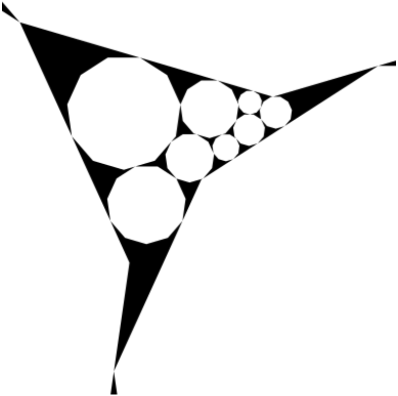
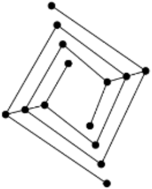
K-gons and Circles

Spring School SGT 2018
Seté, June 11-15, 2018

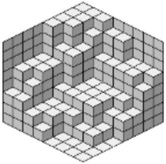
Stefan Felsner
Technische Universität Berlin





Fall School: Order and Geometry

September 14 - 17, 2018



ABOUT:

The school is addressed to advanced undergraduate and graduate students of Mathematics or Computer Science with interest in Discrete Mathematics and in particular in Partially Ordered Sets and Discrete Geometry. Basic knowledge in discrete mathematics is assumed. The fall school consists of four introductory courses. Each course consists of lectures and exercises.

LOCATION:

The fall school will take place at Gutshof Sauen, a remote manor 70 km southeast of Berlin. The costs per participant are EURO 150 and includes full board during the fall school.

Fall School Order and Geometry

LECTURES

- Jean Cardinal, (Université Libre de Bruxelles, Belgium)
  Topics on flip graphs
- Bartosz Walczak, (Jagiellonian University, Kraków, Poland)
- Vida Dujmović, (University of Ottawa, Canada)
- Oswin Aichholzer, (Technische Universität Graz, Austria)
  Crossing numbers of complete and complete bipartite graphs

APPLICATION

- Open until July 31, 2018

`page.math.tu-berlin.de/~felsner/FoC/OGschool2018.html`

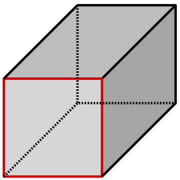
Outline

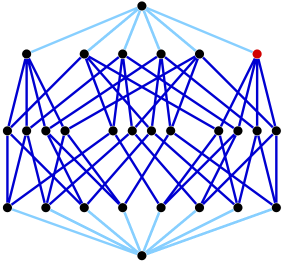
Order Dimension of Planar Graphs – Revisited

Contact Representations with Pentagons

Primal–Dual Contact Representations with Circles

Brightwell–Trotter Theorem I

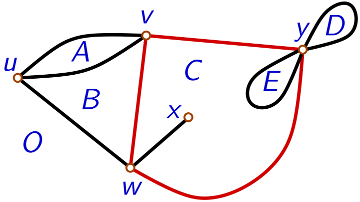


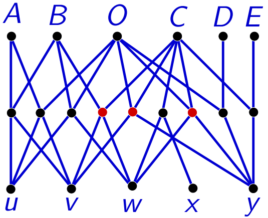


Theorem [Brightwell+Trotter '93].
If G is a 3-connected plane graph with a face F, then

- $\dim(P_{VEF}(G \setminus F)) = 3$
- $\dim(P_{VEF}(G)) = 4$

Brightwell–Trotter Theorem II





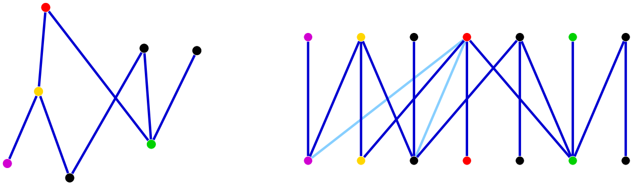
Theorem [Brightwell+Trotter '97].
If G is a plane multi-graph with loops, then

$$\dim(P_{VEF}(G)) \leq 4.$$

Splits and Dimension

The split of $P = (X, <)$ is $\text{split}(P) = (X' \cup X'', <_s)$ with

$$x' <_s y'' \qquad \text{iff} \qquad x \leq y$$

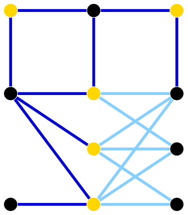


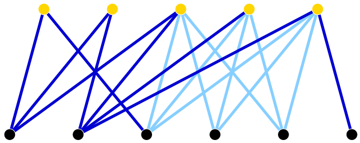
Theorem [Kimble 78].

$$\dim(P) \leq \dim(\text{split}(P)) \leq \dim(P) + 1.$$

Bipartite Orders

A bipartite Graph can be viewed as a height 2 order.

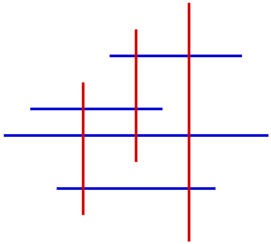


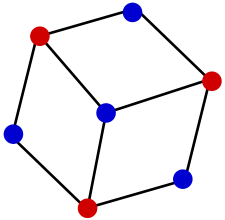


- We can talk about $\dim(G)$ when G is bipartite.

Grid Intersection Graphs

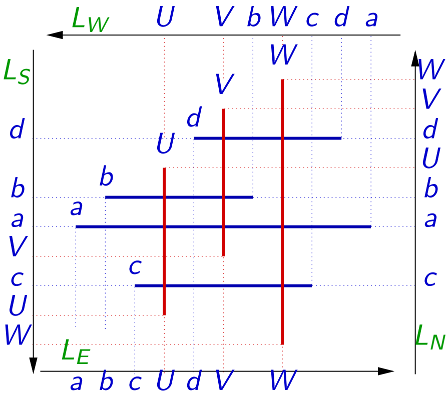
A GIG is an intersection graphs of horizontal and vertical segments (no two on a common line).





- GIGs are bipartite.

Dimension of GIGs



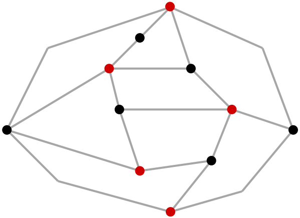
- In each projection minimals (vert) are taken early, maximals (hor) are taken late.

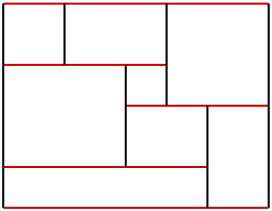
Theorem. G a GIG, then $\dim(G) \leq 4$.

Theorem [generalization]. If a bipartite graph $G = (X, Y; E)$ has a representation as intersection graphs of objects from a t-separable class, then $\dim(G) \leq 2t$.

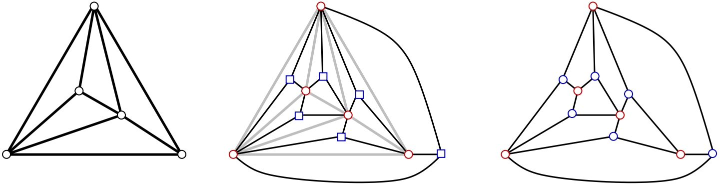
Planar Bipartite Graphs

Theorem. Every planar bipartite graph H admits a contact representation with interiorly disjoint horizontal and vertical segments.





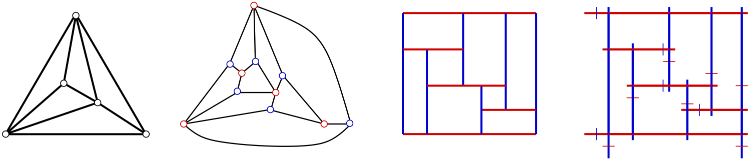
Angle Graphs of Planar Graphs



- Angle graphs are planar bipartite.

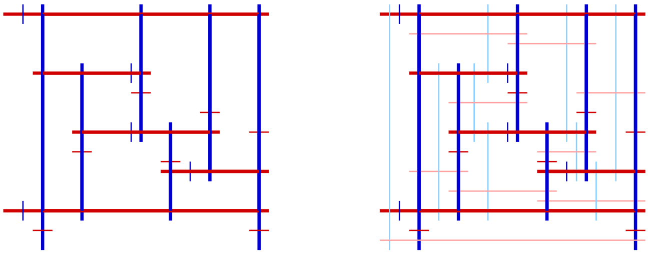
The First Step

G 2-connected plane multi-graph (no loops).



- The order dimension of $P_{VF}(G)$, the incidence order of vertices and faces of a planar multigraph G (no loops) is at most four, moreover $\dim(\mathrm{split}(P_{VF}(G)) \leq 4$.

Adding the Edges



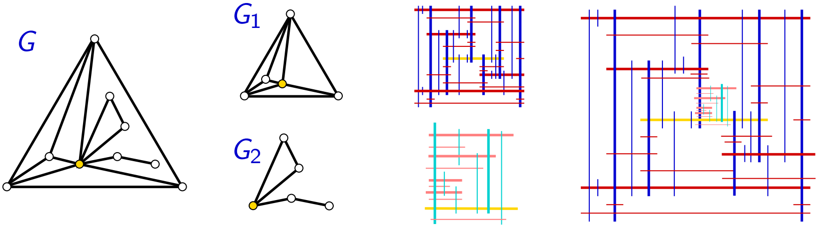
Theorem. If G is a 2-connected and plane multigraph, then $\dim(\mathrm{split}(P_{VEF}(G)) \leq 4$.

Loops

- Break loops by inserting a new vertex. $\mathrm{split}(P_{VEF}(G))$ is a suborder of $\mathrm{split}(P_{VEF}(G^{+}))$

Cut Vertices

- Use induction: break G into G_1 and G_2 at a cut vertex:



Theorem. If G is a plane multigraph -loops allowed-, then $\dim(\mathrm{split}(P_{VEF}(G)) \leq 4$.

Outline

Order Dimension of Planar Graphs – Revisited

Contact Representations with Pentagons

Primal–Dual Contact Representations with Circles

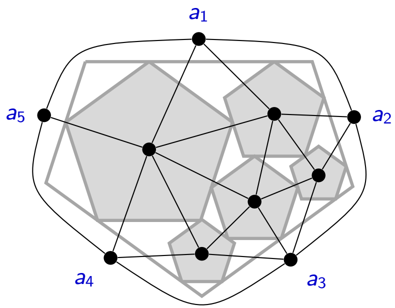
Pentagon contact representations

Stefan Felsner[1] Hendrik Schrezenmaier[1] Raphael Steiner[2]

[1]Technische Universität Berlin

[2]FernUniversität in Hagen

Regular Pentagon Contact Representations



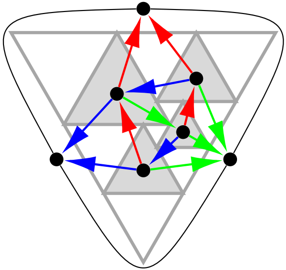
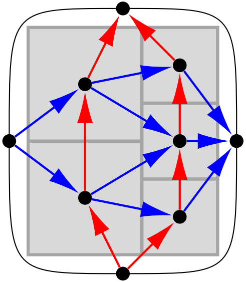
Triangulation G of 5-gon $a_1, \ldots, a_5$

- no multiple edges
- no loops
- no chords

Questions:

- Existence
- Uniqueness
- Combinatorial structure
- Computation

Why Pentagons?

	Triangles	**Squares**
		
Existence	for all 4-connected G [Gonçalves et. al. '11]	for all 5-connected G [Schramm '93]
Uniqueness	open	yes [Schramm '93]
Comb. structure	Schnyder wood	transversal structure
Computation	heuristic [Felsner '09]	heuristic [Felsner '13], iterative [Schramm '93] optimization [Lovász '06]

Existence

Theorem

Each triangulation G of a 5-gon admits a regular pentagon contact representation.

Proof.

Application of Convex Packing Theorem [Schramm '90]. $\square$

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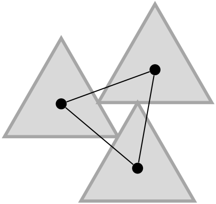
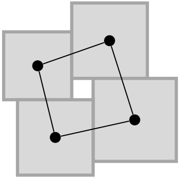
Remember: For triangles G has to be 4-connected, for squares 5-connected.





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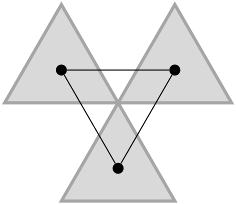

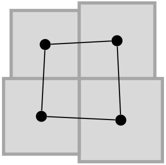

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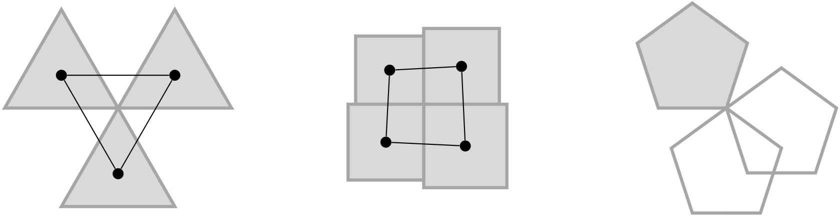
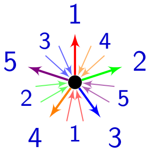
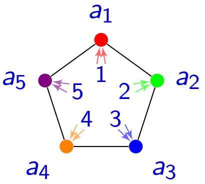


The combinatorial structure: five color forests

Definition (Five color forest)

Orientation and coloring of inner edges of inner triangulation of 5-gon $a_1, \dots, a_5$, s.t.

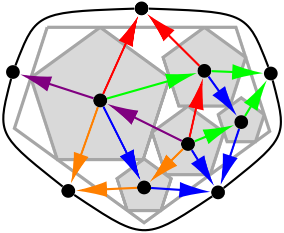


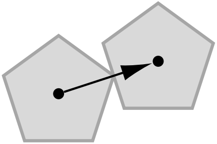


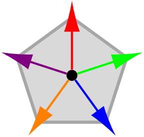
- no incoming edge of color i $\Rightarrow$ outgoing edge of color $i-2$ or $i+2$ exists

Theorem

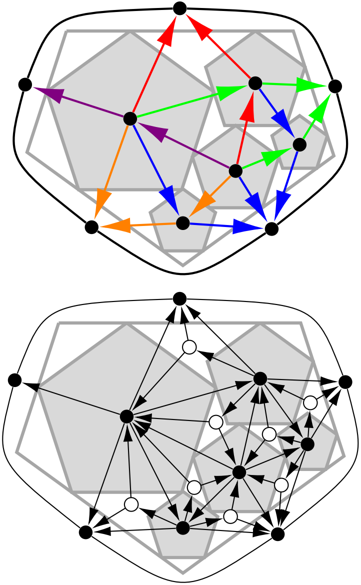
Regular pentagon contact representation induces five color forest on its contact graph.

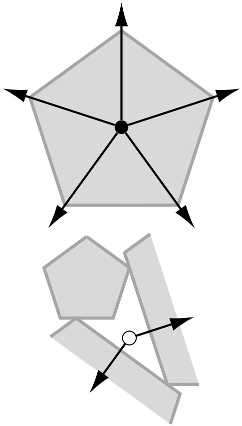






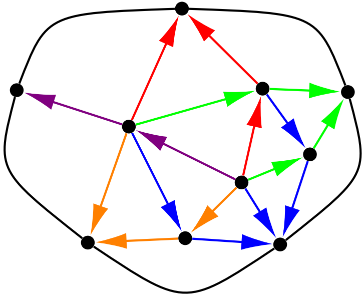
Five color forests $\leftrightarrow$ α-orientations

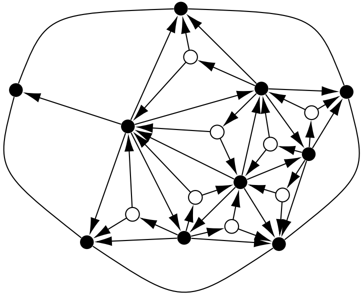




- $\mathrm{outdeg}(\bullet) = 5$
- $\mathrm{outdeg}(\circ) = 2$

Five color forests $\leftrightarrow$ α-orientations



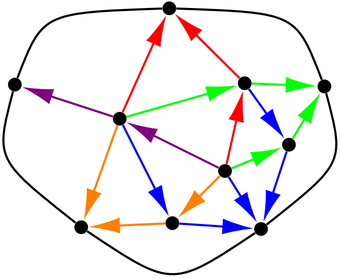


Theorem

There is a bijection between the five color forests and α-orientations of a graph G.

System of linear equations

Computing a regular pentagon contact representation induced by a fixed five color forest



System of linear equations

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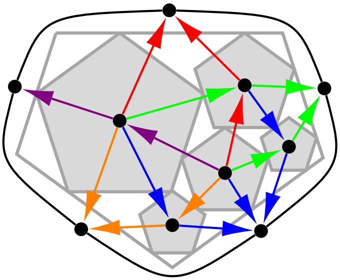

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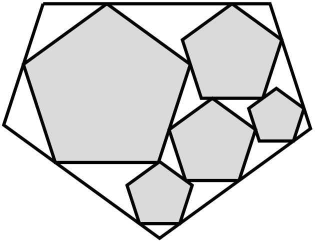

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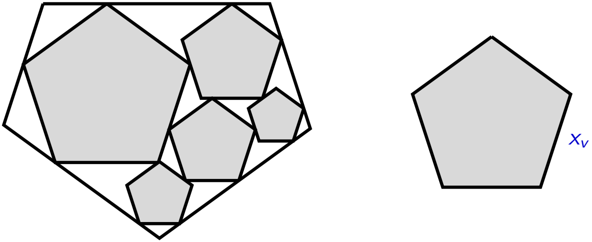

- Variables:
  - one side length for each vertex: x_v

System of linear equations

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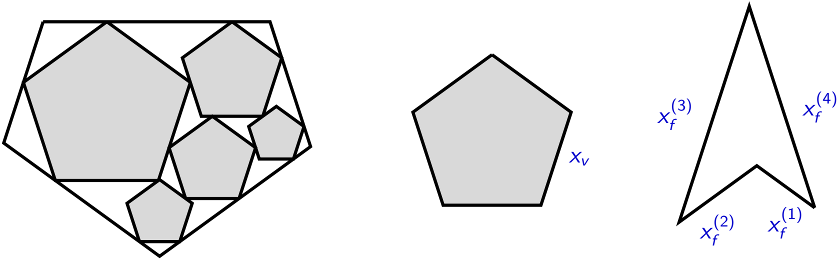

- Variables:
  - one side length for each vertex: x_v
  - four side lengths for each face: $x_f^{(1)}, \ldots, x_f^{(4)}$

System of linear equations

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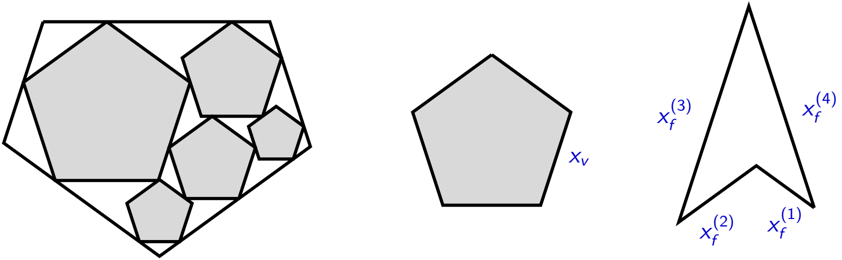

- Variables:
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- Equations:
  - five for each vertex: $x_v =$ sum of touching face side lengths

System of linear equations

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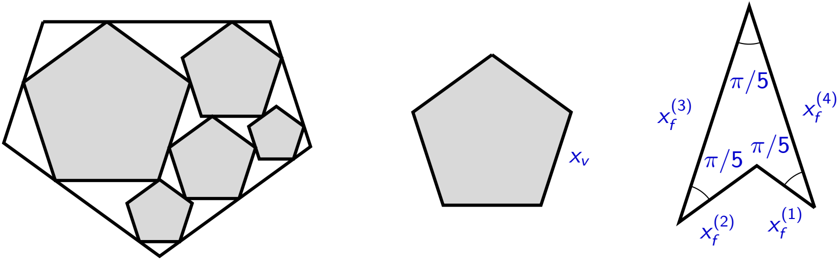

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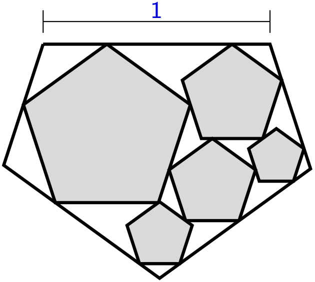
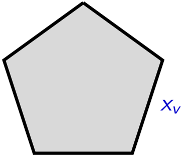
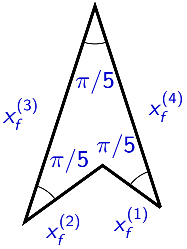
  - two for each face: $x_f^{(3)} = x_f^{(1)} + \phi x_f^{(2)} \;, \quad x_f^{(4)} = \phi x_f^{(1)} + x_f^{(2)}$

System of linear equations

Computing a regular pentagon contact representation induced by a fixed five color forest







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  - two for each face: $x_f^{(3)} = x_f^{(1)} + \phi x_f^{(2)}$, $x_f^{(4)} = \phi x_f^{(1)} + x_f^{(2)}$
  - one inhomogeneous: length of upper segment $= 1$

System of linear equations

Computing a regular pentagon contact representation induced by a fixed five color forest

Lemma

The system $A_F x = \mathbf{e_1}$ is uniquely solvable.

Lemma

$x \geq 0 \;\; \Leftrightarrow$ *there is a regular pentagon contact repr. inducing F*

System of linear equations

Computing a regular pentagon contact representation induced by a fixed five color forest

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Lemma

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Corollary

computing a regular pentagon repr.

can be done by

finding a five color forest F

s.t. $A_F x = \mathbf{e_1}$ *has nonnegative solution*

Heuristic

- Guess the right five color forest F

Heuristic

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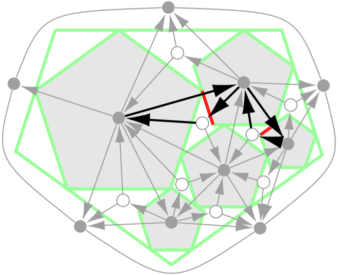
- **Case 1**: solution of $A_F x = \mathbf{e_1}$ is nonnegative
  - construct contact repr. from solution

Heuristic

- Guess the right five color forest F
- **Case 1**: solution of $A_F x = \mathbf{e_1}$ is nonnegative
  - construct contact repr. from solution
- **Case 2**: solution contains negative and nonnegative variables
  - **Lemma**: neg. and nonneg. variables are separated by oriented cycles in the α-orientation



Heuristic

- Guess the right five color forest F
- **Case 1**: solution of $A_F x = \mathbf{e_1}$ is nonnegative
  - construct contact repr. from solution
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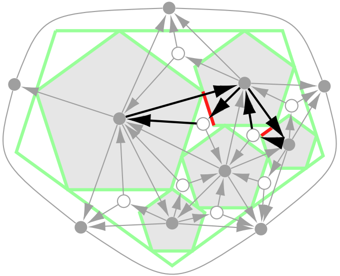
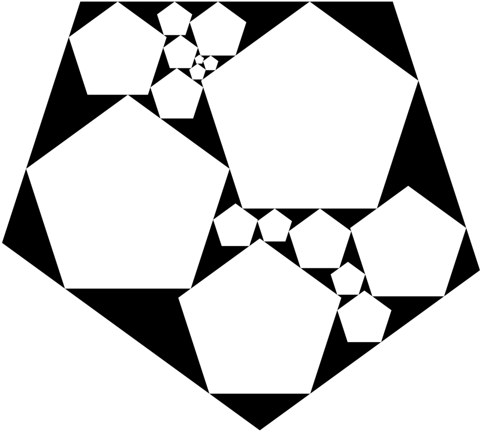


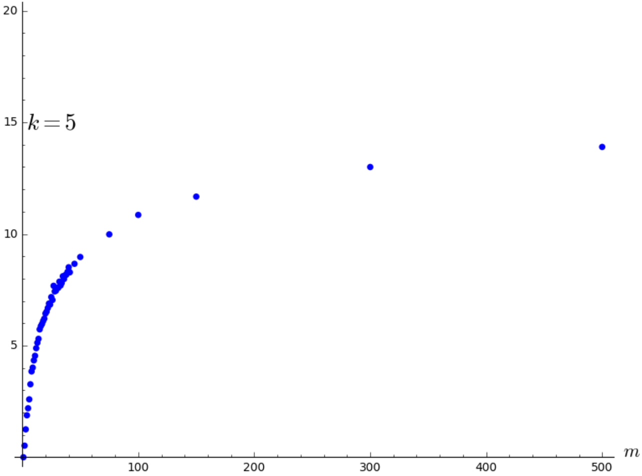
  - change orientation of these cycles
  - restart with new α-orientation, resp. five color forest

Example

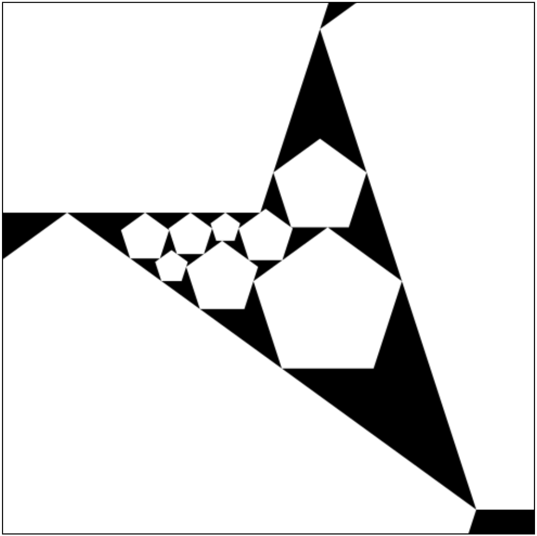


www3.math.tu-berlin.de/diskremath/research/kgon-representations

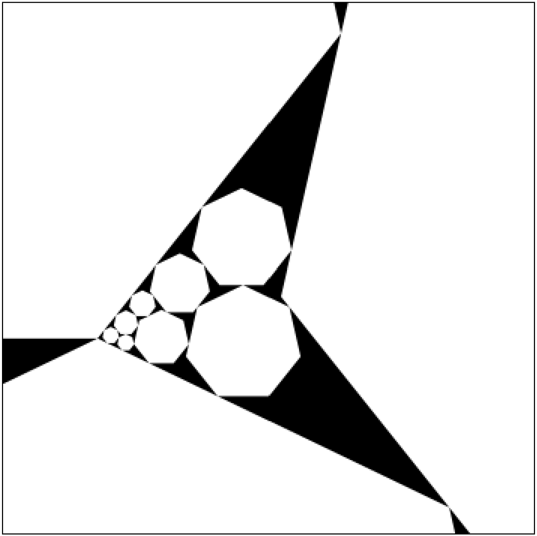
Number of iterations



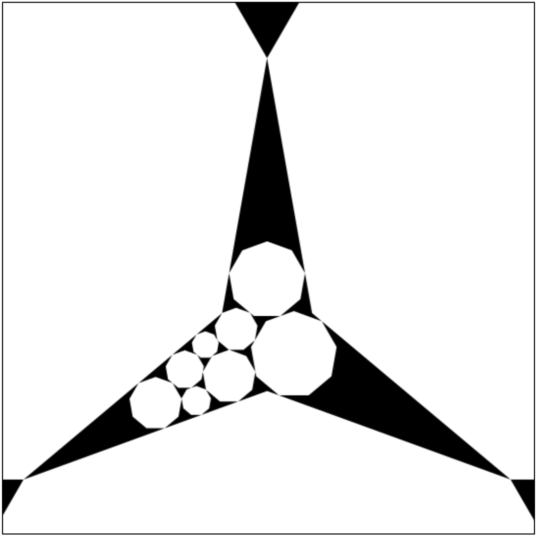
Example



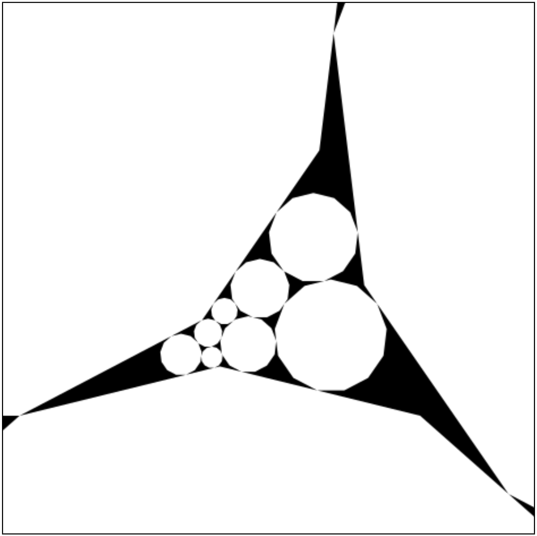
Example



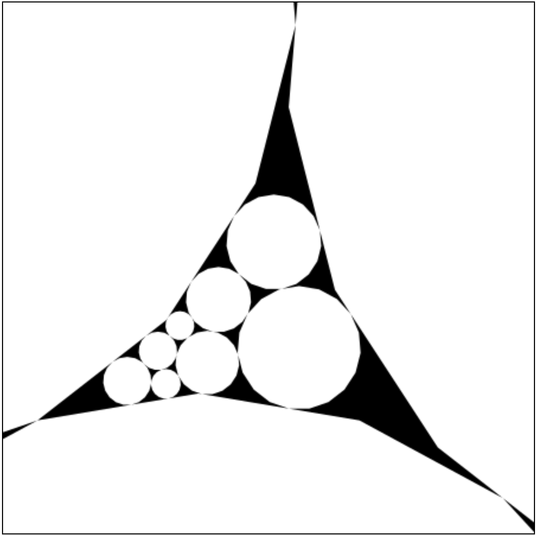
Example



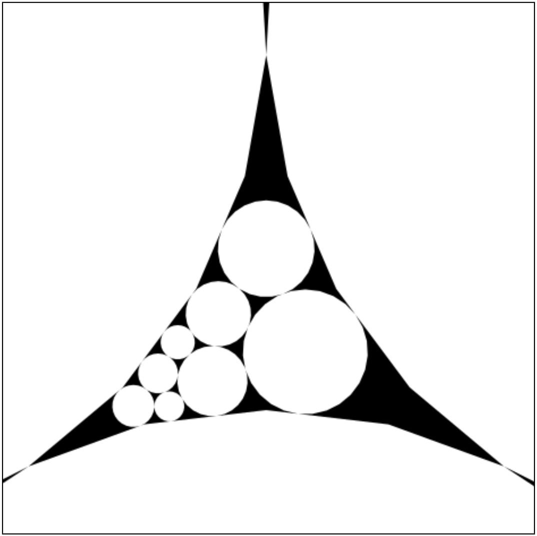
Example



Example



Example



Open problems

Conjecture

The heuristic terminates for each graph and each five color forest to start with.

Conjecture

Regular pentagon contact representations can be computed in polynomial time.

Conjecture

Regular pentagon contact representations are unique.

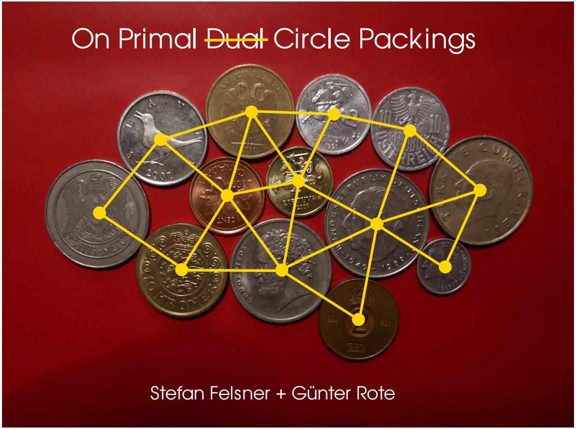
Outline

Order Dimension of Planar Graphs – Revisited

Contact Representations with Pentagons

Primal–Dual Contact Representations with Circles

Coin Graphs



1985 – Ringel's coin graph conjecture.

Circle Packing – History

- Koebe 1936
- Andreev 1970
- Thurston 1978
- Sachs 1991
- Colin de Verdière 1989
- Colin de Verdière 1991
- Brightwell and Scheinerman 1993
- Pulleyblank and Rote 1992
- Mohar 1997

Primal-Dual Circle Representation – The Statement

$G = (V, E, F)$ 3-connected plane.

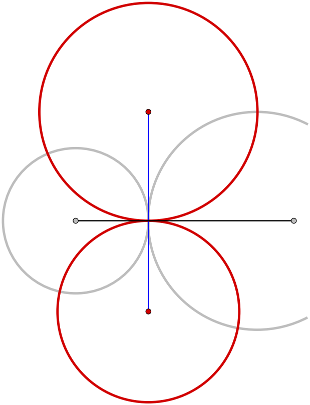
$(C_x : x \in V)$ and $(D_y : y \in F)$ families of circles.

Properties:

(i) primal contact structure.

(ii) dual contact structure.

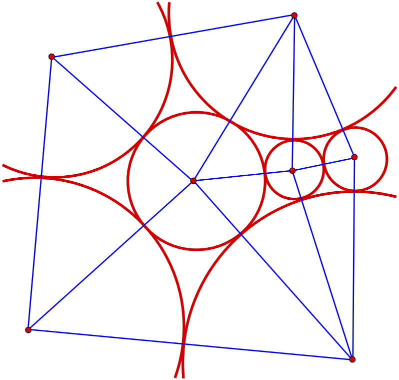
(iii) orthogonality of two straight-line drawings.



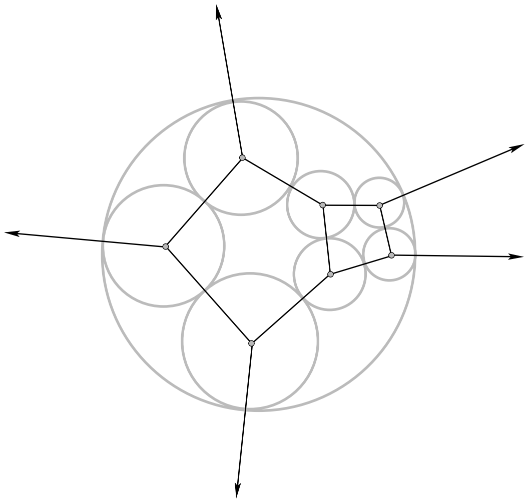
Theorem.
G 3-con. plane $\implies$ G has a primal-dual circle representation.

- The representation is unique up to Möbius transformations.

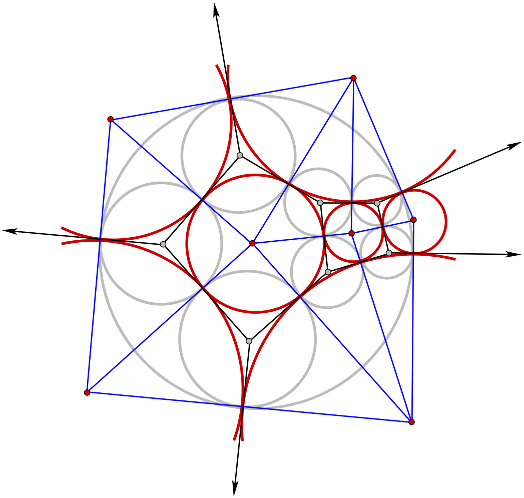
Primal Contact Structure



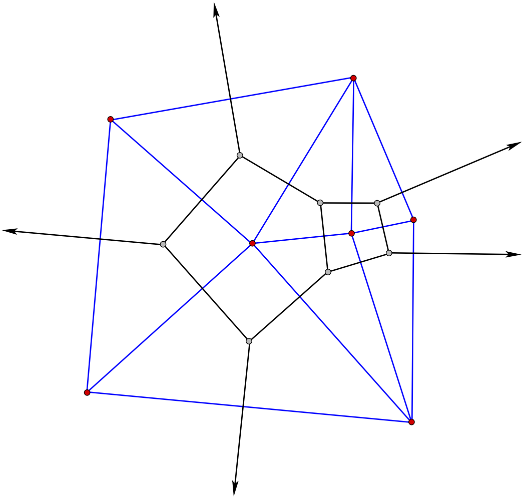
Dual Contact Structure



Orthogonality

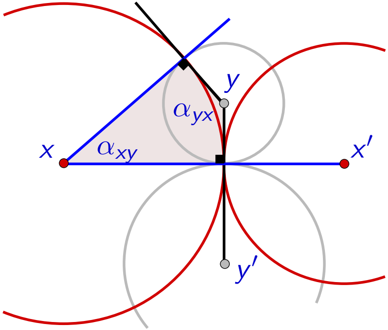


Primal and Dual



Kites

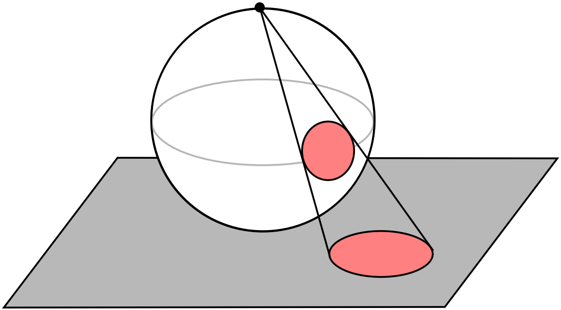
- A kite for each incidence (x, y) with $x \in V$, $y \in F \setminus f_o$.



$$\alpha_{xy} = 2\arctan\frac{r_y}{r_x} \text{ and } \alpha_{yx} = 2\arctan\frac{r_x}{r_y}$$

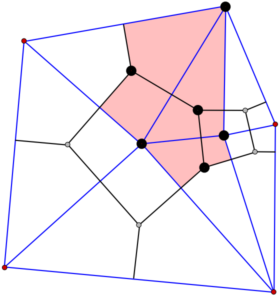
Making the Outer Face a Triangle

- Stereographic projections map between primal-dual circle representation of G in plane and on sphere.
- G or G^* has a vertex of degree 3.



Counting Lemma

- Every subset $S \subseteq V \cup F \setminus f_o$ supports at most $2|S| - 5$ kites.



Target Angles

- wlog. the outer triangle equilateral (Möbius transformation).
- target angles:

$$\beta(u) = \begin{cases} \pi/3 & \text{if } u \text{ is an outer vertex of } G \\ 2\pi & u \text{ some other vertex or face} \neq f_o. \end{cases}$$

$$\sum_u \beta(u) = \big((|V|-3) + (|F|-1)\big)2\pi + 3\frac{\pi}{3} = |K|\pi.$$

For any choice of radii the target angles are attained *on average*

$$\alpha(u) = \sum_{w:\ uw \in K} \alpha_{uw} \quad \Longrightarrow \quad \sum_u \alpha(u) = \sum_{xy \in K} \pi = |K|\pi$$

The Iteration

- Start with any radii $r : U \to \mathbb{R}_+$.
- $U_+ = \{u \in U \colon \alpha(u) > \beta(u)\}$
- If $U_+ = \emptyset$ then $\alpha(u) = \beta(u)$ for all $u \in U$.
- iterate

repeat forever:
  for all $u \in U$:
    if $u \in U_+$ **then**
      increase r_u to make $\alpha(u) = \beta(u)$

Radii Converge I

Radii are only increased $\implies$ if bounded then convergent.

- Let $D \subset U$ be the divergent set.
- If $u \in D$ and $w \in U \setminus D$, then $\alpha_{uw} \to 0$.
- For $\varepsilon > 0$ iterate until for each $u \in D$: $\displaystyle\sum_{w \in U \setminus D:\ uw \in K} \alpha_{uw} \le \frac{\varepsilon}{|U|}$.

$$
\begin{aligned}
\sum_{u \in D} \alpha(u) &\le \varepsilon + \sum_{\text{kite with } x, y \in D} (\alpha_{xy} + \alpha_{yx}) \\
&\le \varepsilon + (2|D| - 5)\pi
\end{aligned}
$$

Radii Converge II

$$\sum_{u \in D} \alpha(u) = \sum_{u \in D \cap U_+} \alpha(u) \quad > \quad \sum_{u \in D} \beta(u) \quad = \quad 2\pi|D| - \frac{5|D_o|}{3}\pi.$$

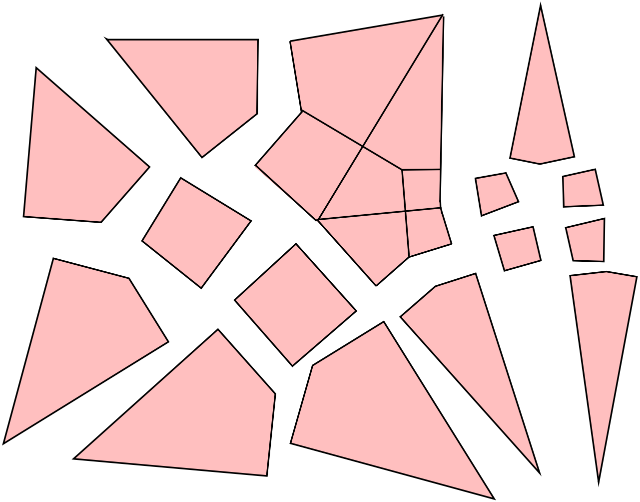
- $2\pi|D| - \dfrac{5|D_o|}{3}\pi \quad < \quad \varepsilon + (2|D| - 5)\pi$

From this we get:

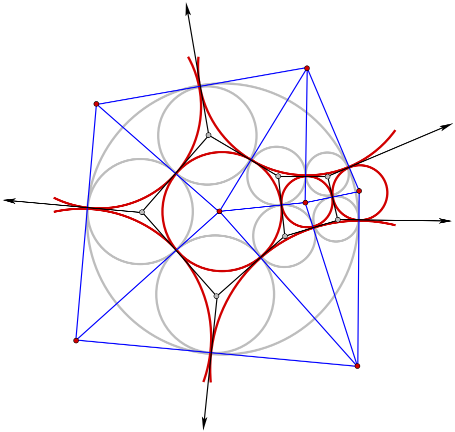
- $|D_o| = 3$.
- $2|D| - 5$ supported kites.

$\implies \quad D = U$ this contradicts $D \subset U$.

Laying Out The Kites



The End



Thank You