Tutorial 08 – Polynomial hierarchy

Exercise 1. *End of the space*

✎ Show that for every space-constructible function $S(n) \geq \log n$, $\mathsf{NSPACE}(S(n)) = \mathsf{coNSPACE}(S(n))$. Hint: Remember Immerman-Szelepcsényi theorem and its proof.

Exercise 2.

1. Show that Σ_i^{P} and Π_i^{P} are closed under polynomial time reductions.
2. Show that Σ_iSat is Σ_i^{P}-complete.
3. Show that $\Sigma_i^{\mathsf{P}} = \bigcup_c \Sigma_i\mathsf{TIME}(n^c)$ and $\Pi_i^{\mathsf{P}} = \bigcup_c \Pi_i\mathsf{TIME}(n^c)$.
4. Show that for $i \geq 1$, $\Sigma_{i+1}^{\mathsf{P}} = \mathsf{NP}^{\Sigma_i\text{Sat}} = \mathsf{NP}^{\Pi_i\text{Sat}}$.
5. Justify the notations $\Sigma_{i+1}^{\mathsf{P}} = \mathsf{NP}^{\Sigma_i^{\mathsf{P}}} = \mathsf{NP}^{\Pi_i^{\mathsf{P}}}$, and $\Pi_{i+1}^{\mathsf{P}} = \mathsf{coNP}^{\Sigma_i^{\mathsf{P}}} = \mathsf{coNP}^{\Pi_i^{\mathsf{P}}}$.

Exercise 3. $\mathsf{NP}^{\mathsf{NP}}$ *?*

Let MinDNF be the set of couples $\langle \phi, k \rangle$ s.t. there exists a DNF formula ϕ' of size $\leq k$ that is equivalent to the DNF formula ϕ.

1. Show that MinDNF $\in \mathsf{NP}^{\mathsf{NP}}$.
2. Show that $\mathsf{NP} = \mathsf{NP}^{\mathsf{NP}}$ implies $\mathsf{NP} = \mathsf{coNP}$.
3. Show that more generally $\Sigma_i^{\mathsf{P}} = \Pi_i^{\mathsf{P}}$ or $\Sigma_i^{\mathsf{P}} = \Sigma_{i+1}^{\mathsf{P}}$ implies the collapse of the polynomial hierarchy.

Exercise 4. $\Sigma\Pi\Delta$

Let $\Delta_{i+1}^{\mathsf{P}} = \mathsf{P}^{\Sigma_i^{\mathsf{P}}} = \mathsf{P}^{\Pi_i^{\mathsf{P}}}$.

1. Show that both definitions are equivalent.
2. Show that $\Sigma_i^{\mathsf{P}} \cup \Pi_i^{\mathsf{P}} \subseteq \Delta_{i+1}^{\mathsf{P}} \subseteq \Sigma_{i+1}^{\mathsf{P}} \cup \Pi_{i+1}^{\mathsf{P}}$.
3. Show that $\Sigma_i^{\mathsf{P}} \cup \Pi_i^{\mathsf{P}} = \Delta_i^{\mathsf{P}}$ implies the collapse of the polynomial hierarchy.
4. Show that Δ_i^{P} is closed under polynomial time reduction, Cook-Turing polynomial time reduction and complementation.
5. Show that $\{\phi(x_1, \ldots, x_n) : \exists!(a_1, \ldots, a_n), \phi(a_1, \ldots, a_n) = 1\}$ belongs to Δ_2^{P}.

Exercise 5. *Let's draw !*

✎ Give all the known relations between the following complexity classes (as a graph for instance), with a complete problem if possible, and with quick justifications: P, NP, coNP, EXP, NEXP, coNEXP, L, NL, coNL, PSPACE, NPSPACE, coNPSPACE, EXPSPACE, NEXPSPACE, coNEXPSPACE, Σ_1^{P}, Π_1^{P}, Δ_2^{P}, Σ_2^{P}, Π_2^{P}, Σ_i^{P}, Π_i^{P}, Δ_i^{P} ($i > 2$), PH.
Note. $\mathsf{EXP} = \bigcup_c \mathsf{DTIME}(2^{n^c})$, and NEXP, EXPSPACE, NEXPSPACE are similarly defined.