Schnyder Woods and Applications

Spring School SGT 2018
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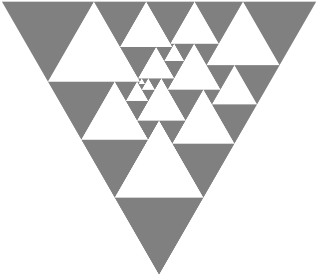


Outline

Introduction to Schnyder Woods

Dimension

Drawings

Triangle representations

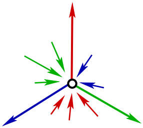
Schnyder Woods

$G = (V, E)$ a plane triangulation,

$F = \{a_1, a_2, a_3\}$ the outer triangle.

A coloring and orientation of the interior edges of G with colors 1,2,3 is a Schnyder wood of G iff

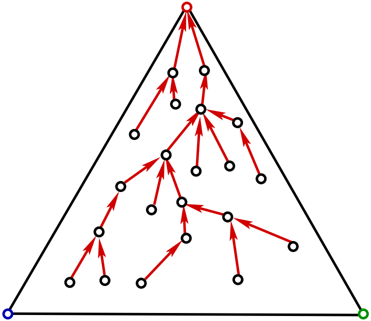
- Inner vertex condition:



- Edges $\{v, a_i\}$ are oriented $v \to a_i$ in color i.

Schnyder Woods - Trees

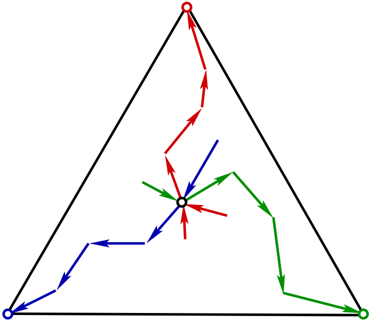
- The set T_i of edges colored i is a tree rooted at a_i.



Proof. Count edges in a cycle — Euler ⚡

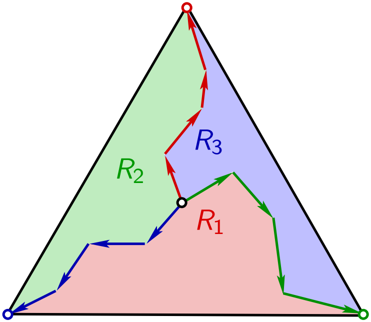
Schnyder Woods - Paths

- Paths of different color have at most one vertex in common.



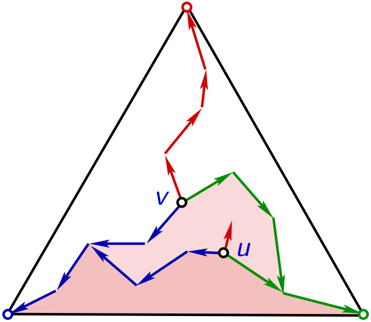
Schnyder Woods - Regions

- Every vertex has three distinguished regions.



Schnyder Woods - Regions

- If $u \in R_i(v)$ then $R_i(u) \subset R_i(v)$.



Outline

Introduction to Schnyder Woods

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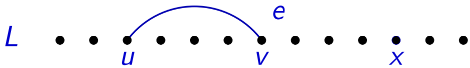
Schnyder's Dimension Theorem

Q_i the inclusion order with respect to $R_i(.)$
L_i a linear extension of Q_i
L_i^+ include edges as low in L_i as possible
$\implies L_1^+, L_2^+, L_3^+$ is a realizer for P_G.

Proof. $e = \{u, v\}$ and edge, $x \notin e$ a vertex.
We need:



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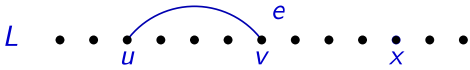
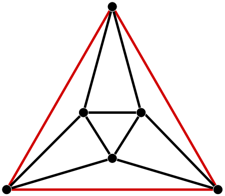


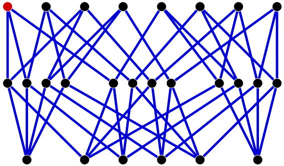
There is an i with $e \in R_i(x)$

$\implies R_i(u) \subset R_i(x)$ and $R_i(v) \subset R_i(x)$

$\implies$ Edge e goes below x in L_i^+. $\square$

Dimension of 3-Polytopes





Theorem [Schnyder 1989].
If G is a plane triangulation with a face F, then

- $\dim(P_{VEF}(G) \setminus F) = 3$
- $\dim(P_{VEF}(G)) = 4$

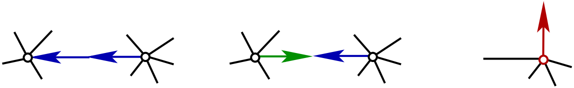
Theorem [Brightwell+Trotter 1993].
If G is a 3-connected plane graph with a face F, then

- $\dim(P_{VEF}(G) \setminus F) = 3$
- $\dim(P_{VEF}(G)) = 4$

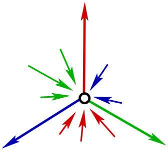
Schnyder Woods

Axioms for 3-coloring and orientation of bi-edges:

(W1 - W2) Rule of edges and half-edges:

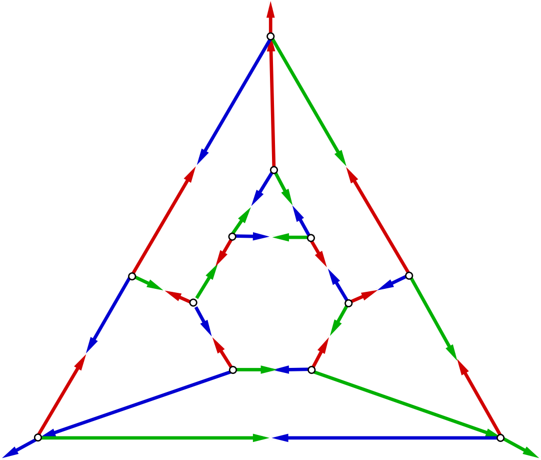


(W3) Rule of vertices:

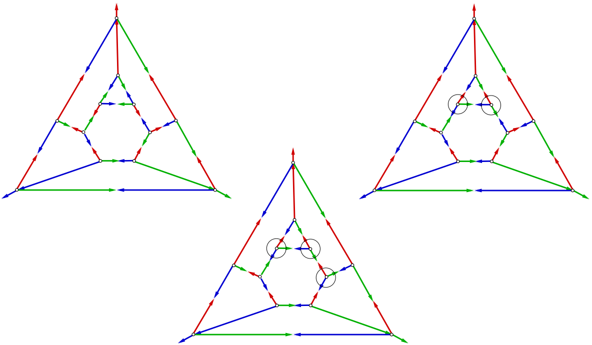


(W4) There is no interior face whose boundary is a directed cycle in one color.

We need W4!

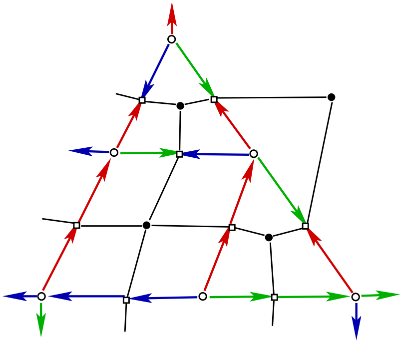


3-Orientations?



Primal and Dual

A Schnyder wood of G induces a Schnyder wood of the dual of G.



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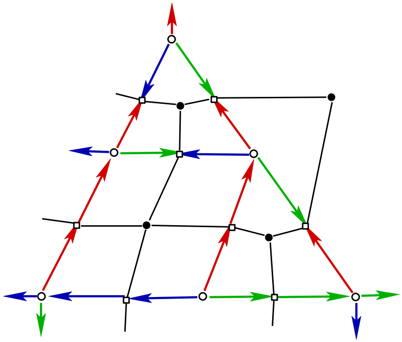

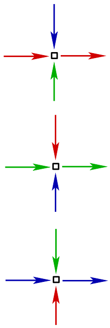

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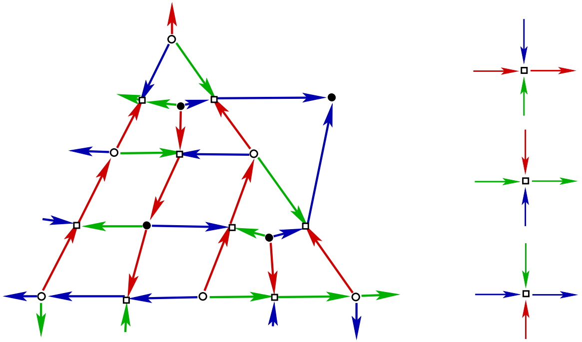
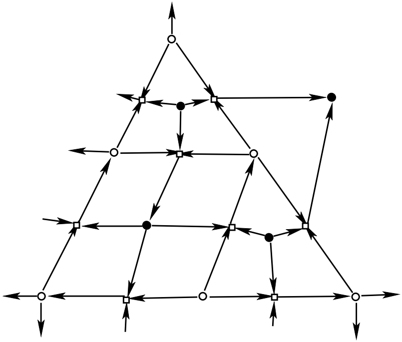


3-Orientations – Rescued

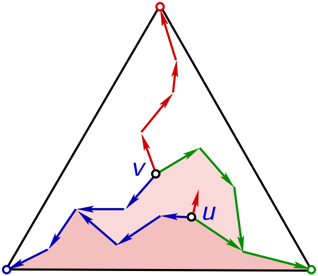
Theorem. Primal dual Schnyder woods are in bijection with primal dual 3-orientations.

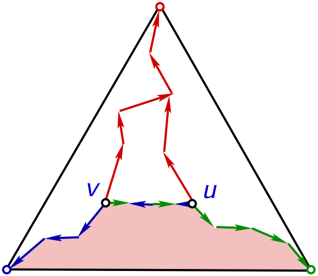


Schnyder Woods - Regions

- If $u \in R_i^o(v)$ then $R_i(u) \subset R_i(v)$.
- If $u \in \partial R_i(v)$ then $R_i(u) \subseteq R_i(v)$

  (equality, iff there is a bi-directed path between u and v.)





Brigthwell Trotter Theorem

- P a 3-polytope with face F.
- G the graph of P with outer face F.
- S a Schnyder wood of G.

Define $<_i$ on V_P for $i = 1, 2, 3$

(1) $u <_i v$ if $R_i(u) \subset R_i(v)$

(2) $u <_i v$ if $R_i(u) || R_i(v)$ and $R_{i+1}(u) \subset R_{i+1}(v)$.

Lemma. The relation $<_i$ is acyclic.

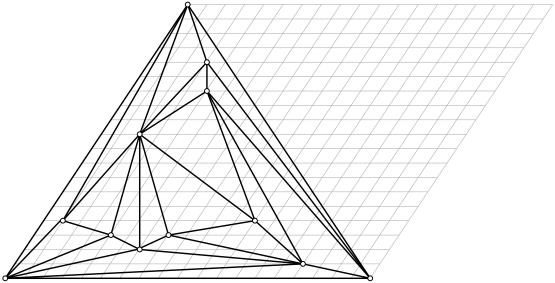
Theorem. L_1, L_2, L_3 is a realizer for $P_{VEF}(G) \setminus F$
(L_i a linear extension of $<_i$).

Outline

Schnyder's Second Theorem

Theorem [Schnyder 1989 **].**
A planar triangulation G admit a straight line drawing on the $(2n-5) \times (2n-5)$ grid.

Example.



Schnyder's Straight Line Embeddings

ϕ : *regions* $\to \mathbb{R}^+$ is S-good iff

(1) $R_i(u) \subset R_i(v) \implies \phi(R_i(u)) < \phi(R_i(v))$

(2) $\phi(R_1(v)) + \phi(R_2(v)) + \phi(R_3(v)) = C$

Theorem. The embedding $v \to f(v) = (\phi(R_1(v), \phi(R_2(v))$ is a straight line embedding in the $C \times C$ square.

- $\{x, y\} \in E$ and $z \notin \{x, y\} \implies f(z) \notin [f(x), f(y)]$

because $x, y \in R_i(z)$ for some $i \in \{1, 2, 3\}$.

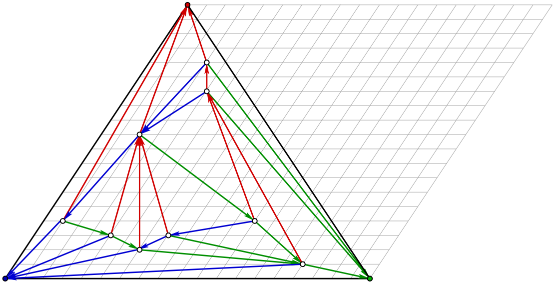
- $\{u, v\}, \{x, y\} \in E$

$x, y \in R_i(u)$, $x, y \in R_j(v)$, $u, v \in R_k(x)$, $u, v \in R_l(y)$
$\implies i = j$ or $k = l$

Grid Embeddings – Faces

The count of faces contained in a region is S-good.

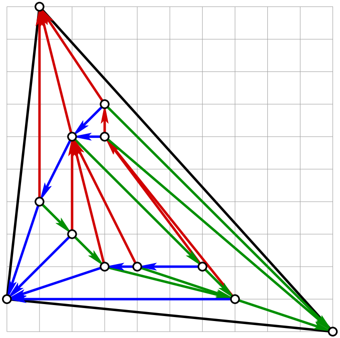
Corollary. Planar triangulations admit a straight line drawing on the $(f-1) \times (f-1)$ grid ($f = 2n - 4$).



Grid Embeddings – Vertices

Counting vertices in regions, including vertices on right path, excluding vertices on left path yields:

Theorem [Schnyder '90]. Planar triangulations admit a straight line drawing on the $(n-2) \times (n-2)$ grid.

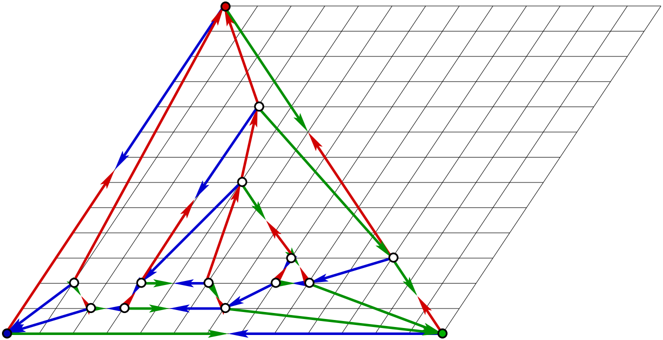


We may have
$R_i(u) \subset R_i(v)$ and
$\phi(R_i(u)) = \phi(R_i(v))$,
still:

- There is no vertex on an edge.
- Edges are separated.

Convex Drawings of 3-Connected Plane Graphs

Theorem. 3-connected planar graphs admit convex drawings on the $(f-1) \times (f-1)$ grid.



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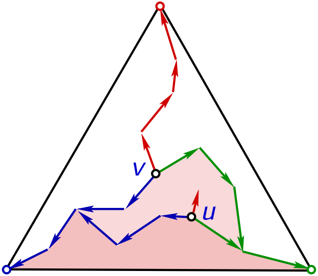
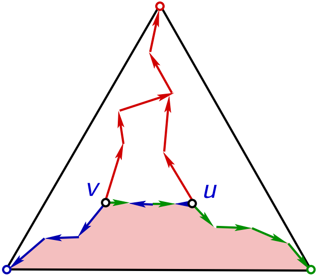
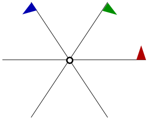
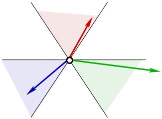




Idea of the Proof

(1) The wedges of a vertex.



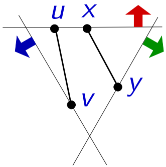


(2) No vertex on an edge.

(3) Edges are disjoint

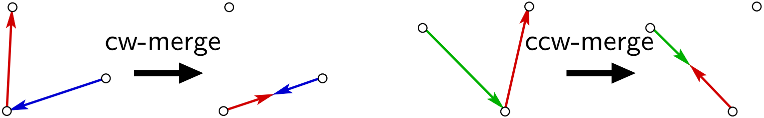
$\{u, v\}, \{x, y\} \in E$
$x, y \in R_i(u)$, $x, y \in R_j(v)$, $u, v \in R_k(x)$, $u, v \in R_l(y)$
does not imply $i = j$ or $k = l$

The hard case:



$x, y \in R_1(u)$ and $u, v \in R_1(x)$

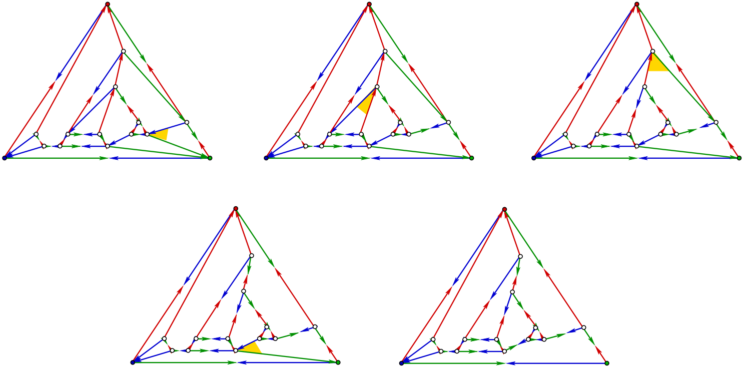
Improving the Area



Merging edges preserves the Schnyder wood properties.

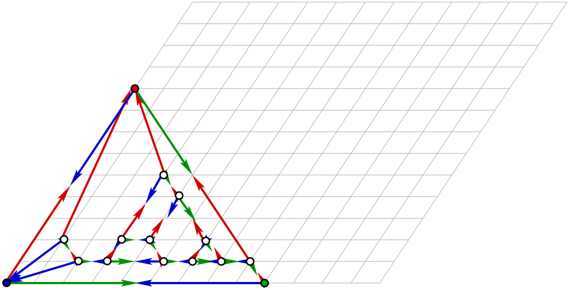
Step I: Reduction

Reduce the face count by merging edges.



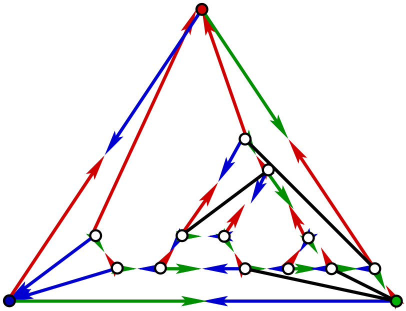
Step II: Drawing

Draw the reduced graph by counting faces on the $(f^{\downarrow} - 1) \times (f^{\downarrow} - 1)$ grid.



Step III: Drawing More

Reinsert the ‘merge edges’.



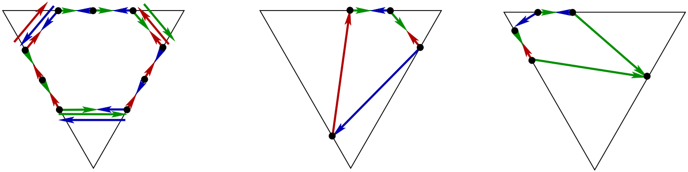
Correctness

Upon reinsertion of merge edges:

- No crossings.

(True when all merges are of same kind.)

- Convex faces.



Area Improvement with Merges

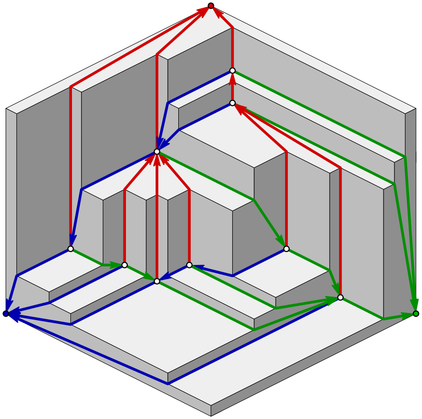
Let $\Delta^{\circlearrowleft}_{S_{\text{Min}}} \geq 0$ be the number of faces with a counterclockwise edge in each color in S_{Min}.

Theorem. A planar triangulation with n vertices has a straight line drawing on a grid of size $(n - 1 - \Delta^{\circlearrowleft}_{S_{\text{Min}}}) \times (n - 1 - \Delta^{\circlearrowleft}_{S_{\text{Min}}})$.

Theorem. A 3-connected planar graph G with n vertices has a convex drawing on a grid of size $(n - 1 - \Delta^{\circlearrowleft}_{S_{\text{Min}}}) \times (n - 1 - \Delta^{\circlearrowleft}_{S_{\text{Min}}})$.

Drawings on Orthogonal Surfaces

Using all three face count coordinates we obtain an embedding of T on an orthogonal surface.



Orthogonal Surfaces and Dimension

A surface S_X is rigid iff

$$e = (x, y) \in G_S \text{ and } z \neq x, y \text{ implies } z \not\leq e = x \vee y.$$

Remark. A rigid orthogonal surface supports a unique Schnyder wood of a 3-connected planar graph.

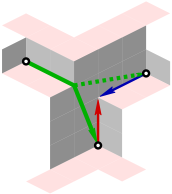
Theorem [Miller '02].
If G is the graph of a rigid orthogonal surface and P a 3-polytope with graph G, then $\dim(\mathcal{F}_P \setminus F_\infty) = 3$.

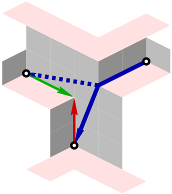
Theorem [Brightwell+Trotter '92].
If P is a 3-polytope and F is any face of P, then

- $\dim(\mathcal{F}_P \setminus F) = 3$
- $\dim(\mathcal{F}_P) = 4$

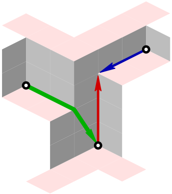
From Graphs to Rigid Surfaces

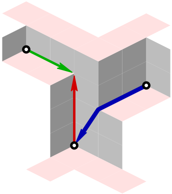
Two types of non-rigid edges





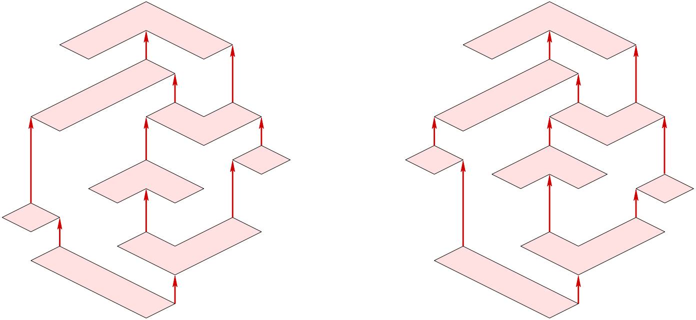
resolved by shifting red flats.





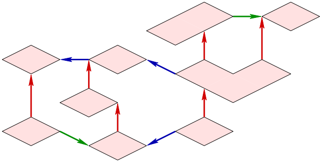
Shifting Flats

Red flats form a 2-dimensional order P_r. Compatible height functions of red flats correspond to order preserving maps $\phi : P_r \to \mathbb{R}$.



Shifting Flats

Adding arcs to prevent non-rigid edges doesn't spoil acyclicity.



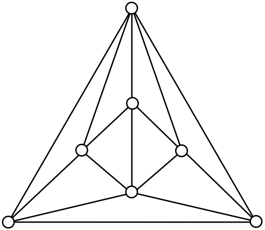
Theorem. There are compatible assignments for the heights of flats of all three colors which together yield a rigid orthogonal surface supporting the original Schnyder wood.

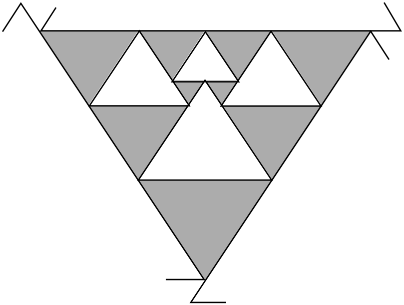
$\Longrightarrow$ the Brightwell–Trotter Theorem.

Outline

Homothetic Triangle Contact Representations

Theorem [Gonçalves, Lévêque, Pinlou (GD 2010) **].**
Every 4-connected triangulation has a triangle contact representation with homothetic triangles.





Triangle Contact Representations

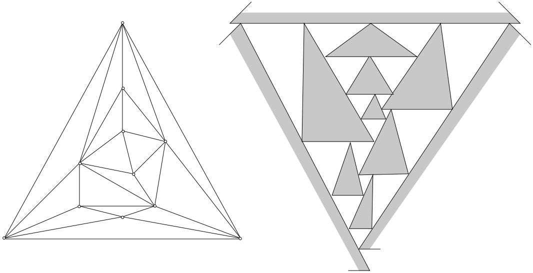
G-L-P observe that the conjecture follows from a corollary of Schramm's "Monster Packing Theorem".

Theorem. Let T be a planar triangulation with outer face $\{a, b, c\}$ and let C be a simple closed curve partitioned into arcs $\{P_a, P_b, P_c\}$. For each interior vertex v of T prescribe a convex set Q_v containing more than one point. Then there is a contact representation of T with homothetic copies.

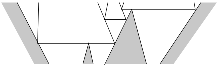
Remark. In general homothetic copies of the Q_v can degenerate to a point. Gonçalves et al. show that this is impossible if T is 4-connected.

Triangle Contact Representation

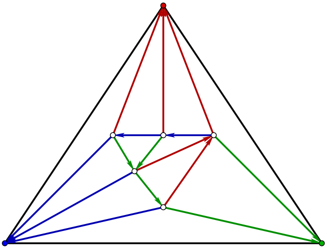
de Fraysseix, de Mendez and Rosenstiehl construct triangle contact representations of triangulations.

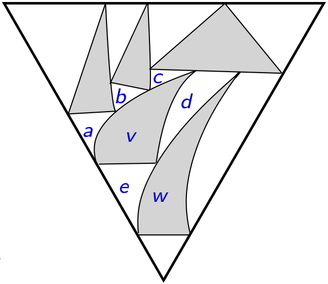


Construct along a good ordering of vertices
${\color{red}T_1} + {\color{green}T_2}^{-1} + {\color{blue}T_1}^{-1}$



Triangle Contacts and Equations





The abstract triangle contact representation implies equations for the sidelength:

$x_a + x_b + x_c = x_v$ and $x_d = x_v$ and $x_e = x_v$ and $x_d + x_e = x_w$ and $\dots$

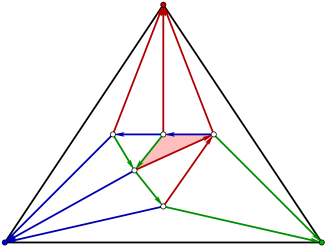
Solving the Equations

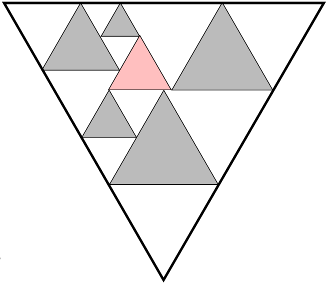
Theorem. The system of equations has a unique solution.

The proof is based on counting matchings.

In the solution some variables may be negative.

Still the solution yields a triangle contact representation.





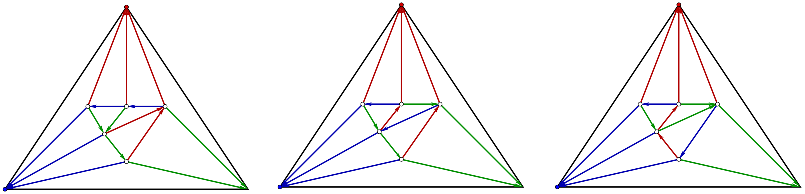
Flipping Cycles

Proposition. The boundary of a negative area is a directed cycle in the underlying Schnyder wood.

From the bijection

Schnyder woods $\Longleftrightarrow$ 3-orientations

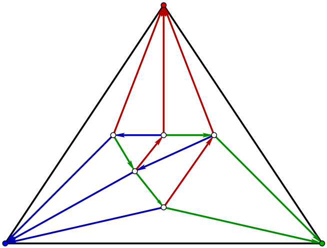
it follows that cycles can be reverted (flipped).

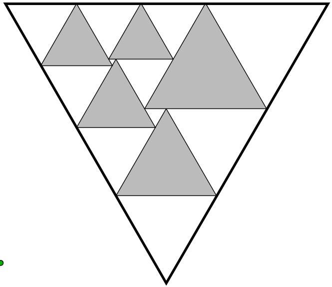


Resolving

A new Schnyder wood yields new equations and a new solution.

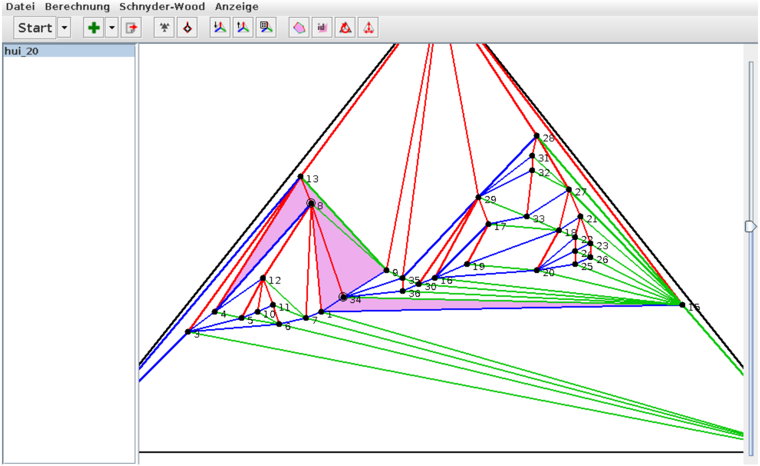
Theorem. A negative triangle becomes positive by flipping.





More Complications

It may be necessary to flip longer cycles.



Status Report and End

- We have no proof that the process always ends with a homothetic triangle representation.

The End