Tutorial 10 – Some advice

Exercise 1.
Reminder: Notations NC^k and NC stand for L-**uniform** classes here.

1. Show that PARITY $\in \mathsf{NC}^1$.
2. Let (a_{ij}) and (b_{ij}) be two $m \times m$ boolean matrices. Their **boolean product** is the matrix (c_{ij}) defined by $c_{ij} = \bigvee_k (a_{ik} \wedge b_{kj})$. Show that the boolean product belongs to FNC^1 (defined as NC^1 but with multiple-output circuits).
3. Show that $\mathsf{NC}^1 \subseteq \mathsf{L}$.
4. Generalize: $\mathsf{NC} \subseteq \mathsf{polyL} = \bigcup_k \mathsf{SPACE}(\log^k n)$.
5. What does this imply for the language TQBF ?
6. Show that $\mathsf{NL} \subseteq \mathsf{NC}^2$.

Exercise 2. *Advice and Oracle*
You learned in a lecture that every tally set belongs to $\mathsf{P/poly}$. A natural generalization of tally sets is given by sparse sets (see Mahaney's theorem for instance).

✎ Show that $\mathsf{P/poly} = \bigcup_{\text{sparse } L} P^L$.

Exercise 3. *Diagonal*
In the same lecture, you proved that there exist undecidable languages in $\mathsf{P/poly}$.

✎ Show that there exist decidable languages outside $\mathsf{P/poly}$.
Hint. Diagonalization on size-$n^{\log n}$ circuits.

Exercise 4. P-Sel*'s back!*
We show in the following that P-$\mathsf{Sel} \subseteq \mathsf{P/poly}$ [Ko, 1983]. The class P-Sel is the set of languages L which have a polynomial-time computable selector f: f outputs one of its both inputs, and $f(x,y) \in L$ whenever it is possible. Let $L \in \mathsf{P}$-Sel and f its selector.

1. Show that f can be supposed to be symmetric: for all x, y, $f(x,y) = f(y,x)$.
2. A **tournament** is a complete graph in which each edge has been directed. Show that if $G = (V,E)$ is a k-vertex tournament, then there exists a subset U of the vertices, of cardinality at most $\lfloor \log k + 1 \rfloor$, s.t. for every $v \in V \setminus U$, there exists $u \in U$ s.t. $(v,u) \in E$.
3. Let $L^{=n} = L \cap \{0,1\}^n$. Show that there exists $A_n \subseteq L^{=n}$, of cardinality at most $(n+1)$, s.t. $x \in L^{=n}$ if and only if there exists $y \in A_n$ s.t. $f(x,y) = x$.
4. Conclude

Note. It is possible to show [1] that P-$\mathsf{Sel} \subseteq \mathsf{NP/lin} \cap \mathsf{coNP/lin}$, where lin is the set of linear functions from $\mathbb{N}$ to $\mathbb{N}$. One can even show that P-$\mathsf{Sel} \subseteq \mathsf{NP}/\{n \mapsto n+1\}$. On the other hand, P-$\mathsf{Sel} \not\subseteq \mathsf{NP}/\{n \mapsto n\}$ (much harder!).

1. And it is a good training! It uses the fact that in a tournament, there exists a vertex s from which every vertex is reachable by a path of length at most two.