Schnyder Woods and Applications

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Outline

Introduction to Schnyder Woods
Dimension
Drawings
Triangle representations
Schnyder Woods

\( G = (V, E) \) a plane triangulation, 
\( F = \{a_1, a_2, a_3\} \) the outer triangle.

A coloring and orientation of the interior edges of \( G \) with colors 1, 2, 3 is a Schnyder wood of \( G \) iff

- Inner vertex condition:

  - Edges \( \{v, a_i\} \) are oriented \( v \rightarrow a_i \) in color \( i \).
The set $T_i$ of edges colored $i$ is a tree rooted at $a_i$. 

**Proof.** Count edges in a cycle — Euler

$\square$
• Paths of different color have at most one vertex in common.
• Every vertex has three distinguished regions.
If $u \in R_i(v)$ then $R_i(u) \subset R_i(v)$. 
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Schnyder’s Dimension Theorem

$Q_i$ the inclusion order with respect to $R_i(.)$  
$L_i$ a linear extension of $Q_i$  
$L_i^+$ include edges as low in $L_i$ as possible  

$\implies$  
$L_1^+, L_2^+, L_3^+$ is a realizer for $P_G$.

**Proof.** $e = \{u, v\}$ and edge, $x \not\in e$ a vertex.  
We need:

![Diagram showing a linear extension $L$ with vertices $u$, $v$, and $x$, and an edge $e$ between $u$ and $v$.]
**Schnyder’s Dimension Theorem**

$Q_i$ the inclusion order with respect to $R_i(.)$
$L_i$ a linear extension of $Q_i$
$L_i^+$ include edges as low in $L_i$ as possible

$\implies L_i^+, L_i^+, L_i^+$ is a realizer for $P_G$.

**Proof.** $e = \{u, v\}$ and edge, $x \notin e$ a vertex.

We need:

There is an $i$ with $e \in R_i(x)$

$\implies R_i(u) \subset R_i(x)$ and $R_i(v) \subset R_i(x)$

$\implies$ Edge $e$ goes below $x$ in $L_i^+$.  

**Dimension of 3-Polytopes**

**Theorem [ Schnyder 1989 ].**
If $G$ is a plane triangulation with a face $F$, then

- $\dim(P_{VEF}(G) \setminus F) = 3$
- $\dim(P_{VEF}(G)) = 4$

**Theorem [ Brightwell+Trotter 1993 ].**
If $G$ is a 3-connected plane graph with a face $F$, then

- $\dim(P_{VEF}(G) \setminus F) = 3$
- $\dim(P_{VEF}(G)) = 4$
Schnyder Woods

Axioms for 3-coloring and orientation of bi-edges:

(W1 - W2) Rule of edges and half-edges:

(W3) Rule of vertices:

(W4) There is no interior face whose boundary is a directed cycle in one color.
We need W4!
3-Orientations?
Primal and Dual

A Schnyder wood of $G$ induces a Schnyder wood of the dual of $G$. 
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3-Orientations – Rescued

**Theorem.** Primal dual Schnyder woods are in bijection with primal dual 3-orientations.
Schnyder Woods - Regions

- If $u \in R_i^o(v)$ then $R_i(u) \subset R_i(v)$.
- If $u \in \partial R_i(v)$ then $R_i(u) \subseteq R_i(v)$

(equality, iff there is a bi-directed path between $u$ and $v$.)
Brigthwell Trotter Theorem

- \( P \) a 3-polytope with face \( F \).
- \( G \) the graph of \( P \) with outer face \( F \).
- \( S \) a Schnyder wood of \( G \).

Define \( <_i \) on \( V_P \) for \( i = 1, 2, 3 \)

1. \( u <_i v \) if \( R_i(u) \subset R_i(v) \)
2. \( u <_i v \) if \( R_i(u) \parallel R_i(v) \) and \( R_{i+1}(u) \subset R_{i+1}(v) \).

**Lemma.** The relation \( <_i \) is acyclic.

**Theorem.** \( L_1, L_2, L_3 \) is a realizer for \( P_{VEF}(G) \setminus F \)
( \( L_i \) a linear extension of \( <_i \)).
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Schnyder’s Second Theorem

**Theorem** [Schnyder 1989]. A planar triangulation $G$ admit a straight line drawing on the $(2n - 5) \times (2n - 5)$ grid.

**Example.**
Schnyder’s Straight Line Embeddings

\[ \phi : \text{regions} \rightarrow \mathbb{R}^+ \text{ is S-good iff} \]

(1) \( R_i(u) \subset R_i(v) \implies \phi(R_i(u)) < \phi(R_i(v)) \)

(2) \( \phi(R_1(v)) + \phi(R_2(v)) + \phi(R_3(v)) = C \)

**Theorem.** The embedding \( v \rightarrow f(v) = (\phi(R_1(v)), \phi(R_2(v))) \) is a straight line embedding in the \( C \times C \) square.

- \( \{x, y\} \in E \) and \( z \not\in \{x, y\} \implies f(z) \not\in [f(x), f(y)] \) because \( x, y \in R_i(z) \) for some \( i \in \{1, 2, 3\} \).

- \( \{u, v\}, \{x, y\} \in E \)

\[ x, y \in R_i(u), x, y \in R_j(v), u, v \in R_k(x), u, v \in R_l(y) \implies i = j \text{ or } k = l \]
Grid Embeddings – Faces

The count of faces contained in a region is S-good.

**Corollary.** Planar triangulations admit a straight line drawing on the \((f - 1) \times (f - 1)\) grid \((f = 2n - 4)\).
Grid Embeddings – Vertices

Counting vertices in regions, including vertices on right path, excluding vertices on left path yields:

**Theorem [Schnyder ’90].** Planar triangulations admit a straight line drawing on the \((n - 2) \times (n - 2)\) grid.

We may have 
\[ R_i(u) \subset R_i(v) \]
and 
\[ \phi(R_i(u)) = \phi(R_i(v)) \],
still:

- There is no vertex on an edge.
- Edges are separated.
**Theorem.** 3-connected planar graphs admit convex drawings on the $(f - 1) \times (f - 1)$ grid.
Schnyder Woods - Regions

- If $u \in R_i^o(v)$ then $R_i(u) \subset R_i(v)$.
- If $u \in \partial R_i(v)$ then $R_i(u) \subseteq R_i(v)$
  (equality, iff there is a bi-directed path between $u$ and $v$.)
Idea of the Proof

(1) The wedges of a vertex.

(2) No vertex on an edge.

(3) Edges are disjoint

\{u, v\}, \{x, y\} \in E

\(x, y \in R_i(u)\), \(x, y \in R_j(v)\), \(u, v \in R_k(x)\), \(u, v \in R_l(y)\)

does not imply \(i = j\) or \(k = l\)

The hard case:

\(x, y \in R_1(u)\) and \(u, v \in R_1(x)\)
Improving the Area

Merging edges preserves the Schnyder wood properties.
Step I: Reduction

Reduce the face count by merging edges.
Step II: Drawing

Draw the reduced graph by counting faces on the \((f↓ - 1) \times (f↓ - 1)\) grid.
Step III: Drawing More

Reinsert the ‘merge edges’.
Correctness

Upon reinsertion of merge edges:

- No crossings.
  (True when all merges are of same kind.)
- Convex faces.
Let $\Delta_{S_{\text{Min}}}^{(2)} \geq 0$ be the number of faces with a counterclockwise edge in each color in $S_{\text{Min}}$.

**Theorem.** A planar triangulation with $n$ vertices has a straight line drawing on a grid of size $(n - 1 - \Delta_{S_{\text{Min}}}^{(2)}) \times (n - 1 - \Delta_{S_{\text{Min}}}^{(2)})$.

**Theorem.** A 3-connected planar graph $G$ with $n$ vertices has a convex drawing on a grid of size $(n - 1 - \Delta_{S_{\text{Min}}}^{(2)}) \times (n - 1 - \Delta_{S_{\text{Min}}}^{(2)})$. 
Using all three face count coordinates we obtain an embedding of $T$ on an orthogonal surface.
Orthogonal Surfaces and Dimension

A surface $S_X$ is rigid iff

$$e = (x, y) \in G_S \text{ and } z \neq x, y \text{ implies } z \not\leq e = x \lor y.$$  

**Remark.** A rigid orthogonal surface supports a unique Schnyder wood of a 3-connected planar graph.

**Theorem [Miller '02].**
If $G$ is the graph of a rigid orthogonal surface and $P$ a 3-polytope with graph $G$, then $\dim(\mathcal{F}_P \setminus F_\infty) = 3$.

**Theorem [Brightwell+Trotter '92].**
If $P$ is a 3-polytope and $F$ is any face of $P$, then

- $\dim(\mathcal{F}_P \setminus F) = 3$
- $\dim(\mathcal{F}_P) = 4$
From Graphs to Rigid Surfaces

Two types of non-rigid edges

resolved by shifting red flats.
Red flats form a 2-dimensional order $P_r$. Compatible height functions of red flats correspond to order preserving maps $\phi : P_r \to \mathbb{R}$. 
Adding arcs to prevent non-rigid edges doesn’t spoil acyclicity.

**Theorem.** There are compatible assignments for the heights of flats of all three colors which together yield a rigid orthogonal surface supporting the original Schnyder wood.

⇒ the Brightwell–Trotter Theorem.
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Homothetic Triangle Contact Representations

**Theorem [ Gonçalves, Lévêque, Pinlou (GD 2010)].**
Every 4-connected triangulation has a triangle contact representation with homothetic triangles.
Triangle Contact Representations

G-L-P observe that the conjecture follows from a corollary of Schramm’s “Monster Packing Theorem”.

**Theorem.** Let $T$ be a planar triangulation with outer face $\{a, b, c\}$ and let $C$ be a simple closed curve partitioned into arcs $\{P_a, P_b, P_c\}$. For each interior vertex $v$ of $T$ prescribe a convex set $Q_v$ containing more than one point. Then there is a contact representation of $T$ with homothetic copies.

**Remark.** In general homothetic copies of the $Q_v$ can degenerate to a point. Gonçalves et al. show that this is impossible if $T$ is 4-connected.
de Fraysséix, de Mendez and Rosenstiehl construct triangle contact representations of triangulations.

Construct along a good ordering of vertices

\[ T_1 + T_2^{-1} + T_1^{-1} \]
The abstract triangle contact representation implies equations for the sidelength:

\[ x_a + x_b + x_c = x_v \quad \text{and} \quad x_d = x_v \quad \text{and} \quad x_e = x_v \quad \text{and} \quad x_d + x_e = x_w \quad \text{and} \quad \ldots \]
Solving the Equations

**Theorem.** The system of equations has a unique solution.
The proof is based on counting matchings.
In the solution some variables may be **negative**.
Still the solution yields a triangle contact representation.
Proposition. The boundary of a negative area is a directed cycle in the underlying Schnyder wood.

From the bijection

Schnyder woods $\iff$ 3-orientations

it follows that cycles can be reverted (flipped).
A new Schnyder wood yields new equations and a new solution.

**Theorem.** A negative triangle becomes positive by flipping.
More Complications

It may be necessary to flip longer cycles.
Status Report and End

- We have no proof that the process always ends with a homothetic triangle representation.

The End