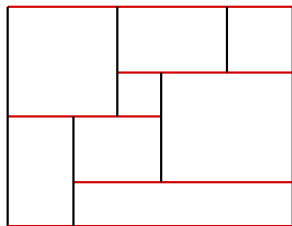


Rectangular Dissections Induce Graphs



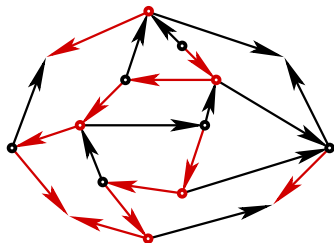
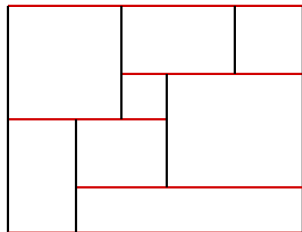
A bipolar graph G induced by R .
(R is a visibility representation of G .)

Rectangular Dissections Induce Graphs



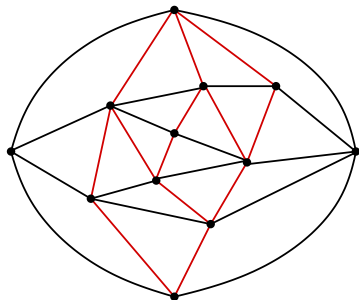
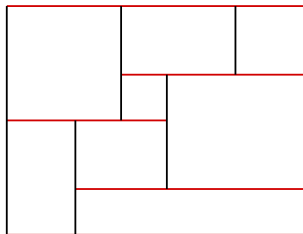
A quadrangulation based on segment contacts.

Rectangular Dissections Induce Graphs



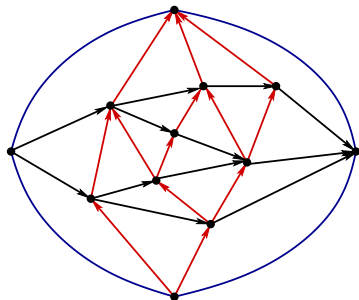
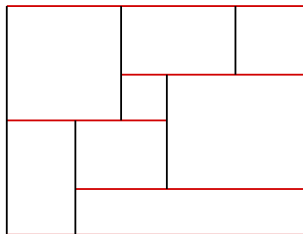
A separating decomposition of the quadrangulation.

Rectangular Dissections Induce Graphs



An inner triangulation of a quadrangle.

Rectangular Dissections Induce Graphs



A transversal structure on the inner triangulation of a quadrangle.

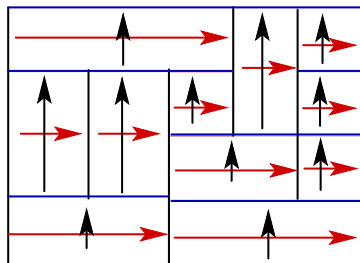
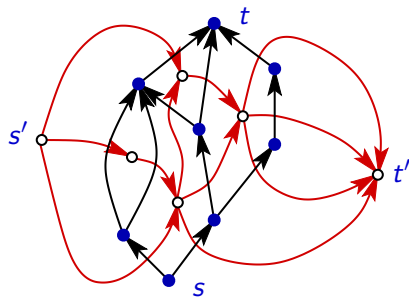
Problems

- Find a representation.
- Find an optimal representation.
- Find representation with additional properties.

Segment contact representations

Sketch: Bipolar Orientation

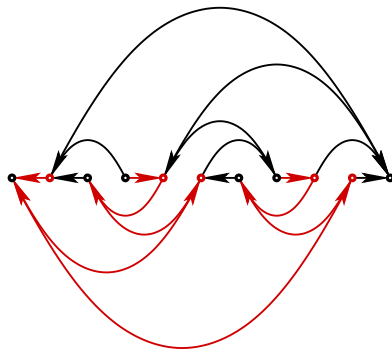
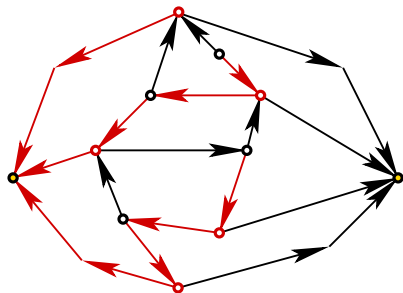
From the bipolar orientation compute its dual orientation.
Together they yield a rectangular dissection.



coordinates from longest paths

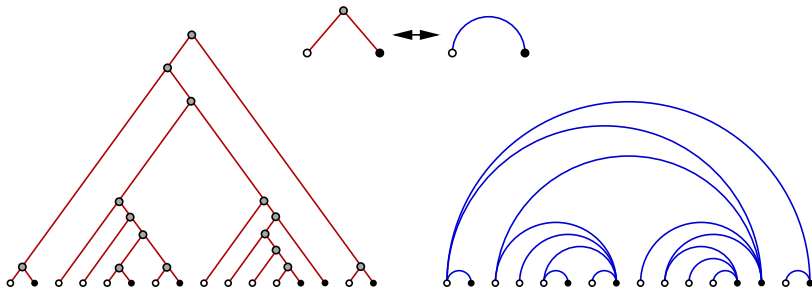
Sketch: Quadrangulation

- Compute a separating decomposition.
- Separate the two alternating trees.



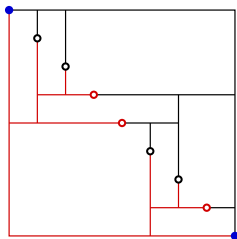
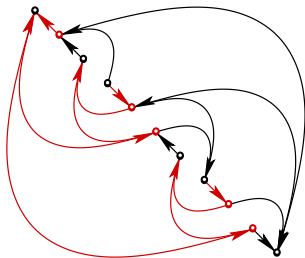
Alternating and Full Binary Trees

Proposition. There is bijection between alternating and binary trees that preserves types (left/right) of nodes.



Sketch: Quadrangulation

- The two binary trees obtained from the separating decomposition fit together.



More Representation Problems

Add some conditions.

- Find a representing rectangulation R in a square such that all rectangles intersect the diagonal.

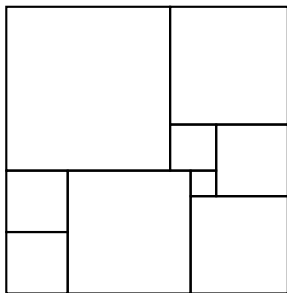
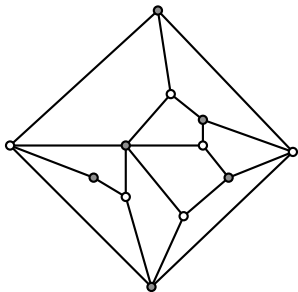
(We just saw a solution)

- Find a representing rectangulation R such that all inner rectangles are squares.

The dissection of rectangles into squares

Brooks, Smith, Stone and Tutte 1940.

Segment Contact Squarings



Squarings a la BSST

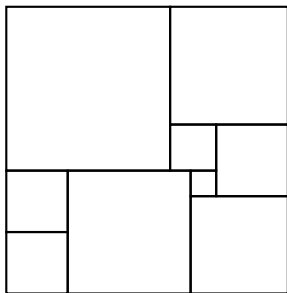
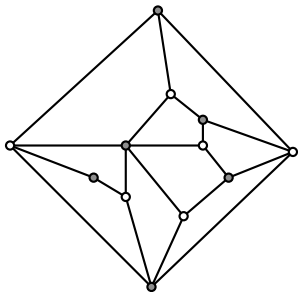
View the bipolar graph as electrical network with edges of resistance 1 Ohm. Consider an $s \rightarrow t$ flow in this network. The distribution of current in edges corresponds to a squaring.

Based on this theory they give explicit solutions:

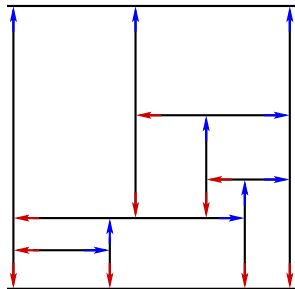
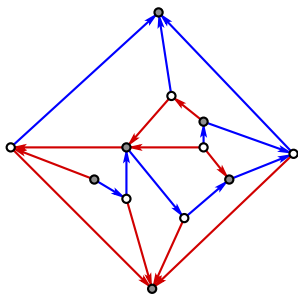
$\text{size}(i, j) =$

- # spanning trees T with (i, j) on the $s \rightarrow t$ path in T
- # spanning trees T with (j, i) on the $s \rightarrow t$ path in T .

Segment Contact Squarings

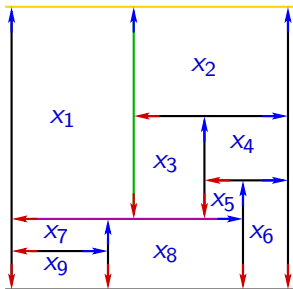


Computing a Segment Contact Squaring



Step I: Compute a separating decomposition on Q .
This yields a rectangular dissection.

Computing a Segment Contact Squaring



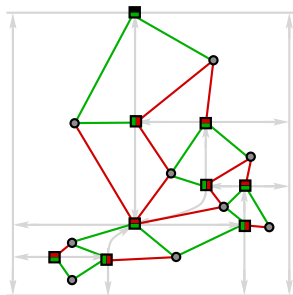
- $x_1 = x_2 + x_3$
- $x_1 + x_3 + x_5 = x_7 + x_8$
- $x_1 + x_2 = 1$

Step II: Set up a linear system of equations: $A_S \cdot x = e_1$

- A_S is a square matrix.

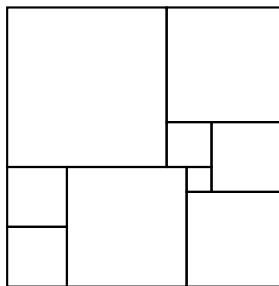
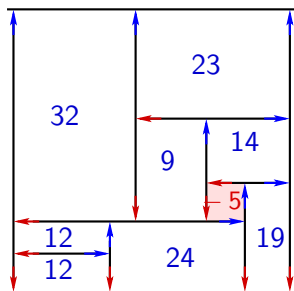
Theorem. $\det(A_S) \neq 0$.

Computing a Segment Contact Squaring



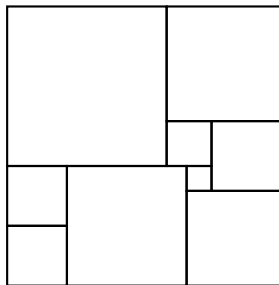
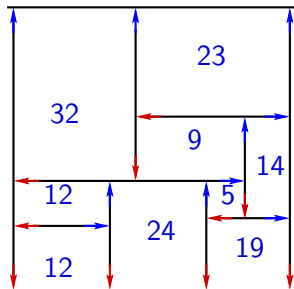
- A_S is a bipartite adjacency matrix of a bipartite graph H .
- $\det(A_S) = \sum_{\pi} \text{sign}(\pi) \prod a_{i,\pi(i)}$
- $\det(A_S) = \sum_M \text{sign}(M)$, with M perfect matching of H .
- $\text{sign}(M) = \text{sign}(M')$ for all M and M' perfect matching.
- H has a perfect matching.

Squarings with Segment Contacts



Step III: Flip negative faces to get the good separating decomposition and the squaring.

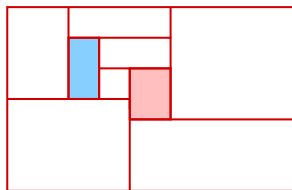
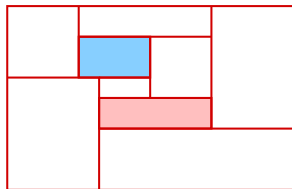
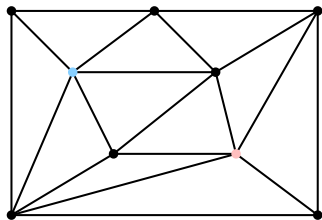
Squarings with Segment Contacts



Step III: Flip negative faces to get the good separating decomposition and the squaring.

Rectangular and square duals

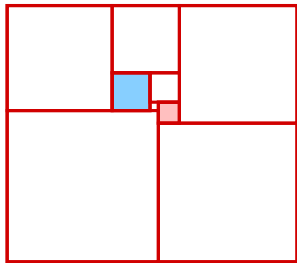
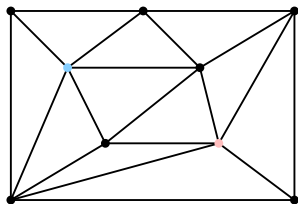
Rectangular Duals



Rectangular duals are not unique.
They exist if the triangulation is 4-connected.

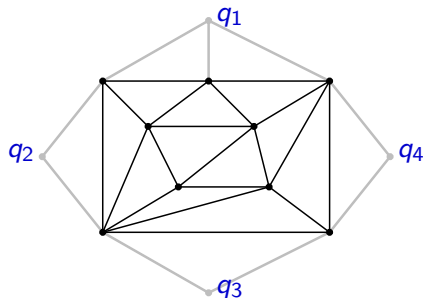
Squarings for Inner Triangulations

The squaring is unique.



O. Schramm *Square Tilings with prescribed Combinatorics*. 1993

A Polyhedral view on Squarings



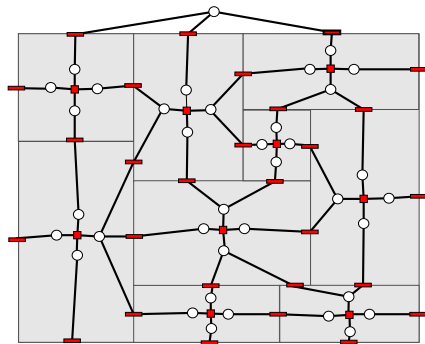
$$P = \{x \in \mathbb{R}_{\geq 0}^V : \sum_{i \in \gamma} x_i \geq 1 \text{ for all } q_1 \rightarrow q_3 \text{ paths } \gamma\}$$

solution vector for $\min(\sum_i x_i^2 : x \in P)$ yields squaring.

L. Lovász *Geometric Representations of Graphs*, 2009, Sec. 6.3.2.

Computing Squarings with Systems of Equations

transversal structure $\Rightarrow$ rectangular dissection.

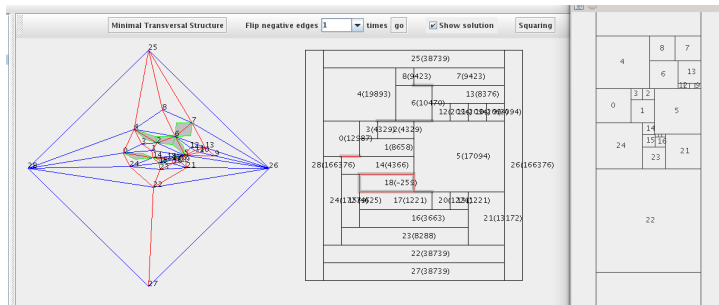


- red rectangles = variables
- white circles = equations

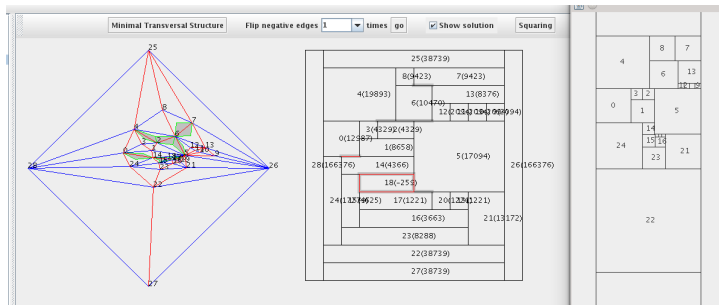
The algorithm

1. T transversal structure on a 4-triangulation G .
2. Calculate the solution x_T of the equation system $Ax = b$.
3. If $x_T \geq 0$
 - ▶ Construct a square contact representation from x_T .
4. Else
 - ▶ T' obtained by 'reverting bad-directed cycle' in T .
 - ▶ Set $T := T'$ and go to 2.

The program



The program



Thank you.